N I a

KUMON

N

Arithmetic Sequences

Name

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A succession of numbers which are arranged in a specific order according [illegible] rule is called a ***sequence***. Each number in a sequence is called a ***term*** and the [illegible] number from the beginning is called the ***n^{th} term***. (For example, the initial term [illegible] called the ***1^{st} term***.)

1. Fill in the blanks with the appropriate number for each given sequence.

(1) 2, 5, 8, $\boxed{11}$, 14, 17, ...

(2) 2, 4, 8, $\boxed{16}$, 32, 64, ...

(3) $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, $\boxed{\frac{5}{6}}$, $\boxed{\frac{6}{7}}$, ...

(4) 4, 2, $\boxed{0}$, −2, $\boxed{-4}$, −6, ...

(5) 1, 4, 9, 16, $\boxed{25}$, 36, ...

2. Find the terms for each given sequence where the n^{th} term is expresse[illegible] follows by substituting $n=1$, 2, 3, ..., 6.

	1^{st} term	2^{nd} term	3^{rd} term	4^{th} term	5^{th} term	6^{th} term
[illegible]-1	2	5	8	11	14	17
[illegible]	2	4	8	16	32	64
[illegible]	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{5}{6}$	$\frac{6}{7}$
[illegible]$2n$	4	2	0	−2	−4	−6
n^2	1	4	9	16	25	36
(6) $(-1)^n$	−1	1	−1	1	−1	1

N Ib

Generally, a sequence is expressed as follows:

$a_1, \quad a_2, \quad a_3, \quad \cdots, \quad a_n, \quad \cdots$

$\uparrow \quad \uparrow \quad \uparrow \quad \quad \uparrow$

1st term 2nd term 3rd term ... nth term ...

This sequence is also expressed as $\{a_n\}$.
When the nth term a_n of a sequence $\{a_n\}$ is expressed in terms of n, it is called the ***general term*** of $\{a_n\}$.

1. Choose the general term of each given sequence $\{a_n\}$ from (A)~(H) and find the 10th term.

Ex. 3, 5, 7, 9, 11, ...

[Sol] a_n: (A) $\quad a_{10}=21$

(1) 7, 4, 1, -2, -5, ...

[Sol] a_n: (E) $\quad a_{10}=-20$

(2) 2, 4, 8, 16, 32, ...

[Sol] a_n: (C) $\quad a_{10}=1024$

(3) 1, -1, 1, -1, 1, ...

[Sol] a_n: (F) $\quad a_{10}=-1$

(4) $\frac{1}{4}, \frac{2}{5}, \frac{3}{6}, \frac{4}{7}, \frac{5}{8}, \cdots$

[Sol] a_n: (H) $\quad a_{10}=\frac{10}{13}$

(A) $2n+1$ (B) $-2n+9$ (C) 2^n (D) $(-1)^n$

(E) $-3n+10$ (F) $-(-1)^n$ (G) $\frac{n}{2n+2}$ (H) $\frac{n}{n+3}$

N2a

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Arithmetic Sequences

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A sequence whose terms are found by successively adding a fixed nu
the 1st term a is called an ***arithmetic sequence*** (or ***arithmetic progres***
d is called the ***common difference*** of the arithmetic sequence.

1. Find the 10th term of the following arithmetic sequence $\{a_n\}$.
   10, 13, 16, 19, 22, ...

[Sol] $a_1 = 10$

$a_2 = 10 + \boxed{1} \cdot 3 = 13$

$a_3 = 10 + \boxed{2} \cdot 3 = 16$

$a_4 = 10 + \boxed{3} \cdot 3 = 19$

$\vdots$

$a_{10} = 10 + \boxed{9} \cdot 3 = \boxed{37}$

2. Find the general term of the following arithmetic sequence $\{a_n\}$.
   8, 4, 0, −4, −8, ...

[Sol] $a_1 = 8$

$a_2 = 8 + \boxed{1} \cdot (-4) = 4$

$a_3 = 8 + \boxed{2} \cdot (-4) = 0$

$a_4 = 8 + \boxed{3} \cdot (-4) = -4$

$\vdots$

$a_{n-1} = 8 + (n-2) \cdot (-4) = -4n + 16$

$a_n = 8 + (\boxed{n-1}) \cdot (-4) = \boxed{-4n+12}$

3. Find the general term of the following arithmetic sequence $\{a_n\}$ wit
   a and common difference d.

[Sol] $a_1 = a$

$a_2 = a + 1\,d$

$a_3 = a + 2\,d$

$a_4 = a + 3\,d$

$\vdots$

$a_n = a + (\boxed{n-1})d$ ←

$$a_1,\ a_2,\ a_3,\ \ldots,\ a_{n-1},\ a_n$$
$$\underbrace{+d\ \ +d\ \ +d\ \ \ldots\ +d\ \ \ +d}$$
The number of $+d$ is $(n-1)$.

N2b

General Term of an Arithmetic Sequence

The general term of an arithmetic sequence $\{a_n\}$ with 1st term a and common difference d is

$$a_n=a+(n-1)d$$

Ex. Find the 1st term a and the general term of the arithmetic sequence $\{a_n\}$ whose common difference is -2 and 6th term is 8.

[Sol] $a_6=a+5\cdot(-2)=8$

$\therefore\ a=18$

Also, $a_n=18+(n-1)\cdot(-2)=-2n+20$

Find the 1st term a and the general term of the arithmetic sequence $\{a_n\}$ whose common difference is -3 and 11th term is -27.

[Sol] $a_{11}=a+10\cdot(-3)=-27$

$\therefore\ a=3$

Also, $a_n=3+(n-1)\cdot(-3)=-3n+6$

Find the common difference d and the general term of the arithmetic sequence $\{a_n\}$ whose 1st term is 20 and 15th term is 90.

[Sol] $a_{15}=20+14d=90$

$\therefore\ d=5$

Also, $a_n=20+(n-1)\cdot 5=5n+15$

Given the arithmetic sequence $\{a_n\}$ whose 1st term is -25 and common difference is 5, find the value of n for which the nth term is -5.

[Sol] $a_n=-25+(n-1)\cdot 5=5n-30$

Since $a_n=-5$,

$5n-30=-5$

$\therefore\ n=5$

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Ex. Find the general term of the arithmetic sequence $\{a_n\}$ whose 5^{th} term is -5 and 9^{th} term is 11.

[Sol] Let a be the 1^{st} term and d be the common difference.

$$\begin{cases} a_5=a+4d=-5 & \cdots① \\ a_9=a+8d=11 & \cdots② \end{cases}$$

From ① and ②,

$a=-21,\ d=4$

$\therefore\ a_n=-21+(n-1)\cdot 4=4n-25$

1. Find the general term of the arithmetic sequence $\{a_n\}$ whose 5^{th} term is -3 and 14^{th} term is 24.

[Sol] Let a be the 1^{st} term and d be the common difference.

$$\begin{cases} a_5=a+4d=-3 & \cdots① \\ a_{14}=a+13d=24 & \cdots② \end{cases}$$

From ① and ②,

$a=-15,\ d=3$

$\therefore\ a_n=-15+(n-1)\cdot 3=3n-18$

2. Find the general term of the arithmetic sequence $\{a_n\}$ whose 3^{rd} term is 7 and 8^{th} term is -8.

[Sol] Let a be the 1^{st} term and d be the common difference.

$$\begin{cases} a_3=a+2d=7 & \cdots① \\ a_8=a+7d=-8 & \cdots② \end{cases}$$

From ① and ②,

$a=13,\ d=-3$

$\therefore\ a_n=13+(n-1)\cdot(-3)=-3n+16$

N3b

3. Find the 55^{th} term of the arithmetic sequence $\{a_n\}$ whose 15^{th} term is 77 and 42^{nd} term is 239.

Sol] Let a be the 1^{st} term and d be the common difference.

$$\begin{cases} a_{15}=a+14d=77 & \cdots ① \\ a_{42}=a+41d=239 & \cdots ② \end{cases}$$

From ① and ②,

$a=-7,\ d=6$

$\therefore\ a_{55}=-7+54\cdot 6=317$

Given the arithmetic sequence $\{a_n\}$ whose 53^{rd} term is -47 and 77^{th} term is -95, find the value of n for which the n^{th} term is -111.

Sol] Let a be the 1^{st} term and d be the common difference.

$$\begin{cases} a_{53}=a+52d=-47 & \cdots ① \\ a_{77}=a+76d=-95 & \cdots ② \end{cases}$$

From ① and ②,

$a=57,\ d=-2$

$\therefore\ a_n=57+(n-1)\cdot(-2)=-2n+59$

Since $a_n=-111$,

$-2n+59=-111$

$\therefore\ n=85$

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Ex. Find the value of x when the sequence 1, x, 8, ... is an arithmet
sequence.

[Sol] Since the sequence 1, x, 8 is an arithmetic sequence,

$$x-1=8-x$$
$$\therefore\ x=\frac{9}{2}$$

1. Find the value of x when the sequence $\frac{5}{6}$, x, $\frac{5}{2}$, ... is an arithmeti
sequence.

[Sol] Since the sequence $\frac{5}{6}$, x, $\frac{5}{2}$ is an arithmetic sequence,

$$x-\frac{5}{6}=\frac{5}{2}-x$$
$$\therefore\ x=\frac{5}{3}$$

2. Find the value of x when the sequence x, 6, $2x$, ... is an arithmetic s

[Sol] Since the sequence x, 6, $2x$ is an arithmetic sequence,

$$6-x=2x-6$$
$$\therefore\ x=4$$

3. Find the values of x and y when the sequence 1, x, y, -11, ... is
arithmetic sequence.

[Sol] Since the sequence 1, x, y, -11 is an arithmetic sequence,

$$\begin{cases} x-1=y-x & \cdots ① \\ y-x=-11-y & \cdots ② \end{cases}$$

From ① and ②,

$$x=-3,\ y=-7$$

N4b

Ex. Prove that the sequence $\{a_n\}$ where $a_n=5n+1$ is an arithmetic sequence. Then, find the 1st term and the common difference.

[Sol] Since $a_n=5n+1$, $a_{n+1}=5(n+1)+1=5n+6$

$\therefore a_{n+1}-a_n=(5n+6)-(5n+1)=5$

Since $a_{n+1}-a_n$ is fixed as 5, $\{a_n\}$ is an arithmetic sequence.

If $a_{n+1}-a_n=d$ (fixed), then it is an arithmetic sequence.

Also, $a_1=5\cdot1+1=6$

Therefore, the 1st term is 6 and the common difference is 5.

4. Prove that the sequence $\{a_n\}$ where $a_n=4n+3$ is an arithmetic sequence. Then, find the 1st term and the common difference.

[Sol] Since $a_n=4n+3$, $a_{n+1}=4(n+1)+3=4n+7$

$\therefore a_{n+1}-a_n=(4n+7)-(4n+3)=4$

Since $a_{n+1}-a_n$ is fixed as 4, $\{a_n\}$ is an arithmetic sequence.

Also, $a_1=4\cdot1+3=7$

Therefore, the 1st term is 7 and the common difference is 4.

5. Given that p and q are constants, prove that the sequence $\{a_n\}$ where $a_n=pn+q$ is an arithmetic sequence. Then, find the 1st term and the common difference.

[Sol] Since $a_n=pn+q$, $a_{n+1}=p(n+1)+q=pn+p+q$

$\therefore a_{n+1}-a_n=(pn+p+q)-(pn+q)=p$

Since $a_{n+1}-a_n$ is fixed as p, $\{a_n\}$ is an arithmetic sequence.

Also, $a_1=p\cdot1+q=p+q$

Therefore, the 1st term is $p+q$ and the common difference is p.

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Ex. Find the three numbers whose sum is 15 and product is 80, which f
arithmetic sequence.

[Sol] Let c be the middle term and d be the common difference.

Since the three numbers are expressed as $c-d,\ c,\ c+d,$

$$\begin{cases}(c-d)+c+(c+d)=15 & \cdots① \\ (c-d)c(c+d)=80 & \cdots②\end{cases}$$

From ① and ②,

From ①, $c=5$
Substituting this in

$c=5,\ d=\pm3$

2, 5, 8 when
and $d=-3$

Therefore, the three numbers are: 2, 5, 8

1. Find the three numbers whose sum is 27 and product is 693, which
arithmetic sequence.

[Sol] Let c be the middle term and d be the common difference.

Since the three numbers are expressed as $c-d,\ c,\ c+d,$

$$\begin{cases}(c-d)+c+(c+d)=27 & \cdots① \\ (c-d)c(c+d)=693 & \cdots②\end{cases}$$

From ① and ②,

$c=9,\ d=\pm2$

Therefore, the three numbers are: 7, 9, 11

N5b

2. Find the 1[st] term and the common difference of the arithmetic sequence $\{a_n\}$, given $a_1+a_2+a_3=-15$ and $a_1a_2a_3=120$.

[Sol] Let c be a_2 and d be the common difference.

Since the three numbers are expressed as $a_1=c-d$, $a_2=c$, $a_3=c+d$,

$$\begin{cases}(c-d)+c+(c+d)=-15 & \cdots① \\ (c-d)c(c+d)=120 & \cdots②\end{cases}$$

From ① and ②,

$c=-5$, $d=\pm7$

When $d=7$, $a_1=-5-7=-12$

When $d=-7$, $a_1=-5-(-7)=2$ ← The 1[st] term varies with d; therefore, consider the different cases of d.

Therefore, the 1[st] term is -12 and the common difference is 7; or the 1[st] term is 2 and the common difference is -7.

3. Find the 1[st] term and the common difference of the arithmetic sequence $\{a_n\}$, given $a_1+a_3+a_5=-12$ and $a_1a_3a_5=80$.

[Sol] Let c be a_3 and d be the common difference.

Since the three numbers are expressed as $a_1=c-2d$, $a_3=c$, $a_5=c+2d$,

$$\begin{cases}(c-2d)+c+(c+2d)=-12 & \cdots① \\ (c-2d)c(c+2d)=80 & \cdots②\end{cases}$$

From ① and ②,

$c=-4$, $d=\pm3$

When $d=3$, $a_1=-4-2\cdot3=-10$

When $d=-3$, $a_1=-4-2\cdot(-3)=2$

Therefore, the 1[st] term is -10 and the common difference is 3; or the 1[st] term is 2 and the common difference is -3.

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Ex. Find the sum S_6 of the arithmetic sequence whose 1st term is 1, commo difference is 2 and number of terms is 6.

[Sol]

$$\begin{array}{rl} S_6 = & 1+3+5+7+9+11 \\ +)\ S_6 = & 11+9+7+5+3+1 \\ \hline 2S_6 = & 12+12+12+12+12+12 \end{array}$$

$\therefore\ 2S_6 = 12 \cdot 6$

$\therefore\ S_6 = 36$

$\leftarrow$ $S_n = a_1 + a_2 + a_3 + \cdots + a$

$\leftarrow$ $S_n = a_n + a_3 + a_4 + \cdots + a$

S_n will be the same eve if the order of terms i reversed.

1. Find the sum S_8 of the arithmetic sequence whose 1st term is 1, common difference is 4 and number of terms is 8.

[Sol]

$$\begin{array}{rl} S_8 = & 1+5+9+13+17+21+25+29 \\ +)\ S_8 = & 29+25+21+17+13+9+5+1 \\ \hline 2S_8 = & 30+30+30+30+30+30+30+30 \end{array}$$

$\therefore\ 2S_8 = 30 \cdot 8$

$\therefore\ S_8 = 120$

Find the sum S_n of the arithmetic sequence with 1st term a, common differenc last term l and number of terms n.

[Sol]

$$\begin{array}{rl} S_n = & a + (a+d) + (a+2d) + \cdots + (l-d) + l \\ +)\ S_n = & l + (l-d) + (l-2d) + \cdots + (a+d) + a \\ \hline 2S_n = & \underbrace{(a+l) + (a+l) + (a+l) + \cdots + (a+l) + (a+l)} \end{array}$$

The number of $(a+l)$ is n.

$\therefore\ 2S_n = \boxed{n}\,(a+l)$

$\therefore\ S_n = \frac{1}{2}\boxed{n}\,(a+l)$

Also, since l is the n^{th} term, $l = a + (n-1)d$

$\therefore\ S_n = \frac{1}{2}\boxed{n[2a+(n-1)d]}$

Answers: n, n, $n[2a+(n-1)d]$

Sum of an Arithmetic Sequence

Let S_n be the sum of an arithmetic sequence with 1st term a, common difference d, last term l and number of terms n.

$$S_n=\frac{1}{2}n(a+l)=\frac{1}{2}n[2a+(n-1)d]$$

2. Using the formula above, find the sum of each given arithmetic sequence.

Ex. 1st term: 3, last term: 21, number of terms: 10

[Sol] $\frac{1}{2}\cdot 10(3+21)=120$

1st term: 5, common difference: 4, number of terms: 12

[Sol] $\frac{1}{2}\cdot 12[2\cdot 5+(12-1)\cdot 4]=324$

(1) 1st term: 2, last term: 34, number of terms: 13

[Sol] $\frac{1}{2}\cdot 13(2+34)=234$

(2) 1st term: -3, last term: -42, number of terms: 20

[Sol] $\frac{1}{2}\cdot 20[-3+(-42)]=-450$

(3) 1st term: 4, common difference: 3, number of terms: 15

[Sol] $\frac{1}{2}\cdot 15[2\cdot 4+(15-1)\cdot 3]=375$

(4) 1st term: -2, common difference: 5, number of terms: 18

[Sol] $\frac{1}{2}\cdot 18[2\cdot(-2)+(18-1)\cdot 5]=729$

Ex. S_7, the sum of the first 7 terms of the arithmetic sequence 5, 9, 13, 17, …

[Sol] $S_7=\frac{1}{2}\cdot 7[2\cdot 5+(7-1)\cdot 4]=119$ ← 1st term: 5, common difference: 4, number of terms: 7

(5) S_{11}, the sum of the first 11 terms of the arithmetic sequence -9, -3, 3, 9, …

[Sol] $S_{11}=\frac{1}{2}\cdot 11[2\cdot(-9)+(11-1)\cdot 6]=231$

(6) S_{20}, the sum of the first 20 terms of the arithmetic sequence 50, 47, 44, 41, …

[Sol] $S_{20}=\frac{1}{2}\cdot 20[2\cdot 50+(20-1)\cdot(-3)]=430$

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Ex. Find the general term of the arithmetic sequence $\{a_n\}$ whose sum of the first 4 terms is 80 and whose sum of the first 8 terms is 320.

[Sol] Let a be the 1st term, d be the common difference and S_n be the sum of the first n terms.

$$\begin{cases} S_4=\frac{1}{2}\cdot 4[2a+(4-1)d]=80 & \cdots① \\ S_8=\frac{1}{2}\cdot 8[2a+(8-1)d]=320 & \cdots② \end{cases}$$

From ① and ②,

$a=5,\ d=10$

$\therefore\ a_n=5+(n-1)\cdot 10=10n-5$

1. Find the general term of the arithmetic sequence $\{a_n\}$ whose sum of the first 5 terms is 20 and whose sum of the first 7 terms is 7.

[Sol] Let a be the 1st term, d be the common difference and S_n be the sum of the first n terms.

$$\begin{cases} S_5=\frac{1}{2}\cdot 5[2a+(5-1)d]=20 & \cdots① \\ S_7=\frac{1}{2}\cdot 7[2a+(7-1)d]=7 & \cdots② \end{cases}$$

From ① and ②,

$a=10,\ d=-3$

$\therefore\ a_n=10+(n-1)\cdot(-3)=-3n+13$

N7b

2. Find the sum S of the first 30 terms of the arithmetic sequence (a_n) whose sum of the first 10 terms is 100 and whose sum of the first 20 terms is 350.

[Sol] Let a be the 1st term, d be the common difference and S_n be the sum of the first n terms.

$$\begin{cases} S_{10}=\frac{1}{2}\cdot 10[2a+(10-1)d]=100 & \cdots ① \\ S_{20}=\frac{1}{2}\cdot 20[2a+(20-1)d]=350 & \cdots ② \end{cases}$$

From ① and ②,

$$a=\frac{13}{4},\ d=\frac{3}{2}$$

$$\therefore S=\frac{1}{2}\cdot 30\left[2\cdot\frac{13}{4}+(30-1)\cdot\frac{3}{2}\right]=750$$

3. Find the sum S from the 11th to the 20th term of the arithmetic sequence (a_n) whose 8th term is 37 and 24th term is 117.

[Sol] Let a be the 1st term, d be the common difference and S_n be the sum of the first n terms.

$$\begin{cases} a_8=a+7d=37 & \cdots ① \\ a_{24}=a+23d=117 & \cdots ② \end{cases}$$

From ① and ②,

$$a=2,\ d=5$$

$$\therefore S_{20}=\frac{1}{2}\cdot 20[2\cdot 2+(20-1)\cdot 5]=990$$

$$S_{10}=\frac{1}{2}\cdot 10[2\cdot 2+(10-1)\cdot 5]=245$$

$$\therefore S=S_{20}-S_{10}=990-245=745 \leftarrow \overbrace{\underbrace{a_1+a_2+\cdots+a_{10}}_{245}+\underbrace{a_{11}+\cdots+a_{20}}_{S}}^{990}$$

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Ex. Given the arithmetic sequence $\{a_n\}$ whose 1st term is 53 and comm difference is -6, find the value of m for which the sum of the first terms is the greatest. Then, find the sum S.

[Sol] $a_n=53+(n-1)\cdot(-6)=-6n+59$

$-6n+59>0$ when

$n<\frac{59}{6}=9.8\ldots$

m is the greatest natural number that satisfies this.

$\therefore\ m=9$

Also, $S=\frac{1}{2}\cdot 9[2\cdot 53+(9-1)\cdot(-6)]=261$

In this arithmetic seque
$\underbrace{53,\ 47,\ \ldots,\ 5}_{a_n>0},\ \underbrace{-1,}_{a_n}$

The sum is the greatest when $53+47+\cdots+5$.

1. Given the arithmetic sequence $\{a_n\}$ whose 1st term is 58 and common di is -3, find the value of m for which the sum of the first m terms is the g Then, find the sum S.

[Sol] $a_n=58+(n-1)\cdot(-3)=-3n+61$

$-3n+61>0$ when

$n<\frac{61}{3}=20.3\ldots$

m is the greatest natural number that satisfies this.

$\therefore\ m=20$

Also, $S=\frac{1}{2}\cdot 20[2\cdot 58+(20-1)\cdot(-3)]=590$

Integers that are greater than or equal to 1 (e.g. 1, 2, 3, ...) are called *natural numbe*

N8b

2. Given the arithmetic sequence $\{a_n\}$ whose 1^{st} term is -120 and common difference is 7, find the value of m for which the sum of the first m terms is the smallest. Then, find the sum S.

[Sol] $a_n=-120+(n-1)\cdot 7=7n-127$

In this arithmetic sequence,

$\underbrace{-120,\ -113,\ \ldots,\ -1}_{a_n<0},\ \underbrace{6,\ 13,\ \ldots}_{a_n>0}$

The sum is the smallest when $(-120)+(-113)+\cdots+(-1)$.

$7n-127<0$ when

$n<\frac{127}{7}=18.1\ldots$

m is the greatest natural number that satisfies this.

$\therefore\ m=18$

Also, $S=\frac{1}{2}\cdot 18[2\cdot(-120)+(18-1)\cdot 7]=-1089$

3. Given the arithmetic sequence $\{a_n\}$ whose sum of the first 5 terms is -445 and whose sum of the first 10 terms is -765, find the value of m for which the sum of the first m terms is the smallest. Then, find the sum S.

[Sol] Let a be the 1^{st} term, d be the common difference and S_n be the sum of the first n terms.

$$\begin{cases} S_5=\frac{1}{2}\cdot 5[2a+(5-1)d]=-445 & \cdots① \\ S_{10}=\frac{1}{2}\cdot 10[2a+(10-1)d]=-765 & \cdots② \end{cases}$$

From ① and ②,

$a=-99,\ d=5$

$\therefore\ a_n=-99+(n-1)\cdot 5=5n-104$

$5n-104<0$ when

$n<\frac{104}{5}=20.8$

m is the greatest natural number that satisfies this.

$\therefore\ m=20$

Also, $S=\frac{1}{2}\cdot 20[2\cdot(-99)+(20-1)\cdot 5]=-1030$

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Ex. Find the sum S of all natural numbers from 10 to 100 which are d
by 3.

[Sol] The general term of the sequence of natural numbers greater
equal to 10 which are divisible by 3 is:

$12+(n-1)\cdot 3=3n+9$ ← 1st term: 12, common differen

$3n+9 \leqslant 100$ when $n \leqslant \frac{91}{3}=30.3\ldots$

The greatest natural number n that satisfies this is $n=30$.

$\therefore\ S=\frac{1}{2}\cdot 30[2\cdot 12+(30-1)\cdot 3]=1665$

1. Find the sum S of all natural numbers from 30 to 100 which are divisi

[Sol] The general term of the sequence of natural numbers greater than
to 30 which are divisible by 6 is:

$30+(n-1)\cdot 6=6n+24$

$6n+24 \leqslant 100$ when $n \leqslant \frac{76}{6}=12.6\ldots$

The greatest natural number n that satisfies this is $n=12$.

$\therefore\ S=\frac{1}{2}\cdot 12[2\cdot 30+(12-1)\cdot 6]=756$

2. Find the sum S of all natural numbers from 10 to 200 which leave a rem
of 4 when divided by 7.

[Sol] The general term of the sequence of natural numbers greater than o
to 10 which leave a remainder of 4 when divided by 7 is:

$11+(n-1)\cdot 7=7n+4$

$7n+4 \leqslant 200$ when $n \leqslant \frac{196}{7}=28$

The greatest natural number n that satisfies this is $n=28$.

$\therefore\ S=\frac{1}{2}\cdot 28[2\cdot 11+(28-1)\cdot 7]=2954$

N9b

3. Find the sum S of all natural numbers from 100 to 200 which are divisible by either 3 or 5.

[Sol] The general term of the sequence of natural numbers greater than or equal to 100 which are divisible by 3 is:

$$102+(n-1)\cdot 3=3n+99$$

$3n+99 \leqslant 200$ when $n \leqslant \frac{101}{3}=33.6\ldots$

The greatest natural number n that satisfies this is $n=33$.
Therefore, the sum of natural numbers from 100 to 200 which are divisible by 3 is:

$$\frac{1}{2}\cdot 33[2\cdot 102+(33-1)\cdot 3]=4950 \quad \cdots①$$

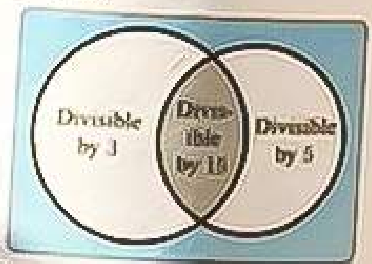
The general term of the sequence of natural numbers greater than or equal to 100 which are divisible by 5 is:

$$100+(n-1)\cdot 5=5n+95$$

$5n+95 \leqslant 200$ when $n \leqslant \frac{105}{5}=21$

The greatest natural number n that satisfies this is $n=21$.
Therefore, the sum of natural numbers from 100 to 200 which are divisible by 5 is:

$$\frac{1}{2}\cdot 21[2\cdot 100+(21-1)\cdot 5]=3150 \quad \cdots②$$



The general term of the sequence of natural numbers greater than or equal to 100 which are divisible by 15 is:

$$105+(n-1)\cdot 15=15n+90$$

$15n+90 \leqslant 200$ when $n \leqslant \frac{110}{15}=7.3\ldots$

The greatest natural number n that satisfies this is $n=7$.
Therefore, the sum of natural numbers from 100 to 200 which are divisible by 15 is:

$$\frac{1}{2}\cdot 7[2\cdot 105+(7-1)\cdot 15]=1050 \quad \cdots③$$

From ①~③,

$$S=4950+3150-1050$$
$$=7050$$

Since ③ is included in both ① and ②, ③ has to be subtracted when finding S.

N10a

KUMON

Arithmetic Sequences

Name

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
0			1	2~

1. Find the general term of the arithmetic sequence $\{a_n\}$ whose 3rd ter and 8th term is 50.

[Sol] Let a be the 1st term and d be the common difference.

$$\begin{cases} a_3 = a + 2d = 10 & \cdots ① \\ a_8 = a + 7d = 50 & \cdots ② \end{cases}$$

From ① and ②,

$a = -6,\ d = 8$

$\therefore\ a_n = -6 + (n-1)\cdot 8 = 8n - 14$

2. Find the three numbers whose sum is 18 and product is 162, which f arithmetic sequence.

[Sol] Let c be the middle term and d be the common difference.

Since the three numbers are expressed as $c-d,\ c,\ c+d,$

$$\begin{cases} (c-d)+c+(c+d)=18 & \cdots ① \\ (c-d)c(c+d)=162 & \cdots ② \end{cases}$$

From ① and ②,

$c = 6,\ d = \pm 3$

Therefore, the three numbers are: 3, 6, 9

3. Find the general term of the arithmetic sequence $\{a_n\}$ whose sum of the first 5 terms is 100 and whose sum of the first 10 terms is 150.

➡ N7

[Sol] Let a be the 1st term, d be the common difference and S_n be the sum of the first n terms.

$$\begin{cases} S_5=\frac{1}{2}\cdot 5[2a+(5-1)d]=100 & \cdots① \\ S_{10}=\frac{1}{2}\cdot 10[2a+(10-1)d]=150 & \cdots② \end{cases}$$

From ① and ②,

$a=24,\ d=-2$

$\therefore a_n=24+(n-1)\cdot(-2)=-2n+26$

4. Given the arithmetic sequence $\{a_n\}$ whose 1st term is 50 and common difference is -3, find the value of m for which the sum of the first m terms is the greatest. Then, find the sum S.

➡ N8

[Sol] $a_n=50+(n-1)\cdot(-3)=-3n+53$

$-3n+53>0$ when

$n<\frac{53}{3}=17.6\ldots$

m is the greatest natural number that satisfies this.

$\therefore m=17$

Also, $S=\frac{1}{2}\cdot 17[2\cdot 50+(17-1)\cdot(-3)]=442$

N II a

KUMON

Geometric Sequences

Name

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
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A sequence whose terms are found by successively multiplying a fixed nur
the 1st term a is called a ***geometric sequence*** (or ***geometric progression***)
r is called the ***common ratio*** of the geometric sequence.

1. For each given geometric sequence $\{a_n\}$, find the common ratio r an
term.

Ex. 3, 6, 12, 24, ...

[Sol] $r=2$, $a_5=48$

(1) 1, 3, 9, 27, ...

[Sol] $r=3$, $a_5=81$

(2) $-\frac{1}{2}$, 1, -2, 4, ...

[Sol] $r=-2$, $a_5=-8$

(3) 4, -2, 1, $-\frac{1}{2}$, ...

[Sol] $r=-\frac{1}{2}$, $a_5=\frac{1}{4}$

(4) 5, -5, 5, -5, ...

[Sol] $r=-1$, $a_5=5$

(5) -3, -3, -3, -3, ...

[Sol] $r=1$, $a_5=-3$

NIIb

2. Find the 10^{th} term of the following geometric sequence $\{a_n\}$.

1, 2, 4, 8, 16, ...

[Sol] $a_1=1$

$a_2=1\cdot 2^{1}=2$

$a_3=1\cdot 2^{2}=4$

$a_4=1\cdot 2^{3}=8$

$\vdots$

$a_{10}=1\cdot 2^{\boxed{9}}=\boxed{512}$

1. Find the general term of the following geometric sequence $\{a_n\}$.

5, -10, 20, -40, 80, ...

Sol] $a_1=5$

$a_2=5(-2)^{1}=-10$

$a_3=5(-2)^{2}=20$

$a_4=5(-2)^{3}=-40$

$\vdots$

$a_{n-1}=5(-2)^{n-2}$

$a_n=5(-2)^{\boxed{n-1}}$

Find the general term of the following geometric sequence $\{a_n\}$ with 1^{st} term a and common ratio r.

ol] $a_1=a$

$a_2=ar^{1}$

$a_3=ar^{2}$

$a_4=ar^{3}$

$\vdots$

$a_n=ar^{\boxed{n-1}}$ ←

$a_1, a_2, a_3, \cdots, a_{n-1}, a_n$

$\times r\ \times r\ \times r \cdots \times r\ \times r$

The number of $\times r$ is $(n-1)$.

Note) $r^0=1$

KUMON

Geometric Sequences

Name

Date / /

Time : to :

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General Term of a Geometric Sequence

The general term of a geometric sequence $\{a_n\}$ with 1^{st} term a and co
ratio r is

$$a_n = ar^{n-1}$$

1. Find the general term and the 6^{th} term of each given geometric sequenc

Ex. 3, 6, 12, 24, …

[Sol] Since the 1^{st} term is 3 and the common ratio is 2,

$$a_n = 3 \cdot 2^{n-1}$$

$$a_6 = 3 \cdot 2^5 = 96$$

(1) 2, 6, 18, 54, …

[Sol] Since the 1^{st} term is 2 and the common ratio is 3,

$$a_n = 2 \cdot 3^{n-1}$$

$$a_6 = 2 \cdot 3^5 = 486$$

(2) -1, -4, -16, -64, …

[Sol] Since the 1^{st} term is -1 and the common ratio is 4,

$$a_n = -1 \cdot 4^{n-1} = -4^{n-1}$$

$$a_6 = -4^5 = -1024$$

(3) -2, 4, -8, 16, …

[Sol] Since the 1^{st} term is -2 and the common ratio is -2,

$$a_n = -2(-2)^{n-1} = (-2)^n$$

$$a_6 = (-2)^6 = 64$$

2. Find the 1^{st} term a of the geometric sequence $\{a_n\}$ whose common ratio is $-\frac{1}{3}$ and 3^{rd} term is 6.

[Sol] $a_3 = a\left(-\frac{1}{3}\right)^2 = 6$

$\therefore a = 54$

3. Find the common ratio r of the geometric sequence $\{a_n\}$ whose 1^{st} term is 3 and 3^{rd} term is 48.

[Sol] $a_3 = 3r^2 = 48$

$r^2 = 16$

$\therefore r = \pm 4$

4. Find the common ratio r of the geometric sequence $\{a_n\}$ whose 1^{st} term is 5 and 4^{th} term is 135. (r is a real number.)

[Sol] $a_4 = 5r^3 = 135$

$r^3 = 27$

$\therefore r = 3$

$r^3 - 27 = 0$
$(r-3)(r^2+3r+9) = 0$
Since r is a real number, $r = 3$

5. Given the geometric sequence $\{a_n\}$ whose 1^{st} term is 16 and common ratio is $\frac{3}{2}$, find the value of n for which the n^{th} term is 81.

[Sol] $a_n = 16\left(\frac{3}{2}\right)^{n-1}$

Since $a_n = 81$,

$16\left(\frac{3}{2}\right)^{n-1} = 81$

$\left(\frac{3}{2}\right)^{n-1} = \left(\frac{3}{2}\right)^4$

Rewriting with base $\frac{3}{2}$
(K192)

$\therefore n = 5$

N I 3a

KUMON

Geometric Sequences

Name

Date / /

Time : to :

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Ex. Find the general term of the geometric sequence $\{a_n\}$ whose 4^{th} [illegible] -24 and 6^{th} term is -96.

[Sol] Let a be the 1^{st} term and r be the common ratio.

$$\begin{cases} a_4 = ar^3 = -24 & \cdots ① \\ a_6 = ar^5 = -96 & \cdots ② \end{cases}$$

From ② , $ar^3 \cdot r^2 = -96$
Substituting ① into this

From ① and ②, $-24r^2 = -96$

$$r^2 = 4$$

$\therefore\ r = \pm 2$

(i) When $r = 2$, from ①, $a = -3$

$\therefore\ a_n = -3 \cdot 2^{n-1}$

(ii) When $r = -2$, from ①, $a = 3$

$\therefore\ a_n = 3(-2)^{n-1}$

From (i) and (ii), $a_n = -3 \cdot 2^{n-1}$ or $a_n = 3(-2)^{n-1}$

1. Find the general term of the geometric sequence $\{a_n\}$ whose 3^{rd} te[illegible] and 5^{th} term is 324.

[Sol] Let a be the 1^{st} term and r be the common ratio.

$$\begin{cases} a_3 = ar^2 = 36 & \cdots ① \\ a_5 = ar^4 = 324 & \cdots ② \end{cases}$$

From ① and ②, $36r^2 = 324$

$$r^2 = 9$$

$\therefore\ r = \pm 3$

(i) When $r = 3$, from ①, $a = 4$

$\therefore\ a_n = 4 \cdot 3^{n-1}$

(ii) When $r = -3$, from ①, $a = 4$

$\therefore\ a_n = 4(-3)^{n-1}$

From (i) and (ii), $a_n = 4 \cdot 3^{n-1}$ or $a_n = 4(-3)^{n-1}$

N13b

2. Find the general term of the geometric sequence $\{a_n\}$ whose 2^{nd} term is 6 and 6^{th} term is $\frac{3}{8}$. (The common ratio is a real number.)

[Sol] Let a be the 1^{st} term and r be the common ratio.

$$\begin{cases} a_2 = ar = 6 & \cdots ① \\ a_6 = ar^5 = \frac{3}{8} & \cdots ② \end{cases}$$

From ① and ②, $6r^4 = \frac{3}{8}$

$$r^4 = \frac{1}{16}$$

$$\therefore r = \pm\frac{1}{2}$$

$r^4 - \frac{1}{16} = 0$
$\left(r^2 + \frac{1}{4}\right)\left(r^2 - \frac{1}{4}\right) = 0$
$\left(r^2 + \frac{1}{4}\right)\left(r + \frac{1}{2}\right)\left(r - \frac{1}{2}\right) = 0$
Since r is a real number, $r = \pm\frac{1}{2}$

(i) When $r = \frac{1}{2}$, from ①, $a = 12$

$$\therefore a_n = 12\left(\frac{1}{2}\right)^{n-1}$$

(ii) When $r = -\frac{1}{2}$, from ①, $a = -12$

$$\therefore a_n = -12\left(-\frac{1}{2}\right)^{n-1}$$

From (i) and (ii), $a_n = 12\left(\frac{1}{2}\right)^{n-1}$ or $a_n = -12\left(-\frac{1}{2}\right)^{n-1}$

3. Given the geometric sequence $\{a_n\}$ whose 4^{th} term is $6\sqrt{3}$ and 7^{th} term is 54, find the value of n for which the n^{th} term is 162. (The common ratio is a real number.)

[Sol] Let a be the 1^{st} term and r be the common ratio.

$$\begin{cases} a_4 = ar^3 = 6\sqrt{3} & \cdots ① \\ a_7 = ar^6 = 54 & \cdots ② \end{cases}$$

From ① and ②, $6\sqrt{3}r^3 = 54$

$$r^3 = 3\sqrt{3}$$

$$\therefore r = \sqrt{3}$$

$r^3 - 3\sqrt{3} = 0$
$(r - \sqrt{3})(r^2 + \sqrt{3}r + 3) = 0$
Since r is a real number, $r = \sqrt{3}$

From ①, $a = 2$

$$\therefore a_n = 2(\sqrt{3})^{n-1}$$

Since $a_n = 162$,

$$2(\sqrt{3})^{n-1} = 162$$

$$3^{\frac{1}{2}(n-1)} = 3^4$$

$$\therefore n = 9$$

Rewriting with base 3 (K192)

$\frac{1}{2}(n-1) = 4$

Geometric Sequences

Name

Date / /

100%	~90%	~80%	~70%	69%~
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Time : to :

Ex. Find the 1st term a and the common ratio r of the geometric sequence $\{a_n\}$ whose 2nd term is 2 and whose sum of the first 3 terms is 7.

[Sol] $\begin{cases} ar=2 & \cdots ① \\ a+ar+ar^2=7 & \cdots ② \end{cases}$

From ②, $a(1+r+r^2)=7$

Multiplying both sides by r,

$ar(1+r+r^2)=7r \quad \cdots ③$

From ① and ③, $2(1+r+r^2)=7r$

$2r^2-5r+2=0$

$(2r-1)(r-2)=0$

$\therefore\ r=\frac{1}{2},\ 2$

(i) When $r=\frac{1}{2}$, from ①, $a=4$

(ii) When $r=2$, from ①, $a=1$

From (i) and (ii), $a=4,\ r=\frac{1}{2}$ or $a=1,\ r=2$

1. Find the 1st term a and the common ratio r of the geometric sequence $\{$ whose 2nd term is 3 and whose sum of the first 3 terms is 13.

[Sol] $\begin{cases} ar=3 & \cdots ① \\ a+ar+ar^2=13 & \cdots ② \end{cases}$

From ②, $a(1+r+r^2)=13$

Multiplying both sides by r,

$ar(1+r+r^2)=13r \quad \cdots ③$

From ① and ③, $3(1+r+r^2)=13r$

$3r^2-10r+3=0$

$(3r-1)(r-3)=0$

$\therefore\ r=\frac{1}{3},\ 3$

(i) When $r=\frac{1}{3}$, from ①, $a=9$

(ii) When $r=3$, from ①, $a=1$

From (i) and (ii), $a=9,\ r=\frac{1}{3}$ or $a=1,\ r=3$

NI4b

2. Find the 1st term a and the common ratio r of the geometric sequence $\{a_n\}$ whose 3rd term is 12 and whose sum of the first 3 terms is 21.

[Sol] $\begin{cases} ar^2=12 & \cdots ① \\ a+ar+ar^2=21 & \cdots ② \end{cases}$

From ②, $a(1+r+r^2)=21$

Multiplying both sides by r^2,

$ar^2(1+r+r^2)=21r^2 \quad \cdots ③$

From ① and ③, $12(1+r+r^2)=21r^2$

$3r^2-4r-4=0$

$(3r+2)(r-2)=0$

$\therefore r=-\frac{2}{3},\ 2$

(i) When $r=-\frac{2}{3}$, from ①, $a=27$

(ii) When $r=2$, from ①, $a=3$

From (i) and (ii), $a=27,\ r=-\frac{2}{3}$ or $a=3,\ r=2$

3. Find the 1st term a and the common ratio r of the geometric sequence $\{a_n\}$ whose sum of the first 3 terms is 35 and whose sum from the 3rd to the 5th term is 140.

[Sol] $\begin{cases} a+ar+ar^2=35 & \cdots ① \\ ar^2+ar^3+ar^4=140 & \cdots ② \end{cases}$

From ②, $r^2(a+ar+ar^2)=140 \quad \cdots ③$

From ① and ③, $35r^2=140$

$r^2=4$

$\therefore r=\pm 2$

(i) When $r=2$, from ①, $a=5$

(ii) When $r=-2$, from ①, $a=\frac{35}{3}$

From (i) and (ii), $a=5,\ r=2$ or $a=\frac{35}{3},\ r=-2$

Geometric Sequences

Name

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~

Ex. Find the value of x when the sequence 2, x, 5, … is a geometric sequence.

[Sol] Since the sequence 2, x, 5 is a geometric sequence,

2, x, 5

$\frac{x}{2}$ $\frac{5}{x}$

$$\frac{x}{2}=\frac{5}{x}$$

$$\therefore\ x^2=10$$

$$\therefore\ x=\pm\sqrt{10}$$

1. Find the value of x when the sequence 3, x, 9, … is a geometric sequen

[Sol] Since the sequence 3, x, 9 is a geometric sequence,

$$\frac{x}{3}=\frac{9}{x}$$

$$\therefore\ x^2=27$$

$$\therefore\ x=\pm 3\sqrt{3}$$

2. Find the values of x and y when the sequence x, -5, x, y, … is a geometric sequence.

[Sol] Since the sequence x, -5, x, y is a geometric sequence,

$$\begin{cases} \dfrac{-5}{x}=\dfrac{x}{-5} & \cdots ① \\ \dfrac{x}{-5}=\dfrac{y}{x} & \cdots ② \end{cases}$$

From ①, $x^2=25$ ⋯③

$\therefore\ x=\pm 5$

From ②, $x^2=-5y$ ⋯④

From ③ and ④, $25=-5y$

$\therefore\ y=-5$

Therefore, $x=5$, $y=-5$ or $x=-5$, $y=-5$

N15b

3. Find the values of x and y when the sequence $x,\ y,\ -4$ is an arithmetic sequence and the sequence $y,\ x,\ -4$ is a geometric sequence.

[Sol] Since the sequence $x,\ y,\ -4$ is an arithmetic sequence,

$$y-x=-4-y \quad \leftarrow \boxed{\text{N4}}$$

$$\therefore\ x=2y+4 \quad \cdots ①$$

Since the sequence $y,\ x,\ -4$ is a geometric sequence,

$$\frac{x}{y}=\frac{-4}{x}$$

$$\therefore\ x^2=-4y \quad \cdots ②$$

From ① and ②, $(2y+4)^2=-4y$

$$y^2+5y+4=0$$

$$(y+4)(y+1)=0$$

$$\therefore\ y=-4,\ -1$$

(i) When $y=-4$, from ①, $x=-4$

(ii) When $y=-1$, from ①, $x=2$

From (i) and (ii), $x=-4,\ y=-4$ or $x=2,\ y=-1$

Geometric Sequences

Name

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
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Ex. Find the sum S_4 of the geometric sequence whose 1st term is 2, commo
ratio is 3 and number of terms is 4.

[Sol]

$$
\begin{array}{rl}
S_4 = & 2 + 2\cdot3 + 2\cdot3^2 + 2\cdot3^3 \\
-)\ 3S_4 = & \phantom{2 +{}} 2\cdot3 + 2\cdot3^2 + 2\cdot3^3 + 2\cdot3^4 \\
\hline
S_4 - 3S_4 = & 2 - 2\cdot3^4
\end{array}
$$

Multiplying both sides of S_4 by the common ratio 3

$\therefore\ -2S_4 = 2(1-3^4)$

$\therefore\ S_4 = 80$

1. Find the sum S_5 of the geometric sequence whose 1st term is 3, common r
is 2 and number of terms is 5.

[Sol]

$$
\begin{array}{rl}
S_5 = & 3 + 3\cdot2 + 3\cdot2^2 + 3\cdot2^3 + 3\cdot2^4 \\
-)\ 2S_5 = & \phantom{3 +{}} 3\cdot2 + 3\cdot2^2 + 3\cdot2^3 + 3\cdot2^4 + 3\cdot2^5 \\
\hline
S_5 - 2S_5 = & 3 - 3\cdot2^5
\end{array}
$$

$\therefore\ -S_5 = 3(1-2^5)$

$\therefore\ S_5 = 93$

Find the sum S_n of the geometric sequence with 1st term a, common ratio r
number of terms n.

[Sol]

$$
\begin{array}{rl}
S_n = & a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1} \\
-)\ rS_n = & \phantom{a +{}} ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1} + ar^n \\
\hline
S_n - rS_n = & a \phantom{+ ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1}} - ar^n
\end{array}
$$

$\therefore\ (1-r)S_n = a(1-r^n)$

(i) When $r \neq 1$,

$$S_n = \frac{a(1-\boxed{r^n})}{1-r} \text{ or } S_n = \frac{a(\boxed{r^n}-1)}{r-1}$$

(ii) When $r = 1$,

$$S_n = \underbrace{a + a + a + \cdots + a}_{n \text{ terms}} = \boxed{na}$$

N16b

Sum of a Geometric Sequence

Let S_n be the sum of a geometric sequence with 1st term a, common ratio r and number of terms n.

When $r \neq 1$, $S_n = \dfrac{a(1-r^n)}{1-r} = \dfrac{a(r^n-1)}{r-1}$

When $r=1$, $S_n = na$ ← When $r=1$, S_n is the sum of n terms of a.

2. Using the formula above, find the sum of each given geometric sequence.

Ex. 1st term: -2, common ratio: 3, number of terms: 4

[Sol] $\dfrac{-2(3^4-1)}{3-1} = -80$

1st term: 3, common ratio: 1, number of terms: 10

[Sol] $10 \cdot 3 = 30$

(1) 1st term: -4, common ratio: 2, number of terms: 5

[Sol] $\dfrac{-4(2^5-1)}{2-1} = -124$

(2) 1st term: 3, common ratio: -1, number of terms: 7

[Sol] $\dfrac{3[1-(-1)^7]}{1-(-1)} = 3$

(3) 1st term: -5, common ratio: $-\dfrac{1}{2}$, number of terms: 6

[Sol] $\dfrac{-5\left[1-\left(-\frac{1}{2}\right)^6\right]}{1-\left(-\frac{1}{2}\right)} = -\dfrac{105}{32}$

(4) 1st term: -2, common ratio: 1, number of terms: 12

[Sol] $12 \cdot (-2) = -24$

Ex. S_5, the sum of the first 5 terms of the geometric sequence 1, 3, 9, 27, …

[Sol] $S_5 = \dfrac{1 \cdot (3^5-1)}{3-1} = 121$ ← 1st term: 1, common ratio: 3, number of terms: 5

(5) S_7, the sum of the first 7 terms of the geometric sequence $\dfrac{1}{2}$, 1, 2, 4, …

[Sol] $S_7 = \dfrac{\frac{1}{2}(2^7-1)}{2-1} = \dfrac{127}{2}$

(6) S_8, the sum of the first 8 terms of the geometric sequence 1, $\sqrt{2}$, 2, $2\sqrt{2}$, …

[Sol] $S_8 = \dfrac{1 \cdot [(\sqrt{2})^8-1]}{\sqrt{2}-1} = \dfrac{15}{\sqrt{2}-1} = \dfrac{15(\sqrt{2}+1)}{(\sqrt{2}-1)(\sqrt{2}+1)} = 15(\sqrt{2}+1)$

N 17a

KUMON

Geometric Sequences

Name

Date / /

Time : to :

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1. Given the geometric sequence whose 1st term is 2 and common ratio is -3, find the value of n for which the sum of the first n terms is 122.

[Sol] Let S_n be the sum of the first n terms.

$$S_n=\frac{2[1-(-3)^n]}{1-(-3)}=122$$

$$(-3)^n=-243$$

$$(-3)^n=(-3)^5$$

Rewriting with base -3 (K 192)

$$\therefore\ n=5$$

2. Find the 1st term a of the geometric sequence $\{a_n\}$ whose common ratio is

and whose sum of the first 4 terms is 40.

[Sol] Let S_n be the sum of the first n terms.

$$S_4=\frac{a\left[1-\left(\frac{1}{3}\right)^4\right]}{1-\frac{1}{3}}=40$$

$$\frac{80}{81}a=\frac{80}{3}$$

$$\therefore\ a=27$$

3. Find the sum S from the 7th to the 10th term of the geometric sequence $\{a_n\}$ whose 1st term is $\frac{1}{3}$ and common ratio is 2.

[Sol] Let S_n be the sum of the first n terms.

$$S_{10}=\frac{\frac{1}{3}(2^{10}-1)}{2-1}=341$$

$$S_6=\frac{\frac{1}{3}(2^6-1)}{2-1}=21$$

$$\therefore\ S=S_{10}-S_6=341-21=320$$

$$\overbrace{\underbrace{a_1+a_2+\cdots+a_6}_{21}+\underbrace{a_7+\cdots+a_{10}}_{S}}^{341}$$

17b

Find the sum S from the 6th to the 9th term of the geometric sequence $\{a_n\}$ whose 5th term is -48 and 8th term is 384. (The common ratio is a real number.)

ol] Let a be the 1st term, r be the common ratio and S_n be the sum of the first n terms.

$$\begin{cases} a_5 = ar^4 = -48 & \cdots ① \\ a_8 = ar^7 = 384 & \cdots ② \end{cases}$$

From ① and ②, $-48r^3 = 384$

$$r^3 = -8$$

$\therefore r = -2$

From ①, $a = -3$

$$\therefore S_9 = \frac{-3[1-(-2)^9]}{1-(-2)} = -513$$

$$S_5 = \frac{-3[1-(-2)^5]}{1-(-2)} = -33$$

$$\therefore S = S_9 - S_5 = -513 - (-33) = -480$$

6. Given the geometric sequence whose 1st term is 2 and common ratio is 4, find the value of n for which the sum of the first n terms becomes greater than 1000 for the first time.

[Sol] Let S_n be the sum of the first n terms.

$$S_n = \frac{2(4^n-1)}{4-1} = \frac{2}{3}(4^n-1)$$

Considering $S_n > 1000$,

$\frac{2}{3}(4^n-1) > 1000$ when

Finding the smallest natural number n that satisfies $4^n > 1501$ by substitutin some values into n

$4^n > 1501$

The smallest natural number n that satisfies this is $n=6$.

Geometric Sequences

Name

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~

Ex. Find the 1st term a and the common ratio r of the geometric sequence $\{a$ whose sum of the first 3 terms is 9 and whose sum of the first 6 te -63. (The common ratio is a real number.)

[Sol] Let S_n be the sum of the first n terms.

(i) When $r=1$, $S_3=3a=9$, $S_6=6a=-63$

No a can satisfy these conditions simultaneously.

(ii) When $r\neq 1$,

$$\begin{cases} S_3=\dfrac{a(r^3-1)}{r-1}=9 & \cdots① \\[2ex] S_6=\dfrac{a(r^6-1)}{r-1}=-63 & \cdots② \end{cases}$$

$S_6=\dfrac{a(r^3-1)(r^3+1)}{r-1}$

From ① and ②, $9(r^3+1)=-63$

$r^3=-8$

$\therefore r=-2$

From ①, $a=3$

From (i) and (ii), $a=3$, $r=-2$

1. Find the 1st term a and the common ratio r of the geometric sequence whose sum of the first 3 terms is 52 and whose sum of the first 6 terms is 14 (The common ratio is a real number.)

[Sol] Let S_n be the sum of the first n terms.

(i) When $r=1$, $S_3=3a=52$, $S_6=6a=1456$

No a can satisfy these conditions simultaneously.

(ii) When $r\neq 1$,

$$\begin{cases} S_3=\dfrac{a(r^3-1)}{r-1}=52 & \cdots① \\[2ex] S_6=\dfrac{a(r^6-1)}{r-1}=1456 & \cdots② \end{cases}$$

From ① and ②, $52(r^3+1)=1456$

$r^3=27$

$\therefore r=3$

From ①, $a=4$

From (i) and (ii), $a=4$, $r=3$

N18b

2. Find the sum S from the 31^{st} to the 60^{th} term of the geometric sequence $\{a_n\}$ whose sum of the first 10 terms is 3 and whose sum from the 11^{th} to the 30^{th} term is 60. (The common ratio is a real number.)

[Sol] Let a be the 1^{st} term, r be the common ratio and S_n be the sum of the first n terms.

(i) When $r=1$, $S_{10}=10a=3$, $S_{30}=30a=3+60=63$

$$\overbrace{\underbrace{a_1+a_2+\cdots+a_{10}}_{3}+\underbrace{a_{11}+\cdots+a_{30}}_{60}}^{S_{30}}$$

No a can satisfy these conditions simultaneously.

(ii) When $r\neq 1$,

$$\begin{cases} S_{10}=\dfrac{a(r^{10}-1)}{r-1}=3 & \cdots ① \\ S_{30}=\dfrac{a(r^{30}-1)}{r-1}=63 & \cdots ② \end{cases}$$

$$S_{30}=\frac{a(r^{10}-1)(r^{20}+r^{10}+1)}{r-1}=63$$

From ① and ②, $3(r^{20}+r^{10}+1)=63$

$$r^{20}+r^{10}-20=0$$

$$(r^{10}-4)(r^{10}+5)=0$$

$$r^{10}+5>0$$

$$r^{10}=4 \quad \cdots ③$$

From ② and ③,

$$S_{60}=\frac{a(r^{60}-1)}{r-1}$$

$$=\frac{a(r^{30}-1)(r^{30}+1)}{r-1}$$

$$=\frac{a(r^{30}-1)}{r-1}\{(r^{10})^3+1\}$$

From ②, $\dfrac{a(r^{30}-1)}{r-1}=63$

$$=63(4^3+1)$$

$$=4095$$

From (i) and (ii), $S=S_{60}-S_{30}$

$$=4095-63$$

$$=4032$$

Geometric Sequences

Name

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
0	—	—	—	1

Ex. You want to save 10000 dollars in 5 years by saving x dollars at the beginning of each year. Given that the annual interest rate is 2%, find x. Use $(1.02)^5=1.1$ and round your answer down to the nearest 10 dollars.

[Sol] After 5 years, x dollars from the 1st year becomes $x\times(1.02)^5$ dollars
After 4 years, x dollars from the 2nd year becomes $x\times(1.02)^4$ dollars
⋮
After 1 year, x dollars from the 5th year becomes $x\times\ 1.02$ dollars
Let S be the amount of savings at the end of the 5th year.

$$S=1.02x+(1.02)^2x+\cdots+(1.02)^5x \leftarrow$$

1st term: $1.02x$, common ratio: 1.02, number of terms: 5

$$=\frac{1.02x[(\boxed{1.02})^5-1]}{\boxed{1.02}-1}$$

$$=\frac{1.02(\boxed{1.1}-1)}{0.02}x$$

$$=5.1x$$

Thus, since $S=10000$,

$5.1x=10000 \quad \therefore\ x=1960.7\ldots$

Rounding down to the nearest 10 dollars, 1960

Ans. 1960 dollars

The method of determining the interest on the initial amount and the accumulated interest of previous periods of a deposit or loan is called the ***compound interest method***. An interest calculated by the above method is called a ***compound interest***.
For example, when you save 100 dollars in the bank with an annual interest rate of 2%, after 1 year, your savings becomes $100\times1.02=102$ (dollars).
After 2 years, the annual interest rate of 2% accrues on the interest of 2 dollars; therefore, savings becomes $100\times(1.02)^2=104.04$ (dollars).

N19b

1. You want to save 20000 dollars in 10 years by saving x dollars at the beginning of each year. Given that the annual interest rate is 8%, find x. Use $(1.08)^{10}=2.1$ and round your answer down to the nearest 10 dollars.

[Sol] After 10 years, x dollars from the 1st year become $x\times(1.08)^{10}$ dollars.

After 9 years, x dollars from the 2nd year become $x\times(1.08)^{9}$ dollars.

$\vdots$

After 1 year, x dollars from the 10th year become $x\times$ 1.08 dollars.

Let S be the amount of savings at the end of the 10th year.

$$S=1.08x+(1.08)^2x+\cdots+(1.08)^{10}x$$

$$=\frac{1.08x[(1.08)^{10}-1]}{1.08-1}$$

$$=\frac{1.08(2.1-1)}{0.08}x$$

$$=14.85x$$

Thus, since $S=20000$,

$14.85x=20000 \quad \therefore\ x=1346.8\ldots$

Rounding down to the nearest 10 dollars, 1340

Ans. **1340 dollars**

Geometric Sequences

Name

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
(mistakes) 0	—	—	1	2~

1. Find the general term of the geometric sequence $\{a_n\}$ whose 2^{nd} term is
and 4^{th} term is 192. ➡ N

[Sol] Let a be the 1^{st} term and r be the common ratio.

$$\begin{cases} a_2 = ar = 12 & \cdots① \\ a_4 = ar^3 = 192 & \cdots② \end{cases}$$

From ① and ②, $12r^2 = 192$

$$r^2 = 16$$

$\therefore\ r = \pm 4$

(i) When $r = 4$, from ①, $a = 3$

$\therefore\ a_n = 3\cdot 4^{n-1}$

(ii) When $r = -4$, from ①, $a = -3$

$\therefore\ a_n = -3(-4)^{n-1}$

From (i) and (ii), $a_n = 3\cdot 4^{n-1}$ or $a_n = -3(-4)^{n-1}$

2. Find the 1^{st} term a and the common ratio r of the geometric sequence
whose 2^{nd} term is -3 and whose sum of the first 3 terms is 7. ➡ N

[Sol] $$\begin{cases} ar = -3 & \cdots① \\ a + ar + ar^2 = 7 & \cdots② \end{cases}$$

From ②, $a(1 + r + r^2) = 7$

Multiplying both sides by r,

$ar(1 + r + r^2) = 7r \ \cdots③$

From ① and ③, $-3(1 + r + r^2) = 7r$

$$3r^2 + 10r + 3 = 0$$

$$(r + 3)(3r + 1) = 0$$

$\therefore\ r = -3,\ -\dfrac{1}{3}$

(i) When $r = -3$, from ①, $a = 1$

(ii) When $r = -\dfrac{1}{3}$, from ①, $a = 9$

From (i) and (ii), $a = 1,\ r = -3$ or $a = 9,\ r = -\dfrac{1}{3}$

Alternative Solution

$$\begin{cases} ar = -3 & \cdots① \\ \dfrac{a(r^3 - 1)}{r - 1} = 7 & \cdots② \end{cases}$$

r^3-1
$=(r-1)(r^2+r$

From ②, $a(r^2 + r + 1) = 7$

Multiplying both sides by r,

$ar(r^2 + r + 1) = 7r \ \cdots③$

From ① and ③, $-3(r^2 + r + 1) = 7r$

$$3r^2 + 10r + 3 = 0$$

$$(r + 3)(3r + 1) = 0$$

$\therefore\ r = -3,\ -\dfrac{1}{3}$

(i) When $r = -3$, from ①, $a = 1$

(ii) When $r = -\dfrac{1}{3}$, from ①, $a = 9$

From (i) and (ii),

$a = 1,\ r = -3$ or $a = 9,\ r = -\dfrac{1}{3}$

N20b

3. Given the geometric sequence whose 1st term is 6 and common ratio is -2, find the value of n for which the sum of the first n terms is -30. ➡ N17

[Sol] Let S_n be the sum of the first n terms.

$$S_n=\frac{6[1-(-2)^n]}{1-(-2)}=-30$$

$$(-2)^n=16$$

$$(-2)^n=(-2)^4$$

$$\therefore n=4$$

4. Find the 1st term a and the common ratio r of the geometric sequence $\{a_n\}$ whose sum of the first 3 terms is 6 and whose sum of the first 6 terms is -42. (The common ratio is a real number.) ➡ N18

[Sol] Let S_n be the sum of the first n terms.

(i) When $r=1$, $S_3=3a=6$, $S_6=6a=-42$

No a can satisfy these conditions simultaneously.

(ii) When $r\neq 1$,

$$\begin{cases} S_3=\dfrac{a(r^3-1)}{r-1}=6 & \cdots① \\ S_6=\dfrac{a(r^6-1)}{r-1}=-42 & \cdots② \end{cases}$$

From ① and ②, $6(r^3+1)=-42$

$$r^3=-8$$

$$\therefore r=-2$$

From ①, $a=2$

From (i) and (ii), $a=2$, $r=-2$

Alternative Solution

$$\begin{cases} a+ar+ar^2=6 & \cdots① \\ a+ar+ar^2+ar^3+ar^4+ar^5=-42 & \cdots② \end{cases}$$

From ②,

$$a+ar+ar^2+r^3(a+ar+ar^2)=-42 \quad \cdots③$$

From ① and ③, $6+6r^3=-42$

$$r^3=-8$$

$$\therefore r=-2$$

From ①, $a=2$

$$\therefore a=2,\ r=-2$$

Various Sequences 1

Name

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
(mistakes) 0	1	2~3	4~5	6~

The summation symbol Σ is used to denote the sum of a sequence.
For example, $a_3+a_4+a_5+a_6$, the sum from the 3rd to the 6th term of the seque
$\{a_n\}$ is denoted by $\sum_{k=3}^{6} a_k$.
Generally, $\sum_{k=p}^{q} a_k$ denotes the sum from the pth to the qth term of a sequence $\{a$
(Note that other letters may be used in place of k.)

1. Let $\{a_n\}$ be the sequence

   1, 3, 5, 7, 9, 11, ...

   Find the following sums.

Ex. $\sum_{k=2}^{4} a_k = a_2+a_3+a_4 = 3+5+7 = 15$ ← $\sum_{k=2}^{4} a_k = a_2+a_3+a_4$

(1) $\sum_{k=3}^{5} a_k = a_3+a_4+a_5 = 5+7+9 = 21$

(2) $\sum_{m=3}^{6} a_m = a_3+a_4+a_5+a_6 = 5+7+9+11 = 32$

(3) $\sum_{n=1}^{5} a_n = a_1+a_2+a_3+a_4+a_5 = 1+3+5+7+9 = 25$

(4) $\sum_{k=1}^{3} a_{k+1} = a_{1+1}+a_{2+1}+a_{3+1} = a_2+a_3+a_4 = 3+5+7 = 15$

(5) $\sum_{k=1}^{3} a_{2k} = a_{2\cdot1}+a_{2\cdot2}+a_{2\cdot3} = a_2+a_4+a_6 = 3+7+11 = 21$

(6) $\sum_{k=1}^{2} a_{2k-1} = a_{2\cdot1-1}+a_{2\cdot2-1} = a_1+a_3 = 1+5 = 6$

(7) $\sum_{k=2}^{2} a_k = a_2 = 3$

Σ is read as "sigma."

N2Ib

2. Find the following sums.

Ex. $\sum_{k=2}^{4}(3k-1)=(3\cdot2-1)+(3\cdot3-1)+(3\cdot4-1)=5+8+11=24$

(1) $\sum_{k=3}^{5}2k^2=2\cdot3^2+2\cdot4^2+2\cdot5^2=18+32+50=100$

(2) $\sum_{k=0}^{2}5^k=5^0+5^1+5^2=1+5+25=31$

(3) $\sum_{n=3}^{4}(2n-1)^2=(2\cdot3-1)^2+(2\cdot4-1)^2=25+49=74$

(4) $\sum_{m=2}^{3}(3^m-2^{m-1})=(3^2-2^{2-1})+(3^3-2^{3-1})=7+23=30$

(5) $\sum_{k=4}^{6}(k+n)=(\boxed{4}+n)+(\boxed{5}+n)+(\boxed{6}+n)=3n+\boxed{15}$

(6) $\sum_{k=2}^{5}(n-k)=(n-2)+(n-3)+(n-4)+(n-5)=4n-14$

3. Rewrite the following sums using Σ.

Ex. $2+4+6+8+10=\sum_{k=1}^{5}2k$ ← Given $\{a_k\}$: 2, 4, 6, ..., the sum of the first 5 terms of $\{a_n\}$ is $\sum_{k=1}^{5}2k$.

(1) $5+10+15+20=\sum_{k=1}^{4}5k$

(2) $1+4+9+16+25+36=\sum_{k=1}^{6}k^2$

(3) $3+5+7+\cdots+(2n+1)=\sum_{k=1}^{n}(\boxed{2k+1})$

(4) $3+12+27+\cdots+3n^2=\sum_{k=1}^{n}3k^2$

(5) $3+12+27+\cdots+3n^2+3(n+1)^2=\sum_{k=1}^{n+1}\boxed{3k^2}$ ← Given $\{a_n\}$: 3, 12, 27, ... from (4), the sum of the first n terms of $\{a_n\}$ is $\sum_{k=1}^{n}3k^2$. Adding $3(n+1)^2$ that is the $(n+1)^{th}$ term of $\{a_n\}$, $\sum_{k=1}^{n}3k^2+3(n+1)^2=\sum_{k=1}^{n+1}3k^2$

Various Sequences 1

Name

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
0	—	1	2	3~

Since the sum S_n of the arithmetic sequence with 1st term a, last term l and numb
of terms n is

$$S_n=\frac{1}{2}n(a+l), \quad \leftarrow \boxed{\text{N6}}$$

the sum of the natural numbers from 1 to n (1, 2, 3, ..., n) becomes

$$1+2+3+\cdots+n=\frac{1}{2}n(n+1)$$

Also, the sum of a sequence, where $a_1=a_2=a_3=\cdots=a_n=c$ (c is a constant)

$$\underbrace{c+c+\cdots+c}_{n \text{ terms}}=nc$$

Rewriting the sums of these sequences using Σ, the following formulas are obtain

Summation Formulas I

$$\sum_{k=1}^{n}k=\frac{1}{2}n(n+1), \quad \sum_{k=1}^{n}c=nc \quad (c \text{ is a constant})$$

1. Find the following sums.

Ex. $\displaystyle\sum_{k=1}^{10}k=\frac{1}{2}\cdot10(10+1)=55$ | $\displaystyle\sum_{k=1}^{15}2=15\cdot2=30$

(1) $\displaystyle\sum_{k=1}^{15}k=\frac{1}{2}\cdot15(15+1)=120$

(2) $\displaystyle\sum_{k=1}^{10}5=10\cdot5=50$

(3) $\displaystyle\sum_{k=1}^{20}k-3\sum_{k=1}^{8}2=\frac{1}{2}\cdot20(20+1)-3\cdot8\cdot2=162$

(4) $\displaystyle2\sum_{k=1}^{n}k-\sum_{k=1}^{n}1=2\cdot\frac{1}{2}n(n+1)-n\cdot1=n^2$

b

wo sequences

: $a_1, a_2, a_3, \dots, a_n$

: $b_1, b_2, b_3, \dots, b_n$.

the following equations using Σ.

$a_1+b_1)+(a_2+b_2)+(a_3+b_3)+\cdots+(a_n+b_n)$

$a_1+a_2+a_3+\cdots+a_n)+(b_1+b_2+b_3+\cdots+b_n)$

$$\sum_{k=1}^{n}(a_k+b_k)=\sum_{k=1}^{n}a_k+\boxed{\sum_{k=1}^{n}b_k}$$

$a_1+ca_2+ca_3+\cdots+ca_n=c(a_1+a_2+a_3+\cdots+a_n)$ (c is a constant)

$$\sum_{k=1}^{n}ca_k=c\boxed{\sum_{k=1}^{n}a_k}$$

Answers: $\sum_{k=1}^{n}b_k$, $\sum_{k=1}^{n}a_k$

he above, the following equations are true.

Summation Properties

$$\sum_{k=1}^{n}(a_k+b_k)=\sum_{k=1}^{n}a_k+\sum_{k=1}^{n}b_k,\quad \sum_{k=1}^{n}ca_k=c\sum_{k=1}^{n}a_k \text{ (c is a constant)}$$

d the following sums.

$$\sum_{k=1}^{n}(3k+1)=3\sum_{k=1}^{n}k+\sum_{k=1}^{n}1=3\cdot\frac{1}{2}n(n+1)+n=\frac{1}{2}n(3n+5)$$

$$\sum_{k=1}^{n}(4k+3)=4\sum_{k=1}^{n}k+\sum_{k=1}^{n}3=4\cdot\frac{1}{2}n(n+1)+3n=n(2n+5)$$

$$\sum_{k=1}^{n}(-2k+4)=-2\sum_{k=1}^{n}k+\sum_{k=1}^{n}4=-2\cdot\frac{1}{2}n(n+1)+4n=-n(n-3)$$

$$\sum_{k=1}^{n-1}(3k+5)=3\sum_{k=1}^{n-1}k+\sum_{k=1}^{n-1}5=3\cdot\frac{1}{2}(n-1)[(n-1)+1]+5(n-1)$$

$$=\frac{1}{2}(n-1)(3n+10)$$

Various Sequences 1

Name ________

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
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Find $\sum\limits_{k=1}^{n} k^2$.

[Sol] Use $(k+1)^3-k^3=3k^2+3k+1$. ← $(k+1)^3=k^3+3k^2+3k+1$

When $k=1$, $\quad 2^3-1^3=3\cdot\boxed{1^2}+3\cdot\boxed{1}+\boxed{1}$

When $k=2$, $\quad 3^3-2^3=3\cdot\boxed{2^2}+3\cdot\boxed{2}+\boxed{1}$

When $k=3$, $\quad 4^3-3^3=3\cdot\boxed{3^2}+3\cdot\boxed{3}+\boxed{1}$

$\vdots$

When $k=n$, $\quad (n+1)^3-n^3=3\cdot\boxed{n^2}+3\cdot\boxed{n}+\boxed{1}$

Adding up the terms on each side from all these n equations,

$$(n+1)^3-1^3=3(\boxed{1^2+2^2+\cdots+n^2})+3(\boxed{1+2+\cdots+n})+(\boxed{1+1+\cdots+}$$

$$=3\sum_{k=1}^{n}k^2+3\sum_{k=1}^{n}\boxed{k}+n$$

Rearranging into $3\sum\limits_{k=1}^{n}k^2=$

Therefore,

$$3\sum_{k=1}^{n}k^2=(n+1)^3-1-3\sum_{k=1}^{n}\boxed{k}-n$$

$$=(n+1)^3-3\cdot\boxed{\frac{1}{2}n(n+1)}-(n+1)$$

$$=\frac{1}{2}(\boxed{n+1})[2(n+1)^2-3n-2]$$

$$=\boxed{\frac{1}{2}n(n+1)(2n+1)}$$

$$\therefore\ \sum_{k=1}^{n}k^2=\boxed{\frac{1}{6}n(n+1)(2n+1)}$$

23b

Summation Formula II

$$\sum_{k=1}^{n} k^2 = \frac{1}{6} n(n+1)(2n+1)$$

d the following sums.

$$\sum_{k=1}^{4}(k^2+2k) = \sum_{k=1}^{4} k^2 + 2\sum_{k=1}^{4} k$$
$$= \frac{1}{6}\cdot 4(4+1)(2\cdot 4+1) + 2\cdot\frac{1}{2}\cdot 4(4+1)$$
$$= 50$$

) $$\sum_{k=1}^{n}(3k^2-k) = 3\sum_{k=1}^{n} k^2 - \sum_{k=1}^{n} k$$
$$= 3\cdot\frac{1}{6}n(n+1)(2n+1) - \frac{1}{2}n(n+1)$$
$$= \frac{1}{2}n(n+1)[(2n+1)-1]$$
$$= n^2(n+1)$$

3) $$\sum_{k=1}^{n}(6k^2-7k-3) = 6\sum_{k=1}^{n} k^2 - 7\sum_{k=1}^{n} k - \sum_{k=1}^{n} 3$$
$$= 6\cdot\frac{1}{6}n(n+1)(2n+1) - 7\cdot\frac{1}{2}n(n+1) - 3n$$
$$= \frac{1}{2}n[2(n+1)(2n+1) - 7(n+1) - 6]$$
$$= \frac{1}{2}n(4n^2-n-11)$$

N24a

Various Sequences 1

Name

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
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Find $\sum_{k=1}^{n} k^3$.

[Sol] Use $(k+1)^4-k^4=4k^3+6k^2+4k+1$. ← $(k+1)^4=k^4+4k^3+6k^2+4k$

When $k=1$, $\quad 2^4-1^4=4\cdot \boxed{1^3}+6\cdot \boxed{1^2}+4\cdot \boxed{1}+\boxed{1}$

When $k=2$, $\quad 3^4-2^4=4\cdot \boxed{2^3}+6\cdot \boxed{2^2}+4\cdot \boxed{2}+\boxed{1}$

When $k=3$, $\quad 4^4-3^4=4\cdot \boxed{3^3}+6\cdot \boxed{3^2}+4\cdot \boxed{3}+\boxed{1}$

$\vdots$

When $k=n$, $\quad (n+1)^4-n^4=4\cdot \boxed{n^3}+6\cdot \boxed{n^2}+4\cdot \boxed{n}+\boxed{1}$

Adding up the terms on each side from all n equations,

$$(n+1)^4-1^4=4(\boxed{1^3+2^3+\cdots+n^3})+6(\boxed{1^2+2^2+\cdots+n^2})$$
$$+4(\boxed{1+2+\cdots+n})+(\boxed{1+1+\cdots+1})$$
$$=4\sum_{k=1}^{n}k^3+6\sum_{k=1}^{n}\boxed{k^2}+4\sum_{k=1}^{n}k+n$$

Rearranging in
$4\sum_{k=1}^{n}k^3=$

Therefore,

$$4\sum_{k=1}^{n}k^3=(n+1)^4-1-6\sum_{k=1}^{n}\boxed{k^2}-4\sum_{k=1}^{n}k-n$$
$$=(n+1)^4-6\cdot\boxed{\frac{1}{6}n(n+1)(2n+1)}-4\cdot\frac{1}{2}n(n+1)-(n+1)$$
$$=(\boxed{n+1})[(n+1)^3-n(2n+1)-2n-1]$$
$$=\boxed{n^2(n+1)^2}$$

$$\therefore \sum_{k=1}^{n}k^3=\left[\boxed{\frac{1}{2}n(n+1)}\right]^2$$

Ans: k^3, k^2, k, $\frac{1}{6}n(n+1)(2n+1)$, $n+1$, $n^2(n+1)^2$, $\frac{1}{2}n(n+1)$

b

ummation Formula III

$$\sum_{k=1}^{n} k^3=\left[\frac{1}{2}n(n+1)\right]^2$$

e following sums.

$$\sum_{k=1}^{6}(k^3+k)=\sum_{k=1}^{6}k^3+\sum_{k=1}^{6}k$$

$$=\left[\frac{1}{2}\cdot 6(6+1)\right]^2+\frac{1}{2}\cdot 6(6+1)$$

$$=462$$

$$\sum_{k=1}^{n}(2k^3-3k)=2\sum_{k=1}^{n}k^3-3\sum_{k=1}^{n}k$$

$$=2\left[\frac{1}{2}n(n+1)\right]^2-3\cdot\frac{1}{2}n(n+1)$$

$$=\frac{1}{2}n(n+1)[n(n+1)-3]$$

$$=\frac{1}{2}n(n+1)(n^2+n-3)$$

$$\sum_{k=1}^{n}(4k^3-6k^2+3)=4\sum_{k=1}^{n}k^3-6\sum_{k=1}^{n}k^2+\sum_{k=1}^{n}3$$

$$=4\left[\frac{1}{2}n(n+1)\right]^2-6\cdot\frac{1}{6}n(n+1)(2n+1)+3n$$

$$=n[n(n+1)^2-(n+1)(2n+1)+3]$$

$$=n(n^3-2n+2)$$

Various Sequences 1

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Since the sum S_n of the geometric sequence with 1st term a, common ratio $r(r=$
and number of terms n is

$$S_n=\frac{a(1-r^n)}{1-r}=\frac{a(r^n-1)}{r-1},$$ ← N16

when $r\neq1$, the sum of the geometric sequence a, ar, ar^2, ..., ar^{n-1} become

$$a+ar+ar^2+\cdots+ar^{n-1}=\frac{a(1-r^n)}{1-r}=\frac{a(r^n-1)}{r-1}$$

Rewriting the sum of the sequence using Σ, the following formula is obtaine

Summation Formula IV

$$\sum_{k=1}^{n}ar^{k-1}=\frac{a(1-r^n)}{1-r}=\frac{a(r^n-1)}{r-1}\quad(r\neq1)$$

Find the following sums.

Ex. $\displaystyle\sum_{k=1}^{4}2\cdot3^{k-1}=\frac{2(3^4-1)}{3-1}=80$

(1) $\displaystyle\sum_{k=1}^{6}5\cdot2^{k-1}=\frac{5(2^6-1)}{2-1}=315$

(2) $\displaystyle\sum_{k=1}^{3}(-3)^{k-1}=\frac{1-(-3)^3}{1-(-3)}=7$

(3) $\displaystyle\sum_{k=1}^{n}3\cdot4^{k-1}=\frac{3(4^n-1)}{4-1}=4^n-1$

(4) $\displaystyle\sum_{k=1}^{n}6(-5)^{k-1}=\frac{6[1-(-5)^n]}{1-(-5)}=1-(-5)^n$

(5) $\displaystyle\sum_{k=1}^{n} 4^k = \sum_{k=1}^{n} 4 \cdot 4^{\boxed{k-1}} = \frac{4(4^{\boxed{n}} - 1)}{4-1} = \frac{4^{\boxed{n+1}} - 4}{3}$

(6) $\displaystyle\sum_{k=1}^{n} 6^k = \sum_{k=1}^{n} 6 \cdot 6^{k-1} = \frac{6(6^n - 1)}{6-1} = \frac{6^{n+1} - 6}{5}$

(7) $\displaystyle\sum_{k=1}^{n} 3(-2)^k = \sum_{k=1}^{n} [-6(-2)^{k-1}] = \frac{-6[1-(-2)^n]}{1-(-2)} = -2-(-2)^{n+1}$

(8) $\displaystyle\sum_{k=1}^{n} (4 \cdot 3^k - 2^{k+1}) = \sum_{k=1}^{n} 12 \cdot 3^{k-1} - \sum_{k=1}^{n} 4 \cdot 2^{k-1}$

$\displaystyle = \frac{12(3^n - 1)}{3-1} - \frac{4(2^n - 1)}{2-1}$

$= (2 \cdot 3^{n+1} - 6) - (2^{n+2} - 4)$

$= 2 \cdot 3^{n+1} - 2^{n+2} - 2$

$[= 2(3^{n+1} - 2^{n+1} - 1)]$

(9) $\displaystyle\sum_{k=1}^{n-1} (5^k + 3^{k+1}) = \sum_{k=1}^{n-1} 5 \cdot 5^{k-1} + \sum_{k=1}^{n-1} 9 \cdot 3^{k-1}$

$\displaystyle = \frac{5(5^{n-1} - 1)}{5-1} + \frac{9(3^{n-1} - 1)}{3-1}$

$\displaystyle = \left(\frac{5^n}{4} - \frac{5}{4}\right) + \left(\frac{3^{n+1}}{2} - \frac{9}{2}\right)$

$\displaystyle = \frac{5^n}{4} + \frac{3^{n+1}}{2} - \frac{23}{4}$

$\displaystyle \left[= \frac{1}{4}(5^n + 2 \cdot 3^{n+1} - 23)\right]$

Various Sequences 1

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Find the following sums.

(1) $\displaystyle\sum_{k=1}^{n}(2k+3^k)$

$\displaystyle=2\sum_{k=1}^{n}k+\sum_{k=1}^{n}3\cdot 3^{k-1}$

$\displaystyle=2\cdot\frac{1}{2}n(n+1)+\frac{3(3^n-1)}{3-1}$

$\displaystyle=n(n+1)+\frac{3^{n+1}-3}{2}$

(2) $\displaystyle\sum_{k=1}^{n}[3k^2+(-5)^k]$

$\displaystyle=3\sum_{k=1}^{n}k^2+\sum_{k=1}^{n}[-5(-5)^{k-1}]$

$\displaystyle=3\cdot\frac{1}{6}n(n+1)(2n+1)+\frac{-5[1-(-5)^n]}{1-(-5)}$

$\displaystyle=\frac{1}{2}n(n+1)(2n+1)-\frac{5+(-5)^{n+1}}{6}$

(3) $\displaystyle\sum_{k=1}^{n}(6k^2+2k+4^{k+1})$

$\displaystyle=6\sum_{k=1}^{n}k^2+2\sum_{k=1}^{n}k+\sum_{k=1}^{n}16\cdot 4^{k-1}$

$\displaystyle=6\cdot\frac{1}{6}n(n+1)(2n+1)+2\cdot\frac{1}{2}n(n+1)+\frac{16(4^n-1)}{4-1}$

$\displaystyle=n(n+1)[(2n+1)+1]+\frac{4^{n+2}-16}{3}$

$\displaystyle=2n(n+1)^2+\frac{4^{n+2}-16}{3}$

26b

$$\sum_{k=1}^{2n}(2k^3+3k^2+6^{k-1})$$

$$=2\sum_{k=1}^{2n}k^3+3\sum_{k=1}^{2n}k^2+\sum_{k=1}^{2n}6^{k-1}$$

$$=2\left[\frac{1}{2}\cdot 2n(2n+1)\right]^2+3\cdot\frac{1}{6}\cdot 2n(2n+1)(2\cdot 2n+1)+\frac{6^{2n}-1}{6-1}$$

$$=n(2n+1)[2n(2n+1)+(4n+1)]+\frac{6^{2n}-1}{5}$$

$$=n(2n+1)(4n^2+6n+1)+\frac{6^{2n}-1}{5}$$

5) $$\sum_{k=1}^{n+1}\left(4k^{\frac{3}{2}}+3^k\right)\left(4k^{\frac{3}{2}}-3^k\right)$$

$$=\sum_{k=1}^{n+1}(16k^3-9^k)$$

$$=16\sum_{k=1}^{n+1}k^3-\sum_{k=1}^{n+1}9\cdot 9^{k-1}$$

$$=16\left[\frac{1}{2}(n+1)[(n+1)+1]\right]^2-\frac{9(9^{n+1}-1)}{9-1}$$

$$=4(n+1)^2(n+2)^2-\frac{9^{n+2}-9}{8}$$

6) [illegible]

[illegible]

N27a

KUMON

Various Sequences 1

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Find the following sums.

Ex. $1\cdot3+2\cdot4+3\cdot5+\cdots+n(n+2)$

[Sol] Since this is the sum of the first n terms of the sequence whose k^{th} term is $k(k+2)$,

$$\sum_{k=1}^{n} k(k+2)=\sum_{k=1}^{n}(k^2+2k)$$

$$=\sum_{k=1}^{n}k^2+2\sum_{k=1}^{n}k$$

$$=\frac{1}{6}n(n+1)(2n+1)+2\cdot\frac{1}{2}n(n+1)$$

$$=\frac{1}{6}n(n+1)[(2n+1)+6]$$

$$=\frac{1}{6}n(n+1)(2n+7)$$

(1) $2\cdot2+4\cdot5+6\cdot8+\cdots+2n(3n-1)$

[Sol] Since this is the sum of the first n terms of the sequence whose k^{th} term is $2k(3k-1)$,

$$\sum_{k=1}^{n}2k(3k-1)=\sum_{k=1}^{n}(6k^2-2k)$$

$$=6\sum_{k=1}^{n}k^2-2\sum_{k=1}^{n}k$$

$$=6\cdot\frac{1}{6}n(n+1)(2n+1)-2\cdot\frac{1}{2}n(n+1)$$

$$=n(n+1)[(2n+1)-1]$$

$$=2n^2(n+1)$$

27b

) $1^2\cdot 2+2^2\cdot 3+3^2\cdot 4+\cdots+n^2(n+1)$

ol] Since this is the sum of the first n terms of the sequence whose k^{th} term is $k^2(k+1)$,

$$\sum_{k=1}^{n} k^2(k+1)=\sum_{k=1}^{n}(k^3+k^2)$$
$$=\sum_{k=1}^{n}k^3+\sum_{k=1}^{n}k^2$$
$$=\left[\frac{1}{2}n(n+1)\right]^2+\frac{1}{6}n(n+1)(2n+1)$$
$$=\frac{1}{12}n(n+1)[3n(n+1)+2(2n+1)]$$
$$=\frac{1}{12}n(n+1)(n+2)(3n+1)$$

(3) $1^3+3^3+5^3+\cdots+(2n-1)^3$

[Sol] Since this is the sum of the first n terms of the sequence whose k^{th} term is $(2k-1)^3$,

$$\sum_{k=1}^{n}(2k-1)^3$$
$$=\sum_{k=1}^{n}(8k^3-12k^2+6k-1)$$
$$=8\sum_{k=1}^{n}k^3-12\sum_{k=1}^{n}k^2+6\sum_{k=1}^{n}k-\sum_{k=1}^{n}1$$
$$=8\left[\frac{1}{2}n(n+1)\right]^2-12\cdot\frac{1}{6}n(n+1)(2n+1)+6\cdot\frac{1}{2}n(n+1)-n$$
$$=n[2n(n+1)^2-2(n+1)(2n+1)+3(n+1)-1]$$

$$\ldots^2(2n^2-1)$$

Various Sequences 1

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Find the sum S_n of the first n terms of each given sequence.

Ex. $1\cdot2,\ 2\cdot3,\ 3\cdot4,\ 4\cdot5,\ \dots$

[Sol] The k^{th} term is $k(k+1)=k^2+k$.

$$\therefore\ S_n=\sum_{k=1}^{n}(k^2+k)=\sum_{k=1}^{n}k^2+\sum_{k=1}^{n}k$$
$$=\frac{1}{6}n(n+1)(2n+1)+\frac{1}{2}n(n+1)$$
$$=\frac{1}{6}n(n+1)[(2n+1)+3]$$
$$=\frac{1}{3}n(n+1)(n+2)$$

(1) $1\cdot4,\ 2\cdot7,\ 3\cdot10,\ 4\cdot13,\ \dots$

[Sol] The k^{th} term is $k(3k+1)=3k^2+k$.

$$\therefore\ S_n=\sum_{k=1}^{n}(3k^2+k)=3\sum_{k=1}^{n}k^2+\sum_{k=1}^{n}k$$
$$=3\cdot\frac{1}{6}n(n+1)(2n+1)+\frac{1}{2}n(n+1)$$
$$=\frac{1}{2}n(n+1)[(2n+1)+1]$$
$$=n(n+1)^2$$

(2) $1\cdot2\cdot3,\ 2\cdot3\cdot4,\ 3\cdot4\cdot5,\ 4\cdot5\cdot6,\ \dots$

[Sol] The k^{th} term is $k(k+1)(k+2)=k^3+3k^2+2k$.

$$\therefore\ S_n=\sum_{k=1}^{n}(k^3+3k^2+2k)$$
$$=\sum_{k=1}^{n}k^3+3\sum_{k=1}^{n}k^2+2\sum_{k=1}^{n}k$$
$$=\left[\frac{1}{2}n(n+1)\right]^2+3\cdot\frac{1}{6}n(n+1)(2n+1)+2\cdot\frac{1}{2}n(n+1)$$
$$=\frac{1}{4}n(n+1)[n(n+1)+2(2n+1)+4]$$
$$=\frac{1}{4}n(n+1)(n+2)(n+3)$$

b

$1+2,\ 1+2+3,\ 1+2+3+4,\ \cdots$

he k^{th} term is $1+2+3+\cdots+k=\frac{1}{2}k(k+1)=\frac{1}{2}(k^2+k)$.

$\boxed{1+2+3+\cdots+n=\sum_{k=1}^{n}k}$

$$\therefore\ S_n=\sum_{k=1}^{n}\frac{1}{2}(k^2+k)=\frac{1}{2}\left(\sum_{k=1}^{n}k^2+\sum_{k=1}^{n}k\right)$$

$$=\frac{1}{2}\left[\frac{1}{6}n(n+1)(2n+1)+\frac{1}{2}n(n+1)\right]$$

$$=\frac{1}{12}n(n+1)[(2n+1)+3]$$

$$=\frac{1}{6}n(n+1)(n+2)$$

$,\ 1^2+2^2,\ 1^2+2^2+3^2,\ 1^2+2^2+3^2+4^2,\ \cdots$

The k^{th} term is

$$1^2+2^2+3^2+\cdots+k^2=\frac{1}{6}k(k+1)(2k+1)=\frac{1}{6}(2k^3+3k^2+k).$$

$\boxed{1^2+2^2+3^2+\cdots+n^2=\sum_{k=1}^{n}k^2}$

$$\therefore\ S_n=\sum_{k=1}^{n}\frac{1}{6}(2k^3+3k^2+k)$$

$$=\frac{1}{6}\left(2\sum_{k=1}^{n}k^3+3\sum_{k=1}^{n}k^2+\sum_{k=1}^{n}k\right)$$

$$=\frac{1}{6}\left\{2\left[\frac{1}{2}n(n+1)\right]^2+3\cdot\frac{1}{6}n(n+1)(2n+1)+\frac{1}{2}n(n+1)\right\}$$

$$=\frac{1}{12}n(n+1)[n(n+1)+(2n+1)+1]$$

$$=\frac{1}{12}n(n+1)^2(n+2)$$

Various Sequences 1

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1. Find the sum of the first 30 terms of the sequence $\{a_n=\alpha+\beta n+\gamma n^2\}$ whos term is 3, 2[nd] term is 5 and whose sum of the first 3 terms is 17.

[Sol] $\begin{cases} a_1=3 & \cdots① \\ a_2=5 & \cdots② \\ a_1+a_2+a_3=17 & \cdots③ \end{cases}$

From ①~③,

$a_3=9$

$\therefore \begin{cases} a_1=\alpha+\beta+\gamma=3 & \cdots④ \\ a_2=\alpha+2\beta+4\gamma=5 & \cdots⑤ \\ a_3=\alpha+3\beta+9\gamma=9 & \cdots⑥ \end{cases}$

From ④~⑥,

$\alpha=3,\ \beta=-1,\ \gamma=1$

$\therefore\ a_n=3-n+n^2$

Therefore, the sum is

$$\sum_{k=1}^{30} a_k=\sum_{k=1}^{30}(3-k+k^2)$$

$$=\sum_{k=1}^{30}3-\sum_{k=1}^{30}k+\sum_{k=1}^{30}k^2$$

$$=30\cdot3-\frac{1}{2}\cdot30(30+1)+\frac{1}{6}\cdot30(30+1)(2\cdot30+1)$$

$$=9080$$

N29b

2. Given $S_n=\sum_{k=1}^{n} k$ ($n=1, 2, \ldots$), find the following sums.

(1) $T_n=\sum_{k=1}^{n} S_k$

[Sol] $S_n=\sum_{k=1}^{n} k=\frac{1}{2}n(n+1)=\frac{1}{2}(n^2+n)$

$$\therefore T_n=\sum_{k=1}^{n}\frac{1}{2}(k^2+k) \longleftarrow \boxed{S_k=\frac{1}{2}(k^2+k)}$$

$$=\frac{1}{2}\left(\sum_{k=1}^{n}k^2+\sum_{k=1}^{n}k\right)$$

$$=\frac{1}{2}\left[\frac{1}{6}n(n+1)(2n+1)+\frac{1}{2}n(n+1)\right]$$

$$=\frac{1}{12}n(n+1)[(2n+1)+3]$$

$$=\frac{1}{6}n(n+1)(n+2)$$

(2) $U_n=\sum_{k=1}^{n} T_k$

[Sol] From (1), $T_n=\frac{1}{6}(n^3+3n^2+2n)$

$$\therefore U_n=\sum_{k=1}^{n}\frac{1}{6}(k^3+3k^2+2k) \longleftarrow \boxed{T_k=\frac{1}{6}(k^3+3k^2+2k)}$$

$$=\frac{1}{6}\left(\sum_{k=1}^{n}k^3+3\sum_{k=1}^{n}k^2+2\sum_{k=1}^{n}k\right)$$

$$=\frac{1}{6}\left\{\left[\frac{1}{2}n(n+1)\right]^2+3\cdot\frac{1}{6}n(n+1)(2n+1)+2\cdot\frac{1}{2}n(n+1)\right\}$$

$$=\frac{1}{24}n(n+1)[n(n+1)+2(2n+1)+4]$$

$$=\frac{1}{24}n(n+1)(n+2)(n+3)$$

N30a

KUMON

Various Sequences 1

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Time : to :

1. Find the following sums.

(1) $\sum_{k=1}^{n}(6k^2-2k+3)=6\sum_{k=1}^{n}k^2-2\sum_{k=1}^{n}k+\sum_{k=1}^{n}3$

$$=6\cdot\frac{1}{6}n(n+1)(2n+1)-2\cdot\frac{1}{2}n(n+1)+3n$$

$$=n[(n+1)(2n+1)-(n+1)+3]$$

$$=n(2n^2+2n+3)$$

(2) $\sum_{k=1}^{n}(k^3+2k)=\sum_{k=1}^{n}k^3+2\sum_{k=1}^{n}k$

$$=\left[\frac{1}{2}n(n+1)\right]^2+2\cdot\frac{1}{2}n(n+1)$$

$$=\frac{1}{4}n(n+1)[n(n+1)+4]$$

$$=\frac{1}{4}n(n+1)(n^2+n+4)$$

(3) $\sum_{k=1}^{n}(2^k+4\cdot5^{k-1})=\sum_{k=1}^{n}2\cdot2^{k-1}+\sum_{k=1}^{n}4\cdot5^{k-1}$

$$=\frac{2(2^n-1)}{2-1}+\frac{4(5^n-1)}{5-1}$$

$$=(2^{n+1}-2)+(5^n-1)$$

$$=2^{n+1}+5^n-3$$

(4) $2^2 \cdot 2 + 4^2 \cdot 3 + 6^2 \cdot 4 + \cdots + (2n)^2(n+1)$

➡ N27

[Sol] Since this is the sum of the first n terms of the sequence whose k^{th} term is $(2k)^2(k+1)$,

$$\begin{aligned}\sum_{k=1}^{n}(2k)^2(k+1) &= \sum_{k=1}^{n}4(k^3+k^2)\\ &= 4\left(\sum_{k=1}^{n}k^3+\sum_{k=1}^{n}k^2\right)\\ &= 4\left\{\left[\frac{1}{2}n(n+1)\right]^2+\frac{1}{6}n(n+1)(2n+1)\right\}\\ &= \frac{1}{3}n(n+1)[3n(n+1)+2(2n+1)]\\ &= \frac{1}{3}n(n+1)(n+2)(3n+1)\end{aligned}$$

2. Find the sum S_n of the first n terms of the following sequence.

➡ N28

$1^2 \cdot 3,\ 2^2 \cdot 5,\ 3^2 \cdot 7,\ 4^2 \cdot 9,\ \cdots$

[Sol] The k^{th} term is $k^2(2k+1) = 2k^3 + k^2$.

$$\begin{aligned}\therefore S_n = \sum_{k=1}^{n}(2k^3+k^2) &= 2\sum_{k=1}^{n}k^3+\sum_{k=1}^{n}k^2\\ &= 2\left[\frac{1}{2}n(n+1)\right]^2+\frac{1}{6}n(n+1)(2n+1)\\ &= \frac{1}{6}n(n+1)[3n(n+1)+(2n+1)]\\ &= \frac{1}{6}n(n+1)(3n^2+5n+1)\end{aligned}$$

N31a

Various Sequences 2

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1. Find the sequence by writing out the difference between each two consec
terms in each given sequence.

Ex. 6, 10, 15, 21, 28, 36, …

[Sol] 4, 5, 6, 7, 8, … ←

(1) 1, 2, 4, 7, 11, 16, …

[Sol] 1, 2, 3, 4, 5, …

(2) 3, 6, 11, 18, 27, 38, …

[Sol] 3, 5, 7, 9, 11, …

(3) 1, 4, 11, 22, 37, 56, …

[Sol] 3, 7, 11, 15, 19, …

(4) 18, 17, 14, 9, 2, −7, …

[Sol] −1, −3, −5, −7, −9, …

(5) 2, 4, 6, 8, 10, 12, …

[Sol] 2, 2, 2, 2, 2, …

The term which is derived by taking the difference between two consecutive terms in the sequence $\{a_n\}$ is

$b_n = a_{n+1} - a_n \quad (n = 1, 2, 3, \dots)$ ←

Generally, the sequence $\{b_n\}$ is called the ***sequence of differences*** of $\{a_n\}$.

2. Given that the sequence $\{a_n\}$ is 1, 3, 6, 10, 15, ..., its sequence of differences is 2, 3, 4, 5, ... Using this, find the general term of $\{a_n\}$. $(n \geqq 2)$

[Sol]

$$\begin{aligned} 3-1 &= 2 \\ 6-3 &= 3 \\ 10-6 &= 4 \\ &\vdots \\ a_n - a_{n-1} &= n \end{aligned}$$

Adding up the terms on each side from all $(n-1)$ equalities.

$$a_n - \boxed{1} = 2+3+4+\cdots+n$$

$$\therefore a_n = \boxed{1} + (2+3+4+\cdots+n)$$

$$= \boxed{\frac{1}{2}n(n+1)}$$

3. Let $\{b_n\}$ be the sequence of differences of the sequence $\{a_n\}$. Using $\{b_n\}$, find the general term of $\{a_n\}$. $(n \geqq 2)$

[Sol]

$$\begin{aligned} a_2 - a_1 &= b_1 \\ a_3 - a_2 &= b_2 \\ a_4 - a_3 &= b_3 \\ &\vdots \\ a_n - a_{n-1} &= b_{n-1} \end{aligned}$$

Adding up the terms on each side from all $(n-1)$ equalities.

$$a_n - \boxed{a_1} = b_1 + b_2 + b_3 + \cdots + b_{n-1}$$

$$\therefore a_n = \boxed{a_1} + (b_1 + b_2 + b_3 + \cdots + b_{n-1})$$

$$= \boxed{a_1} + \sum_{k=1}^{n-1} b_k$$

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Various Sequences 2

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Sequence of Differences and General Term

Let $\{b_n\}$ be the sequence of differences of the sequence $\{a_n\}$.

When $n \geqq 2$, $a_n = a_1 + \sum_{k=1}^{n-1} b_k$

Find the general term of each given sequence $\{a_n\}$.

Ex. 2, 6, 12, 20, 30, 42, ...

[Sol] Let $\{b_n\}$ be the sequence of differences of $\{a_n\}$. Then, $\{b_n\}$

4, 6, 8, 10, 12, ...

$\therefore\ b_n = 4 + (n-1) \cdot 2 = 2n + 2$ ← $\{b_n\}$ is the arithm sequence with 1st and common diffe

When $n \geqq 2$,

$$a_n = a_1 + \sum_{k=1}^{n-1}(2k+2)$$

$$= 2 + 2 \cdot \frac{1}{2}(n-1)n + 2(n-1)$$ ← $\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$,

$$= n(n+1) \quad \cdots ①$$

Since $a_1 = 2$, ① is also true when $n = 1$. ← Substituting $n=1$ i $a_1 = 2$ and it coinci with the 1st term of given sequence.

$\therefore\ a_n = n(n+1)$

(1) 2, 10, 24, 44, 70, 102, ...

[Sol] Let $\{b_n\}$ be the sequence of differences of $\{a_n\}$. Then, $\{b_n\}$ is

8, 14, 20, 26, 32, ...

$\therefore\ b_n = 8 + (n-1) \cdot 6 = 6n + 2$

When $n \geqq 2$,

$$a_n = a_1 + \sum_{k=1}^{n-1}(6k+2)$$

$$= 2 + 6 \cdot \frac{1}{2}(n-1)n + 2(n-1)$$

$$= n(3n-1) \quad \cdots ①$$

Since $a_1 = 2$, ① is also true when $n = 1$.

$\therefore\ a_n = n(3n-1)$

N32b

(2) 10, 10, 9, 7, 4, 0, …

[Sol] Let $\{b_n\}$ be the sequence of differences of $\{a_n\}$. Then, $\{b_n\}$ is

$$0,\ -1,\ -2,\ -3,\ -4,\ \dots$$

$$\therefore\ b_n=0+(n-1)\cdot(-1)=-n+1$$

When $n \geqq 2$,

$$a_n=a_1+\sum_{k=1}^{n-1}(-k+1)$$

$$=10-\frac{1}{2}(n-1)n+(n-1)$$

$$=-\frac{1}{2}(n+3)(n-6)\ \cdots①$$

Since $a_1=10$, ① is also true when $n=1$.

$$\therefore\ a_n=-\frac{1}{2}(n+3)(n-6)$$

(3) $1,\ \frac{9}{2},\ \frac{20}{2},\ \frac{35}{2},\ \frac{54}{2},\ \frac{77}{2},\ \dots$

[Sol] Let $\{b_n\}$ be the sequence of differences of $\{a_n\}$. Then, $\{b_n\}$ is

$$\frac{7}{2},\ \frac{11}{2},\ \frac{15}{2},\ \frac{19}{2},\ \frac{23}{2},\ \dots$$

$$\therefore\ b_n=\frac{7}{2}+(n-1)\cdot 2=2n+\frac{3}{2}$$

When $n \geqq 2$,

$$a_n=a_1+\sum_{k=1}^{n-1}\left(2k+\frac{3}{2}\right)$$

$$=1+2\cdot\frac{1}{2}(n-1)n+\frac{3}{2}(n-1)$$

$$=\frac{1}{2}(n+1)(2n-1)\ \cdots①$$

$a_1=1$, ① is also true when $n=1$.

$$a_n=\frac{1}{2}(n+1)(2n-1)$$

Various Sequences 2

Name

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Find the general term of each given sequence $\{a_n\}$.

(1) 1, 4, 10, 22, 46, 94, …

[Sol] Let $\{b_n\}$ be the sequence of differences of $\{a_n\}$. Then, $\{b_n\}$ is

3, 6, 12, 24, 48, …

$\therefore\ b_n = 3\cdot 2^{n-1}$

$\{b_n\}$ is the geometric sequence with 1st term 3 and common ratio 2.

When $n \geqq 2$,

$$a_n = a_1 + \sum_{k=1}^{n-1} 3\cdot 2^{k-1}$$

$$= 1 + \frac{3(2^{n-1}-1)}{2-1}$$

$$\sum_{k=1}^{n} ar^{k-1} = \frac{a(r^n-1)}{r-1} \quad (r \neq 1)$$

$$= 3\cdot 2^{n-1} - 2 \quad \cdots ①$$

Since $a_1 = 1$, ① is also true when $n = 1$.

$\therefore\ a_n = 3\cdot 2^{n-1} - 2$

(2) 1, 3, 7, 15, 31, 63, …

[Sol] Let $\{b_n\}$ be the sequence of differences of $\{a_n\}$. Then, $\{b_n\}$ is

2, 4, 8, 16, 32, …

$\therefore\ b_n = 2\cdot 2^{n-1} = 2^n$

When $n \geqq 2$,

$$a_n = a_1 + \sum_{k=1}^{n-1} 2^k$$

$$= 1 + \sum_{k=1}^{n-1} 2\cdot 2^{k-1}$$

$$= 1 + \frac{2(2^{n-1}-1)}{2-1}$$

$$= 2^n - 1 \quad \cdots ①$$

Since $a_1 = 1$, ① is also true when $n = 1$.

$\therefore\ a_n = 2^n - 1$

N33b

(3) 3, 4, 1, 10, −17, 64, …

[Sol] Let $\{b_n\}$ be the sequence of differences of $\{a_n\}$. Then, $\{b_n\}$ is

1, −3, 9, −27, 81, …

$\therefore\ b_n=1\cdot(-3)^{n-1}=(-3)^{n-1}$

When $n \geqq 2$,

$$a_n=a_1+\sum_{k=1}^{n-1}(-3)^{k-1}$$
$$=3+\frac{1-(-3)^{n-1}}{1-(-3)}$$
$$=\frac{13-(-3)^{n-1}}{4}\ \cdots①$$

Since $a_1=3$, ① is also true when $n=1$.

$$\therefore\ a_n=\frac{13-(-3)^{n-1}}{4}$$

(4) $\frac{1}{6}$, $\frac{1}{9}$, $\frac{1}{14}$, $\frac{1}{21}$, $\frac{1}{30}$, $\frac{1}{41}$, …

(Let $b_n=\frac{1}{a_n}$ and $\{c_n\}$ be the sequence of differences of $\{b_n\}$.)

[Sol] Let $b_n=\frac{1}{a_n}$. $\{b_n\}$ is 6, 9, 14, 21, 30, 41, …

Let $\{c_n\}$ be the sequence of differences of $\{b_n\}$. Then, $\{c_n\}$ is

3, 5, 7, 9, 11, …

$\therefore\ c_n=3+(n-1)\cdot 2=2n+1$

When $n \geqq 2$,

$$b_n=b_1+\sum_{k=1}^{n-1}(2k+1)$$
$$=6+2\cdot\frac{1}{2}(n-1)n+(n-1)$$
$$=n^2+5\ \cdots①$$

Since $b_1=6$, ① is also true when $n=1$.

$b_n=n^2+5$

$$a_n=\frac{1}{b_n}=\frac{1}{n^2+5}$$

Various Sequences 2

Name

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Given that S_n is the sum of the first n terms of the sequence $\{a_n\}$, fill in following blanks and find the relationship between S_n and a_n.

[Sol] When $n \geqq 2$, $S_n = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n$ …①

$S_{n-1} = a_1 + a_2 + a_3 + \cdots + a_{n-1}$ …②

From ①−②, $S_n - \boxed{S_{n-1}} = a_n$

Also, $S_1 = \boxed{a_1}$

From the above, the following is true.

Sum of a Sequence and General Term

Let S_n be the sum of the first n terms of the sequence $\{a_n\}$.

The 1st term a_1 is $a_1 = S_1$.

When $n \geqq 2$, $a_n = S_n - S_{n-1}$

Find the general term of each given sequence $\{a_n\}$ whose sum of the first n terms is S_n.

Ex. $S_n = 2n^2 - 3n$

[Sol] The 1st term a_1 is $a_1 = S_1 = 2\cdot 1^2 - 3\cdot 1 = -1$.

When $n \geqq 2$,

$a_n = S_n - \boxed{S_{n-1}}$

$= (2n^2 - 3n) - \{2(\boxed{n-1})^2 - 3(\boxed{n-1})\}$

$= \boxed{4n-5}$ …①

Since $a_1 = -1$, ① is also true when $n = 1$.

$\therefore a_n = \boxed{4n-5}$

N34b

(1) $S_n = n^2 + 4n$

[Sol] The 1st term a_1 is $a_1 = S_1 = 1^2 + 4 \cdot 1 = 5$.

When $n \geqq 2$,

$$a_n = S_n - S_{n-1}$$
$$= (n^2 + 4n) - \{(n-1)^2 + 4(n-1)\}$$
$$= 2n + 3 \quad \cdots ①$$

Since $a_1 = 5$, ① is also true when $n = 1$.

$\therefore\ a_n = 2n + 3$

(2) $S_n = 2^n - 1$

[Sol] The 1st term a_1 is $a_1 = S_1 = 2^1 - 1 = 1$.

When $n \geqq 2$,

$$a_n = S_n - S_{n-1}$$
$$= (2^n - 1) - (2^{n-1} - 1)$$
$$= 2^{n-1} \quad \cdots ①$$

$$\boxed{\begin{aligned} &2^n - 2^{n-1} \\ &= 2 \cdot 2^{n-1} - 2^{n-1} \\ &= (2-1)2^{n-1} \end{aligned}}$$

Since $a_1 = 1$, ① is also true when $n = 1$.

$\therefore\ a_n = 2^{n-1}$

N35a

Various Sequences 2

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Find the sum S for each given question.

Ex. $S=2\cdot1+4\cdot3+6\cdot3^2+8\cdot3^3+\cdots+2n\cdot3^{n-1}$ $\cdots$①

[Sol] Multiplying both sides of ① by 3,

Since ① is
$S=2\cdot1+4\cdot3+\cdots+2(n-1)\cdot3^{n-2}+2n\cdot3^{n-1}$

$$3S=2\cdot3+4\cdot3^2+6\cdot3^3+\cdots+2(n-1)\cdot3^{n-1}+2n\cdot3^n \quad\cdots②$$

From ①−②,

$$-2S=(2+2\cdot3+2\cdot3^2+2\cdot3^3+\cdots+2\cdot3^{n-1})-2n\cdot3^n$$

$S=2\cdot1+4\cdot3+6\cdot3^2+8\cdot3^3+\cdots+2n\cdot3^{n-1}$
$-)\ 3S=\quad 2\cdot3+4\cdot3^2+6\cdot3^3+\cdots+2(n-1)\cdot3^{n-1}+2n\cdot3^n$
$S-3S=2+2\cdot3+2\cdot3^2+2\cdot3^3+\cdots+2\cdot3^{n-1}-2n\cdot3^n$

$$=\frac{2(3^n-1)}{3-1}-2n\cdot3^n$$

$$=-(2n-1)\cdot3^n-1$$

$$\therefore\ S=\frac{(2n-1)\cdot3^n+1}{2}$$

(1) $S=1\cdot1+2\cdot5+3\cdot5^2+4\cdot5^3+\cdots+n\cdot5^{n-1}$ $\cdots$①

[Sol] Multiplying both sides of ① by 5,

$$5S=1\cdot5+2\cdot5^2+3\cdot5^3+\cdots+(n-1)\cdot5^{n-1}+n\cdot5^n \quad\cdots②$$

From ①−②,

$$-4S=(1+5+5^2+5^3+\cdots+5^{n-1})-n\cdot5^n$$

$$=\frac{5^n-1}{5-1}-n\cdot5^n$$

$$=-\frac{(4n-1)\cdot5^n+1}{4}$$

$$\therefore\ S=\frac{(4n-1)\cdot5^n+1}{16}$$

N35b

(2) $S = n\cdot 1 + (n-1)\cdot 3 + (n-2)\cdot 3^2 + (n-3)\cdot 3^3 + \cdots + 2\cdot 3^{n-2} + 1\cdot 3^{n-1}$ ···①

[Sol] Multiplying both sides of ① by 3,

$$3S = n\cdot 3 + (n-1)\cdot 3^2 + (n-2)\cdot 3^3 + \cdots + 3\cdot 3^{n-2} + 2\cdot 3^{n-1} + 3^n \quad \cdots②$$

From ①−②,

$$-2S = n - (3 + 3^2 + 3^3 + \cdots + 3^{n-2} + 3^{n-1} + 3^n)$$

$$= n - \frac{3(3^n-1)}{3-1}$$

$$= \frac{2n - 3^{n+1} + 3}{2}$$

$$\therefore S = \frac{2n - 3^{n+1} + 3}{4}$$

(3) $S = 4\cdot 1 + 7\cdot 2 + 10\cdot 2^2 + 13\cdot 2^3 + \cdots + (3n+1)\cdot 2^{n-1}$ ···①

[Sol] Multiplying both sides of ① by 2,

$$2S = 4\cdot 2 + 7\cdot 2^2 + 10\cdot 2^3 + \cdots + (3n-2)\cdot 2^{n-1} + (3n+1)\cdot 2^n \quad \cdots②$$

From ①−②,

$$-S = 4 + (3\cdot 2 + 3\cdot 2^2 + 3\cdot 2^3 + \cdots + 3\cdot 2^{n-1}) - (3n+1)\cdot 2^n$$

$$= 4 + \frac{6(2^{n-1}-1)}{2-1} - (3n+1)\cdot 2^n$$

$$= -(3n-2)\cdot 2^n - 2$$

$$\therefore S = (3n-2)\cdot 2^n + 2$$

Alternative Solution

Multiplying both sides of ① by 2,

$$2S = 4\cdot 2 + 7\cdot 2^2 + 10\cdot 2^3 + \cdots + (3n-2)\cdot 2^{n-1} + (3n+1)\cdot 2^n \quad \cdots②$$

①−②,

$$+ (3 + 3\cdot 2 + 3\cdot 2^2 + \cdots + 3\cdot 2^{n-1}) - (3n+1)\cdot 2^n$$

$$+ \frac{3(2^n-1)}{2-1} - (3n+1)\cdot 2^n$$

$$3n-2)\cdot 2^n - 2$$

$$n-2)\cdot 2^n + 2$$

Various Sequences 2

Name

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Time : to :

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Find the sum S for each given question.

Ex. $S=\dfrac{1}{\sqrt{1}+\sqrt{2}}+\dfrac{1}{\sqrt{2}+\sqrt{3}}+\dfrac{1}{\sqrt{3}+\sqrt{4}}+\cdots+\dfrac{1}{\sqrt{n}+\sqrt{n+1}}$

[Sol] $\dfrac{1}{\sqrt{k}+\sqrt{k+1}}=\dfrac{\sqrt{k}-\sqrt{k+1}}{(\sqrt{k}+\sqrt{k+1})(\sqrt{k}-\sqrt{k+1})}$

$=-\sqrt{k}+\sqrt{k+1}$

$$\begin{array}{rl} & -\sqrt{1}+\sqrt{2} \\ & -\sqrt{2}+\sqrt{3} \\ & -\sqrt{3}+\sqrt{4} \\ +) & -\sqrt{n}+\sqrt{n+1} \\ \hline & -1+\sqrt{n+1} \end{array}$$

Therefore,

$S=(-\sqrt{1}+\sqrt{2})+(-\sqrt{2}+\sqrt{3})+(-\sqrt{3}+\sqrt{4})+\cdots+(-\sqrt{n}+\sqrt{n+1})$

$=-1+\sqrt{n+1}$

(1) $S=\dfrac{2}{\sqrt{1}+\sqrt{3}}+\dfrac{2}{\sqrt{3}+\sqrt{5}}+\dfrac{2}{\sqrt{5}+\sqrt{7}}+\cdots+\dfrac{2}{\sqrt{2n-1}+\sqrt{2n+1}}$

[Sol] $\dfrac{2}{\sqrt{2k-1}+\sqrt{2k+1}}=\dfrac{2(\sqrt{2k-1}-\sqrt{2k+1})}{(\sqrt{2k-1}+\sqrt{2k+1})(\sqrt{2k-1}-\sqrt{2k+1})}$

$=-\sqrt{2k-1}+\sqrt{2k+1}$

Therefore,

$S=(-\sqrt{1}+\sqrt{3})+(-\sqrt{3}+\sqrt{5})+(-\sqrt{5}+\sqrt{7})+\cdots+(-\sqrt{2n-1}+\sqrt{2n}$

$=-1+\sqrt{2n+1}$

N36b

(2) $S=\dfrac{1}{\sqrt{2}+\sqrt{5}}+\dfrac{1}{\sqrt{5}+\sqrt{8}}+\dfrac{1}{\sqrt{8}+\sqrt{11}}+\cdots+\dfrac{1}{\sqrt{3n-1}+\sqrt{3n+2}}$

[Sol] $\dfrac{1}{\sqrt{3k-1}+\sqrt{3k+2}}=\dfrac{\sqrt{3k-1}-\sqrt{3k+2}}{(\sqrt{3k-1}+\sqrt{3k+2})(\sqrt{3k-1}-\sqrt{3k+2})}$

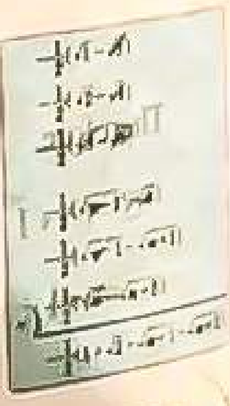
$$=-\frac{1}{3}(\sqrt{3k-1}-\sqrt{3k+2})$$

Therefore,

$$S=-\frac{1}{3}(\sqrt{2}-\sqrt{5})-\frac{1}{3}(\sqrt{5}-\sqrt{8})-\frac{1}{3}(\sqrt{8}-\sqrt{11})-\cdots-\frac{1}{3}(\sqrt{3n-1}-\sqrt{3n+2})$$

$$=-\frac{1}{3}(\sqrt{2}-\sqrt{3n+2})$$

(3) $S=\dfrac{1}{\sqrt{1}+\sqrt{3}}+\dfrac{1}{\sqrt{2}+\sqrt{4}}+\dfrac{1}{\sqrt{3}+\sqrt{5}}+\cdots+\dfrac{1}{\sqrt{n}+\sqrt{n+2}}$



[Sol] $\dfrac{1}{\sqrt{k}+\sqrt{k+2}}=\dfrac{\sqrt{k}-\sqrt{k+2}}{(\sqrt{k}+\sqrt{k+2})(\sqrt{k}-\sqrt{k+2})}$

$$=-\frac{1}{2}(\sqrt{k}-\sqrt{k+2})$$

Therefore,

$$S=-\frac{1}{2}(\sqrt{1}-\sqrt{3})-\frac{1}{2}(\sqrt{2}-\sqrt{4})-\frac{1}{2}(\sqrt{3}-\sqrt{5})-\cdots$$

$$\cdots-\frac{1}{2}(\sqrt{n-2}-\sqrt{n})-\frac{1}{2}(\sqrt{n-1}-\sqrt{n+1})-\frac{1}{2}(\sqrt{n}-\sqrt{n+2})$$

$$=-\frac{1}{2}(1+\sqrt{2}-\sqrt{n+1}-\sqrt{n+2})$$

N37a

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Various Sequences 2

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Find the sum S for each given question.

Ex. $S=\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\dfrac{1}{3\cdot 4}+\cdots+\dfrac{1}{n(n+1)}$

[Sol] Let $\dfrac{1}{k(k+1)}=\dfrac{a}{k}-\dfrac{b}{k+1}$.

Multiplying both sides by $k(k+1)$

$1=a(k+1)-bk$

$=(a-b)k+a$

Coefficient Comparison Method (J181)

$\therefore \begin{cases} a-b=0 \\ a=1 \end{cases}$

$\therefore\ a=1,\ b=1$

Therefore,

$S=\left(\dfrac{1}{1}-\dfrac{1}{2}\right)+\left(\dfrac{1}{2}-\dfrac{1}{3}\right)+\left(\dfrac{1}{3}-\dfrac{1}{4}\right)+\cdots+\left(\dfrac{1}{n}-\dfrac{1}{n+1}\right)$

$=1-\dfrac{1}{n+1}=\dfrac{n}{n+1}$

(1) $S=\dfrac{2}{1\cdot 3}+\dfrac{2}{3\cdot 5}+\dfrac{2}{5\cdot 7}+\cdots+\dfrac{2}{(2n-1)(2n+1)}$

[Sol] Let $\dfrac{2}{(2k-1)(2k+1)}=\dfrac{a}{2k-1}-\dfrac{b}{2k+1}$.

$2=a(2k+1)-b(2k-1)$

$=2(a-b)k+(a+b)$

$\therefore \begin{cases} 2(a-b)=0 \\ a+b=2 \end{cases}$

$\therefore\ a=1,\ b=1$

Therefore,

$S=\left(\dfrac{1}{1}-\dfrac{1}{3}\right)+\left(\dfrac{1}{3}-\dfrac{1}{5}\right)+\left(\dfrac{1}{5}-\dfrac{1}{7}\right)+\cdots+\left(\dfrac{1}{2n-1}-\dfrac{1}{2n+1}\right)$

$=1-\dfrac{1}{2n+1}=\dfrac{2n}{2n+1}$

N37b

(2) $S=\frac{1}{2\cdot4}+\frac{1}{4\cdot6}+\frac{1}{6\cdot8}+\cdots+\frac{1}{2n(2n+2)}$

[Sol] Let $\frac{1}{2k(2k+2)}=\frac{a}{2k}-\frac{b}{2k+2}$.

$$1=a(2k+2)-b\cdot2k$$
$$=2(a-b)k+2a$$
$$\therefore\ \begin{cases}2(a-b)=0\\2a=1\end{cases}$$
$$\therefore\ a=\frac{1}{2},\ b=\frac{1}{2}$$

Therefore,

$$S=\frac{1}{2}\left(\frac{1}{2}-\frac{1}{4}\right)+\frac{1}{2}\left(\frac{1}{4}-\frac{1}{6}\right)+\frac{1}{2}\left(\frac{1}{6}-\frac{1}{8}\right)+\cdots+\frac{1}{2}\left(\frac{1}{2n}-\frac{1}{2n+2}\right)$$
$$=\frac{1}{2}\left(\frac{1}{2}-\frac{1}{2n+2}\right)=\frac{n}{4(n+1)}$$

(3) $S=\frac{1}{1\cdot3}+\frac{1}{2\cdot4}+\frac{1}{3\cdot5}+\cdots+\frac{1}{n(n+2)}$

[Sol] Let $\frac{1}{k(k+2)}=\frac{a}{k}-\frac{b}{k+2}$.

$$1=a(k+2)-bk$$
$$=(a-b)k+2a$$
$$\therefore\ \begin{cases}a-b=0\\2a=1\end{cases}$$
$$\therefore\ a=\frac{1}{2},\ b=\frac{1}{2}$$

Therefore,

$$\frac{1}{2}\left(\frac{1}{1}-\frac{1}{3}\right)+\frac{1}{2}\left(\frac{1}{2}-\frac{1}{4}\right)+\frac{1}{2}\left(\frac{1}{3}-\frac{1}{5}\right)+\cdots$$
$$\cdots+\frac{1}{2}\left(\frac{1}{n-2}-\frac{1}{n}\right)+\frac{1}{2}\left(\frac{1}{n-1}-\frac{1}{n+1}\right)+\frac{1}{2}\left(\frac{1}{n}-\frac{1}{n+2}\right)$$
$$\left(1+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}\right)=\frac{n(3n+5)}{4(n+1)(n+2)}$$

Various Sequences 2

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Ex. Partitioning the sequence of positive odd numbers into groups as shown below, assume that the n^{th} group contains n terms.

$$1 \mid 3,\ 5 \mid 7,\ 9,\ 11 \mid 13,\ 15,\ 17,\ 19 \mid 21,\ \cdots$$

1st group 2nd group 3rd group 4th group

(1) The m^{th} term is the first odd number in the 10^{th} group. Find m.

[Sol] The number of terms from the 1st to the 9^{th} group is:

$1 \mid 3,\ 5 \mid 7,\ 9,\ 11 \mid \cdots \mid 9^{\text{th}}$ group; 1 term, 2 terms, 3 terms, …, 9 terms

$$1+2+3+\cdots+9=\sum_{k=1}^{9}k=\frac{1}{2}\cdot 9(9+1)=45$$

$\leftarrow$ $\sum_{k=1}^{n}k=\frac{1}{2}n(n+1)$

$$\therefore\ m=\boxed{46}$$

$\leftarrow$ Since the last term in the 9^{th} group is the 45^{th} term, the first term in the 10^{th} group is $45+1=46^{\text{th}}$.

(2) Find the first odd number x in the n^{th} group.

[Sol] The n^{th} group contains n terms.

When $n \geqq 2$, the number of terms from the 1st to the $(n-1)^{\text{th}}$ group

$$1+2+3+\cdots+(n-1)=\sum_{k=1}^{n-1}k=\frac{1}{2}(n-1)n$$

Therefore, the first odd number in the n^{th} group is the $\left[\boxed{\frac{1}{2}(n-1)n}+1\right]^{\text{th}}$ term.

$$\therefore\ x=2\left[\boxed{\frac{1}{2}(n-1)n}+1\right]-1=\boxed{n^2-n+1}\ \cdots①$$

$\leftarrow$ Since the sequence of positive odd numbers is the arithmetic sequence with 1st term 1, common difference 2, the k^{th} term is $1+(k-1)\cdot 2=2k-1$

Since $x=1$ when $n=1$, ① is also true when $n=1$.

$$\therefore\ x=\boxed{n^2-n+1}$$

(3) Find the sum S of all odd numbers in the n^{th} group.

[Sol] Since S is the sum of the arithmetic sequence with 1st term $\boxed{n^2-n+1}$, common difference 2 and number of terms n,

$$S=\frac{1}{2}n[2(\boxed{n^2-n+1})+(n-1)\cdot 2]=\boxed{n^3}$$

$\leftarrow$ $S_n=\frac{1}{2}n[2a+(n-1)d]$

46, $\frac{1}{2}(n-1)n$, $\frac{1}{2}(n-1)n$, n^2-n+1, n^2-n+1, n^2-n+1, n^2-n+1, n^3

A set of numbers partitioned under a certain rule is called a *group*.

N38b

Partitioning the sequence of natural numbers into groups as shown below, assume that the n^{th} group contains $2n$ terms.

$$\underbrace{1,\ 2}_{1^{st}\text{ group}}\ \Big|\ \underbrace{3,\ 4,\ 5,\ 6}_{2^{nd}\text{ group}}\ \Big|\ \underbrace{7,\ 8,\ 9,\ 10,\ 11,\ 12}_{3^{rd}\text{ group}}\ \Big|\ 13,\ \ldots$$

(1) The m^{th} term is the first natural number in the 8^{th} group. Find m.

[Sol] The number of terms from the 1^{st} to the 7^{th} group is:

$$2+4+6+\cdots+14=\sum_{k=1}^{7}2k=2\cdot\frac{1}{2}\cdot 7(7+1)=56$$

$\therefore\ m=57$

1, 2 | 3, 4, 5, 6 | … | 7^{th}
2 … 4 … 14 …

(2) Find the first natural number x in the n^{th} group.

[Sol] The n^{th} group contains $2n$ terms.

When $n \geqslant 2$, the number of terms from the 1^{st} to the $(n-1)^{th}$ group is:

$$2+4+6+\cdots+2(n-1)=\sum_{k=1}^{n-1}2k=2\cdot\frac{1}{2}(n-1)n=(n-1)n$$

Therefore, the first natural number in the n^{th} group is the $[(n-1)n+1]^{th}$ term.

$$\therefore\ x=(n-1)n+1=n^2-n+1 \quad \cdots ①$$

Since $x=1$ when $n=1$, ① is also true when $n=1$.

$$\therefore\ x=n^2-n+1$$

(3) Find the sum S of all natural numbers in the n^{th} group.

[Sol] Since S is the sum of the arithmetic sequence with 1^{st} term n^2-n+1, common difference 1 and number of terms $2n$,

$$S=\frac{1}{2}\cdot 2n[2(n^2-n+1)+(2n-1)\cdot 1]=n(2n^2+1)$$

N39a

KUMON

Various Sequences 2

Name

Date / /

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Time : to :

1. Find the general term of the following sequence.

$$6,\ 24,\ 60,\ 120,\ 210,\ 336,\ 504,\ \cdots$$

[Sol] Let $\{a_n\}$ be the given sequence, $\{b_n\}$ be the sequence of differences of $\{a_n\}$, and $\{c_n\}$ be the sequence of the differences of $\{b_n\}$.

$\{a_n\}: 6,\ 24,\ 60,\ 120,\ 210,\ 336,\ 504,\ \cdots$

$\{b_n\}: 18,\ 36,\ 60,\ 90,\ 126,\ 168,\ \cdots$

$\{c_n\}: 18,\ 24,\ 30,\ 36,\ 42,\ \cdots$

$\therefore\ c_n=18+(n-1)\cdot 6=6n+12$

When $n \geqq 2$,

$$b_n=b_1+\sum_{k=1}^{n-1}(6k+12)$$
$$=18+6\cdot\frac{1}{2}(n-1)n+12(n-1)$$
$$=3n^2+9n+6\ \cdots①$$

Since $b_1=18$, ① is also true when $n=1$.

$\therefore\ b_n=3n^2+9n+6$

When $n \geqq 2$,

$$a_n=a_1+\sum_{k=1}^{n-1}(3k^2+9k+6)$$
$$=6+3\cdot\frac{1}{6}(n-1)n(2n-1)+9\cdot\frac{1}{2}(n-1)n+6(n-1)$$
$$=n(n+1)(n+2)\ \cdots②$$

Since $a_1=6$, ② is also true when $n=1$.

$\therefore\ a_n=n(n+1)(n+2)$

N39b

2. Given the sequence 1, $\frac{1}{2}$, $\frac{2}{2}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{3}{3}$, $\frac{1}{4}$, $\frac{2}{4}$, $\frac{3}{4}$, $\frac{4}{4}$, ..., the m^{th} term is $\frac{5}{11}$. Find m. Also, find the 48^{th} term.

[Sol] Partitioning the sequence into groups by the common denominator,

$$1 \left| \frac{1}{2}, \frac{2}{2} \right| \frac{1}{3}, \frac{2}{3}, \frac{3}{3} \left| \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4} \right| \cdots$$

Therefore, $\frac{5}{11}$ is the 5^{th} term in the 11^{th} group.

Since the n^{th} group contains n terms, the number of terms from the 1^{st} to the 10^{th} group is:

The 11^{th} group contains the fractions with denominator 11.

$$\cdots \left| \underset{1^{st}}{\frac{1}{11}}, \frac{2}{11}, \frac{3}{11}, \frac{4}{11}, \underset{5^{th}}{\frac{5}{11}}, \cdots \frac{11}{11} \right| \cdots$$

$$1+2+3+\cdots+10=\sum_{k=1}^{10} k=\frac{1}{2}\cdot 10(10+1)=55$$

$\therefore\ m=60$

Since $\frac{5}{11}$ is the 5^{th} term in the 11^{th} group, $55+5=60$

Also, let the 48^{th} term be in the n^{th} group.

$$\frac{1}{2}(n-1)n<48\leq\frac{1}{2}n(n+1)$$

$$(n-1)n<96\leq n(n+1)$$

The number of terms from the 1^{st} to the $(n-1)^{th}$ group is: $1+2+3+\cdots+(n-1)=\sum_{k=1}^{n-1} k$

The number of terms from the 1^{st} to the n^{th} group is: $1+2+3+\cdots+n=\sum_{k=1}^{n} k$

The natural number n that satisfies this is $n=10$.

Finding the natural number n that satisfies $(n-1)n<96\leq n(n+1)$ by substituting some values into n

The number of terms from the 1^{st} to the 9^{th} group is:

$$1+2+3+\cdots+9=\sum_{k=1}^{9} k=\frac{1}{2}\cdot 9(9+1)=45$$

Therefore, the 48^{th} term is $\frac{3}{10}$

Since $48-45=3$, the 48^{th} term is the 3^{rd} term in the 10^{th} group.

Various Sequences 2

Name

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Time : to :

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1. Find the general term of the following sequence $\{a_n\}$. ➡ N

$$7,\ 19,\ 37,\ 61,\ 91,\ 127,\ \ldots$$

[Sol] Let $\{b_n\}$ be the sequence of differences of $\{a_n\}$. Then, $\{b_n\}$ is

$$12,\ 18,\ 24,\ 30,\ 36,\ \ldots$$

$$\therefore\ b_n=12+(n-1)\cdot 6=6n+6$$

When $n \geqq 2$,

$$a_n=a_1+\sum_{k=1}^{n-1}(6k+6)$$

$$=7+6\cdot\frac{1}{2}(n-1)n+6(n-1)$$

$$=3n^2+3n+1\ \cdots①$$

Since $a_1=7$, ① is also true when $n=1$.

$$\therefore\ a_n=3n^2+3n+1$$

2. Find the general term of the sequence $\{a_n\}$ whose sum of the first n t
is S_n. ➡

$$S_n=5n-n^2$$

[Sol] The 1st term a_1 is $a_1=S_1=5\cdot 1-1^2=4$.

When $n \geqq 2$,

$$a_n=S_n-S_{n-1}$$

$$=(5n-n^2)-[5(n-1)-(n-1)^2]$$

$$=-2n+6\ \cdots①$$

Since $a_1=4$, ① is also true when $n=1$.

$$\therefore\ a_n=-2n+6$$

N40b

3. Find the sum S for each given question.

(1) $S=1\cdot1+2\cdot3+3\cdot3^2+4\cdot3^3+\cdots+n\cdot3^{n-1}$ —①　　➡ N35

[Sol] Multiplying both sides of ① by 3,

$$3S=1\cdot3+2\cdot3^2+3\cdot3^3+\cdots+(n-1)\cdot3^{n-1}+n\cdot3^n \quad —②$$

From ①−②,

$$-2S=(1+3+3^2+3^3+\cdots+3^{n-1})-n\cdot3^n$$

$$=\frac{3^n-1}{3-1}-n\cdot3^n$$

$$=-\frac{(2n-1)\cdot3^n+1}{2}$$

$$\therefore\ S=\frac{(2n-1)\cdot3^n+1}{4}$$

(2) $S=\frac{3}{1\cdot4}+\frac{3}{4\cdot7}+\frac{3}{7\cdot10}+\cdots+\frac{3}{(3n-2)(3n+1)}$　　➡ N37

[Sol] Let $\frac{3}{(3k-2)(3k+1)}=\frac{a}{3k-2}-\frac{b}{3k+1}$

$$3=a(3k+1)-b(3k-2)$$

$$=3(a-b)k+(a+2b)$$

$$\therefore\ \begin{cases}3(a-b)=0\\ a+2b=3\end{cases}$$

$$\therefore\ a=1,\ b=1$$

Therefore,

$$S=\left(\frac{1}{1}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{7}\right)+\left(\frac{1}{7}-\frac{1}{10}\right)+\cdots+\left(\frac{1}{3n-2}-\frac{1}{3n+1}\right)$$

$$=1-\frac{1}{3n+1}=\frac{3n}{3n+1}$$

Recurrence Relations

Name

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Given that the sequence $\{a_n\}$ satisfies the following two conditions:

(i) $a_1=2$

(ii) $a_{n+1}=3a_n-1 \quad (n=1, 2, 3, \ldots)$,

fill in the following blanks.

[Sol] $a_2=3a_1-1=3\cdot\boxed{2}-1=\boxed{5}$

$a_3=3a_2-1=3\cdot\boxed{5}-1=\boxed{14}$

$a_4=3a_3-1=3\cdot\boxed{14}-1=\boxed{41}$

2, 5, 5, 14, 14, 41

From the above, each term is successively determined from a_1 to a_2, a_3, a one after another, then one sequence $\{a_n\}$ will be defined. A condition suc (ii), where the expression defines the relationship between two or more consecu terms in a sequence, is called a ***recurrence relation***.

Find the 4th term of each given sequence $\{a_n\}$ defined by the following condi

Ex. $a_1=4,\ a_{n+1}=2a_n-3$

[Sol] $a_2=2a_1-3=5$

$a_3=2a_2-3=7$

$\therefore\ a_4=2a_3-3=11$

(1) $a_1=1,\ a_{n+1}=3a_n+2$

[Sol] $a_2=3a_1+2=5$

$a_3=3a_2+2=17$

$\therefore\ a_4=3a_3+2=53$

N41b

(2) $a_1=-2,\ a_{n+1}=a_n-5$

[Sol] $a_2=a_1-5=-7$

$a_3=a_2-5=-12$

$\therefore a_4=a_3-5=-17$

(3) $a_1=2,\ a_{n+1}=3a_n$

[Sol] $a_2=3a_1=6$

$a_3=3a_2=18$

$\therefore a_4=3a_3=54$

(4) $a_1=1,\ a_{n+1}=2-\dfrac{1}{a_n}$

[Sol] $a_2=2-\dfrac{1}{a_1}=1$

$a_3=2-\dfrac{1}{a_2}=1$

$\therefore a_4=2-\dfrac{1}{a_3}=1$

(5) $a_1=5,\ a_{n+1}=a_n+n$

[Sol] $a_2=a_1+1=6$

$a_3=a_2+2=8$

$\therefore a_4=a_3+3=11$

(6) $a_2=2,\ a_{n+2}=a_{n+1}+a_n$

[S $=3$

$+a_3=5$

Recurrence Relations

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Find the general term of each given sequence $\{a_n\}$ defined by the following conditions.

Ex. $a_1=1,\ a_{n+1}=a_n+5$

[Sol] $a_1=1$

$a_2=a_1+5=6$

$a_3=a_2+5=11$

$\vdots$

1, 6, 11, …
+5 +5

Therefore, $\{a_n\}$ is the arithmetic sequence with 1^{st} term 1 and common difference 5.

$\therefore\ a_n=1+(n-1)\cdot 5$
$=5n-4$

The general term of an arithmetic sequence is $a_n=a+(n-1)d$.

$a_1=3,\ a_{n+1}=2a_n$

[Sol] $a_1=3$

$a_2=2a_1=6$

$a_3=2a_2=12$

$\vdots$

3, 6, 12, …
×2 ×2

Therefore, $\{a_n\}$ is the geometric sequence with 1^{st} term 3 and common ratio 2.

$\therefore\ a_n=3\cdot 2^{n-1}$

The general term of a geometric sequence is $a_n=ar^{n-1}$.

(1) $a_1=4,\ a_{n+1}=a_n+8$

[Sol] $a_1=4$

$a_2=a_1+8=12$

$a_3=a_2+8=20$

$\vdots$

Therefore, $\{a_n\}$ is the arithmetic sequence with 1^{st} term 4 and common difference 8.

$\therefore\ a_n=4+(n-1)\cdot 8$
$=8n-4$

(2) $a_1=a,\ a_{n+1}=a_n+d$

[Sol] $a_1=a$

$a_2=a_1+d=a+d$

$a_3=a_2+d=a+2d$

$\vdots$

Therefore, $\{a_n\}$ is the arithmetic sequence with 1^{st} term a and common difference d.

$\therefore\ a_n=a+(n-1)d$

(3) $a_1=5,\ a_{n+1}=4a_n$

[Sol] $a_1=5$

$a_2=4a_1=20$

$a_3=4a_2=80$

$\vdots$

Therefore, $\{a_n\}$ is the geometric sequence with 1^{st} term 5 and common ratio 4.

$\therefore\ a_n=5\cdot 4^{n-1}$

(4) $a_1=a,\ a_{n+1}=ra_n$

[Sol] $a_1=a$

$a_2=ra_1=ar$

$a_3=ra_2=ar^2$

$\vdots$

Therefore, $\{a_n\}$ is the geometric sequence with 1^{st} term a and common ratio r.

$\therefore\ a_n=ar^{n-1}$

N42b

Based on the results on side a, the following statements are true.
The sequence $\{a_n\}$ defined by $a_1=a$, $a_{n+1}=a_n+d$ is an arithmetic sequence with 1st term a and common difference d.
The sequence $\{a_n\}$ defined by $a_1=a$, $a_{n+1}=ra_n$ is a geometric sequence with 1st term a and common ratio r.

Ex.

$a_1=2$, $a_{n+1}=a_n+3$

[Sol] $a_n=2+(n-1)\cdot 3$
$=3n-1$

[illegible]

$a_1=3$, $a_{n+1}=7a_n$

[Sol] $a_n=3\cdot 7^{n-1}$

[illegible]

(5) $a_1=-7$, $a_{n+1}=a_n+5$

[Sol] $a_n=-7+(n-1)\cdot 5$
$=5n-12$

(8) $a_1=2$, $a_{n+1}=3a_n$

[Sol] $a_n=2\cdot 3^{n-1}$

(6) $a_1=3$, $a_{n+1}=a_n-7$

[Sol] $a_n=3+(n-1)\cdot(-7)$
$=-7n+10$

(9) $a_1=-5$, $a_{n+1}=-2a_n$

[Sol] $a_n=-5(-2)^{n-1}$

7) $a_1=-4$, $a_{n+1}=a_n-9$

ol] $a_n=-4+(n-1)\cdot(-9)$
$=-9n+5$

(10) $a_1=-7$, $a_{n+1}=7a_n$

[Sol] $a_n=-7\cdot 7^{n-1}$
$=-7^n$

N43a

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Recurrence Relations

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Find the general term of each given sequence $\{a_n\}$ defined by the following conditions.

Ex. $a_1=1,\ a_{n+1}=a_n+3n$

[Sol] $a_{n+1}-a_n=3n$

Since the general term of the sequence of differences of $\{a_n\}$ is $3n$,

when $n \geqq 2$,

$$a_n=a_1+\sum_{k=1}^{n-1}3k=1+3\cdot\frac{1}{2}(n-1)n$$

$$=\frac{1}{2}(3n^2-3n+2) \cdots ①$$

Since $a_1=1$, ① is also true when $n=1$.

$$\therefore\ a_n=\frac{1}{2}(3n^2-3n+2)$$

Let $\{b_n\}$ be the sequence of differences of $\{a_n\}$.

$\{a_n\}$ a_1 a_2 a_3 … a_n a_{n+1}

$\{b_n\}$ 3 6 … $3n$

Therefore, the general term of $\{b_n\}$ is $3n$.

When $n \geqq 2$,

$$a_n=a_1+\sum_{k=1}^{n-1}b_k$$

(N32)

(1) $a_1=3,\ a_{n+1}=a_n+6n$

[Sol] $a_{n+1}-a_n=6n$

Since the general term of the sequence of differences of $\{a_n\}$ is $6n$,

when $n \geqq 2$,

$$a_n=a_1+\sum_{k=1}^{n-1}6k=3+6\cdot\frac{1}{2}(n-1)n$$

$$=3(n^2-n+1) \cdots ①$$

Since $a_1=3$, ① is also true when $n=1$.

$$\therefore\ a_n=3(n^2-n+1)$$

(2) $a_1=2,\ a_{n+1}=a_n+n^2$

[Sol] $a_{n+1}-a_n=n^2$

Since the general term of the sequence of differences of $\{a_n\}$ is n^2,

when $n \geqq 2$,

$$a_n=a_1+\sum_{k=1}^{n-1}k^2=2+\frac{1}{6}(n-1)n(2n-1)$$

$\sum_{k=1}^{n}k^2=\frac{1}{6}n(n+1)(2$

$$=\frac{1}{6}(2n^3-3n^2+n+12) \cdots ①$$

Since $a_1=2$, ① is also true when $n=1$.

$$\therefore\ a_n=\frac{1}{6}(2n^3-3n^2+n+12)$$

N43b

(3) $a_1=4,\ a_{n+1}=a_n+4n^3$

[Sol] $a_{n+1}-a_n=4n^3$

Since the general term of the sequence of differences of $\{a_n\}$ is $4n^3$,

when $n \geqq 2$,

$$a_n=a_1+\sum_{k=1}^{n-1}4k^3=4+4\left[\frac{1}{2}(n-1)n\right]^2 \quad \leftarrow \quad \boxed{\sum_{k=1}^{n}k^3=\left[\frac{1}{2}n(n+1)\right]^2}$$

$$=n^4-2n^3+n^2+4 \quad \cdots①$$

Since $a_1=4$, ① is also true when $n=1$.

$\therefore\ a_n=n^4-2n^3+n^2+4$

(4) $a_1=3,\ a_{n+1}=a_n+2^{n-1}$

[Sol] $a_{n+1}-a_n=2^{n-1}$

Since the general term of the sequence of differences of $\{a_n\}$ is 2^{n-1},

when $n \geqq 2$,

$$a_n=a_1+\sum_{k=1}^{n-1}2^{k-1}=3+\frac{2^{n-1}-1}{2-1} \quad \leftarrow \quad \boxed{\sum_{k=1}^{n}ar^{k-1}=\frac{a(r^n-1)}{r-1}\ (r\neq 1)}$$

$$=2^{n-1}+2 \quad \cdots①$$

Since $a_1=3$, ① is also true when $n=1$.

$\therefore\ a_n=2^{n-1}+2$

(5) $a_1=4,\ a_{n+1}=a_n+5^n$

[Sol] $a_{n+1}-a_n=5^n$

Since the general term of the sequence of differences of $\{a_n\}$ is 5^n,

when $n \geqq 2$,

$$a_n=a_1+\sum_{k=1}^{n-1}5^k=4+\sum_{k=1}^{n-1}5\cdot 5^{k-1}$$

$$=4+\frac{5(5^{n-1}-1)}{5-1}$$

$$=\frac{5^n+11}{4} \quad \cdots①$$

Since $a_1=4$, ① is also true when $n=1$.

$\therefore\ a_n=\frac{5^n+11}{4}$

Recurrence Relations

Name

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Time : to :

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Assume that the recurrence relation $a_{n+1}=pa_n+q$ ⋯① is true when p and q are constants ($p\neq0$, $p\neq1$). By using x which satisfies $x=px+q$ where a_{n+1} and a_n are both replaced with x, prove that the recurrence relation ① can be rearranged into $a_{n+1}-x=p(a_n-x)$.

[Sol] Since $x=px+q$, $x=\dfrac{\boxed{q}}{1-p}$ ⋯②

Subtracting the RHS of ② from the both sides of ①,

$$\text{LHS}=a_{n+1}-\frac{\boxed{q}}{1-p}\ \cdots③$$

$$\text{RHS}=pa_n+q-\frac{\boxed{q}}{1-p}=pa_n+\frac{q(1-p)-\boxed{q}}{1-p}$$

$$=pa_n-\frac{\boxed{pq}}{1-p}=p\left(a_n-\frac{\boxed{q}}{1-p}\right)\ \cdots④$$

From ③ and ④, $a_{n+1}-x=p(a_n-x)$

Find the general term of each given sequence $\{a_n\}$ defined by the following conditions.

Ex. $a_1=3$, $a_{n+1}=3a_n-2$

[Sol] Since $a_{n+1}=3a_n-2$, $a_{n+1}-1=\boxed{3}(a_n-1)$ ← Replacing both a_{n+1} and a_n with x, $x=3x-2$ $\therefore\ x=1$

Let $b_n=a_n-1$.

$$\begin{cases} b_{n+1}=\boxed{3}\,b_n & \cdots① \\ b_1=a_1-1=\boxed{2} & \cdots② \end{cases}$$

$\{b_n\}$ is the geometric sequence with 1st term $b_1=2$ and common ratio 3.

From ① and ②, $b_n=\boxed{2\cdot3^{n-1}}$

$\therefore\ a_n=b_n+1=\boxed{2\cdot3^{n-1}+1}$

4b

$a_1=4,\ a_{n+1}=2a_n+1$

Since $a_{n+1}=2a_n+1$, $a_{n+1}+1=2(a_n+1)$

Let $b_n=a_n+1$

$$\begin{cases} b_{n+1}=2b_n & \cdots ① \\ b_1=a_1+1=5 & \cdots ② \end{cases}$$

From ① and ②, $b_n=5\cdot 2^{n-1}$

$\therefore\ a_n=b_n-1=5\cdot 2^{n-1}-1$

Replacing both a_{n+1} and a_n with x,
$x=2x+1$
$\therefore\ x=-1$

$a_1=1,\ a_{n+1}=2a_n+3$

[Sol] Since $a_{n+1}=2a_n+3$, $a_{n+1}+3=2(a_n+3)$

Let $b_n=a_n+3$.

$$\begin{cases} b_{n+1}=2b_n & \cdots ① \\ b_1=a_1+3=4 & \cdots ② \end{cases}$$

From ① and ②, $b_n=4\cdot 2^{n-1}=2^{n+1}$

$\therefore\ a_n=b_n-3=2^{n+1}-3$

Replacing both a_{n+1} and a_n with x,
$x=2x+3$
$\therefore\ x=-3$

(3) $a_1=1,\ 3a_{n+1}-2a_n=3$

[Sol] Since $3a_{n+1}-2a_n=3$, $a_{n+1}=\frac{2}{3}a_n+1$; therefore,

$$a_{n+1}-3=\frac{2}{3}(a_n-3)$$

Let $b_n=a_n-3$.

$$\begin{cases} b_{n+1}=\frac{2}{3}b_n & \cdots ① \\ b_1=a_1-3=-2 & \cdots ② \end{cases}$$

From ① and ②, $b_n=-2\left(\frac{2}{3}\right)^{n-1}$

$$\therefore\ a_n=b_n+3=-2\left(\frac{2}{3}\right)^{n-1}+3$$

Replacing both a_{n+1} and a_n with x,
$x=\frac{2}{3}x+1$
$\therefore\ x=3$

Note Summary

...urrence relation $a_{n+1}=pa_n+q$ can be rearranged into ... $=p(a_n-x)$ by using x, which satisfies $x=px+q$. If $b_n=a_n-x$, ... sequence $\{b_n\}$ is a geometric sequence. Therefore, the general term ... sequence $\{a_n\}$ can be determined.

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Recurrence Relations

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Find the general term of each given sequence $\{a_n\}$ defined by the following conditions.

Ex. $a_1=1,\ a_{n+1}=\dfrac{a_n}{4a_n+2}$

[Sol] Since $a_{n+1}=\dfrac{a_n}{4a_n+2}$, $\dfrac{1}{a_{n+1}}=\dfrac{4a_n+2}{a_n}=\dfrac{2}{a_n}+4$

Substituting $\dfrac{1}{a_n}=b_n$ and $\dfrac{1}{a_{n+1}}=b_{n+1}$ into $\dfrac{1}{a_{n+1}}=\dfrac{2}{a_n}+4$

Let $b_n=\dfrac{1}{a_n}$. Since $b_{n+1}=2b_n+4$,

Replacing both b_{n+1} and b_n with x, $x=2x+4$ $\therefore\ x=-4$

$$\begin{cases} b_{n+1}+4=2(b_n+4) & \cdots① \\ b_1+4=\dfrac{1}{a_1}+4=5 & \cdots② \end{cases}$$

From ① and ②, $b_n+4=5\cdot 2^{n-1}$

$b_n=5\cdot 2^{n-1}-4$

$\{b_n+4\}$ is the geometric sequence with 1st term $b_1+4=5$ and common ratio 2.

$\therefore\ a_n=\dfrac{1}{b_n}=\dfrac{1}{5\cdot 2^{n-1}-4}$

(1) $a_1=2,\ a_{n+1}=\dfrac{a_n}{2a_n+3}$

[Sol] Since $a_{n+1}=\dfrac{a_n}{2a_n+3}$, $\dfrac{1}{a_{n+1}}=\dfrac{2a_n+3}{a_n}=\dfrac{3}{a_n}+2$

Let $b_n=\dfrac{1}{a_n}$. Since $b_{n+1}=3b_n+2$,

$$\begin{cases} b_{n+1}+1=3(b_n+1) & \cdots① \\ b_1+1=\dfrac{1}{a_1}+1=\dfrac{3}{2} & \cdots② \end{cases}$$

From ① and ②, $b_n+1=\dfrac{3}{2}\cdot 3^{n-1}=\dfrac{3^n}{2}$

$b_n=\dfrac{3^n}{2}-1$

$\therefore\ a_n=\dfrac{1}{b_n}=\dfrac{1}{\dfrac{3^n}{2}-1}=\dfrac{2}{3^n-2}$

N45b

(2) $a_1=1,\ a_{n+1}=\dfrac{2a_n}{a_n+10}$

[Sol] Since $a_{n+1}=\dfrac{2a_n}{a_n+10}$, $\dfrac{1}{a_{n+1}}=\dfrac{a_n+10}{2a_n}=\dfrac{5}{a_n}+\dfrac{1}{2}$

Let $b_n=\dfrac{1}{a_n}$. Since $b_{n+1}=5b_n+\dfrac{1}{2}$,

$$\begin{cases} b_{n+1}+\dfrac{1}{8}=5\left(b_n+\dfrac{1}{8}\right) & \cdots① \\ b_1+\dfrac{1}{8}=\dfrac{1}{a_1}+\dfrac{1}{8}=\dfrac{9}{8} & \cdots② \end{cases}$$

From ① and ②, $b_n+\dfrac{1}{8}=\dfrac{9}{8}\cdot 5^{n-1}$

$$b_n=\frac{9}{8}\cdot 5^{n-1}-\frac{1}{8}$$

$$\therefore\ a_n=\frac{1}{b_n}=\frac{1}{\frac{9}{8}\cdot 5^{n-1}-\frac{1}{8}}=\frac{8}{9\cdot 5^{n-1}-1}$$

(3) $a_1=3,\ a_{n+1}=\dfrac{2a_n}{3a_n+5}$

[Sol] Since $a_{n+1}=\dfrac{2a_n}{3a_n+5}$, $\dfrac{1}{a_{n+1}}=\dfrac{3a_n+5}{2a_n}=\dfrac{5}{2a_n}+\dfrac{3}{2}$

Let $b_n=\dfrac{1}{a_n}$. Since $b_{n+1}=\dfrac{5}{2}b_n+\dfrac{3}{2}$,

$$\begin{cases} b_{n+1}+1=\dfrac{5}{2}(b_n+1) & \cdots① \\ b_1+1=\dfrac{1}{a_1}+1=\dfrac{4}{3} & \cdots② \end{cases}$$

From ① and ②, $b_n+1=\dfrac{4}{3}\left(\dfrac{5}{2}\right)^{n-1}$

$$b_n=\frac{4}{3}\left(\frac{5}{2}\right)^{n-1}-1$$

$$\therefore\ a_n=\frac{1}{b_n}=\frac{1}{\frac{4}{3}\left(\frac{5}{2}\right)^{n-1}-1}=\frac{3\cdot 2^{n-1}}{4\cdot 5^{n-1}-3\cdot 2^{n-1}}$$

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N 4

Recurrence Relations

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Find the general term of each given sequence $\{a_n\}$ defined by the following conditions.

Ex. $a_1=1,\ a_{n+1}=2a_n-3^n$

[Sol] Dividing both sides of $a_{n+1}=2a_n-3^n$ by 3^{n+1},

$$\frac{a_{n+1}}{3^{n+1}}=\frac{2}{3}\cdot\frac{a_n}{3^n}-\frac{1}{3}$$

Let $b_n=\frac{a_n}{3^n}$. Since $b_{n+1}=\frac{2}{3}b_n-\frac{1}{3}$,

$$\begin{cases} b_{n+1}+1=\frac{2}{3}(b_n+1) & \cdots① \\ b_1+1=\frac{a_1}{3^1}+1=\frac{4}{3} & \cdots② \end{cases}$$

From ① and ②, $b_n+1=\frac{4}{3}\left(\frac{2}{3}\right)^{n-1}=2\left(\frac{2}{3}\right)^n$

$$b_n=2\left(\frac{2}{3}\right)^n-1$$

$$\therefore\ a_n=3^nb_n=3^n\left[2\left(\frac{2}{3}\right)^n-1\right]=2^{n+1}-3^n$$

Substituting $\frac{a_n}{3^n}=b$ and $\frac{a_{n+1}}{3^{n+1}}=b_{n+1}$ int

$$\frac{a_{n+1}}{3^{n+1}}=\frac{2}{3}\cdot\frac{a_n}{3^n}-\frac{1}{3}$$

Replacing both b_{n+1} an b_n with x, $x=\frac{2}{3}x-\frac{1}{3}$

$\therefore\ x=-1$

$\{b_n+1\}$ is the geometric sequen with 1^{st} term $b_1+1=\frac{4}{3}$ and common ratio $\frac{2}{3}$

(1) $a_1=1,\ a_{n+1}=3a_n+2^n$

[Sol] Dividing both sides of $a_{n+1}=3a_n+2^n$ by 2^{n+1},

$$\frac{a_{n+1}}{2^{n+1}}=\frac{3}{2}\cdot\frac{a_n}{2^n}+\frac{1}{2}$$

Let $b_n=\frac{a_n}{2^n}$. Since $b_{n+1}=\frac{3}{2}b_n+\frac{1}{2}$,

$$\begin{cases} b_{n+1}+1=\frac{3}{2}(b_n+1) & \cdots① \\ b_1+1=\frac{a_1}{2^1}+1=\frac{3}{2} & \cdots② \end{cases}$$

From ① and ②, $b_n+1=\frac{3}{2}\left(\frac{3}{2}\right)^{n-1}=\left(\frac{3}{2}\right)^n$

$$b_n=\left(\frac{3}{2}\right)^n-1$$

$$\therefore\ a_n=2^nb_n=2^n\left[\left(\frac{3}{2}\right)^n-1\right]=3^n-2^n$$

N46b

(2) $a_1=\frac{1}{8}$, $a_{n+1}=3^{n+1}-5a_n$

[Sol] Dividing both sides of $a_{n+1}=3^{n+1}-5a_n$ by 3^{n+1},

$$\frac{a_{n+1}}{3^{n+1}}=1-\frac{5}{3}\cdot\frac{a_n}{3^n}$$

Let $b_n=\frac{a_n}{3^n}$. Since $b_{n+1}=1-\frac{5}{3}b_n$,

$$\begin{cases} b_{n+1}-\frac{3}{8}=-\frac{5}{3}\left(b_n-\frac{3}{8}\right) & \cdots① \\ b_1-\frac{3}{8}=\frac{a_1}{3^1}-\frac{3}{8}=-\frac{1}{3} & \cdots② \end{cases}$$

From ① and ②, $b_n-\frac{3}{8}=-\frac{1}{3}\left(-\frac{5}{3}\right)^{n-1}$

$$b_n=-\frac{1}{3}\left(-\frac{5}{3}\right)^{n-1}+\frac{3}{8}$$

$$\therefore\ a_n=3^n b_n=3^n\left[-\frac{1}{3}\left(-\frac{5}{3}\right)^{n-1}+\frac{3}{8}\right]=-(-5)^{n-1}+\frac{3^{n+1}}{8}$$

(3) $a_1=1$, $a_{n+1}=a_n+\frac{1}{2^n}$

[Sol] Multiplying both sides of $a_{n+1}=a_n+\frac{1}{2^n}$ by 2^{n+1},

$$2^{n+1}a_{n+1}=2\cdot 2^n a_n+2$$

Let $b_n=2^n a_n$. Since $b_{n+1}=2b_n+2$,

$$\begin{cases} b_{n+1}+2=2(b_n+2) & \cdots① \\ b_1+2=2^1 a_1+2=4 & \cdots② \end{cases}$$

From ① and ②, $b_n+2=4\cdot 2^{n-1}=2^{n+1}$

$$b_n=2^{n+1}-2$$

$$\therefore\ a_n=\frac{b_n}{2^n}=\frac{1}{2^n}(2^{n+1}-2)=2-\frac{1}{2^{n-1}}$$

N47a

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Assume that the recurrence relation $a_{n+2}+pa_{n+1}+qa_n=0$ ···① is true whe p and q are non-zero constants. By using two solutions α and β of quadrat equation $x^2+px+q=0$ where a_{n+2}, a_{n+1} and a_n are replaced with x^2 and 1 respectively, prove that the recurrence relation ① can be rearranged in $a_{n+2}-\alpha a_{n+1}=\beta(a_{n+1}-\alpha a_n)$.

[Sol] Since two solutions of $x^2+px+q=0$ are α and β, from root-coefficient relationships,

$$\alpha+\beta=\boxed{-p} \cdots② ,\ \alpha\beta=\boxed{q} \cdots③ \leftarrow$$

> From root-coefficient relationshi let two solutions of ax^2+bx+c $(a\neq0)$ be α and β.
> $\alpha+\beta=-\frac{b}{a}$, $\alpha\beta=\frac{c}{a}$ (J131)

Substituting ② and ③ into ①,

$$a_{n+2}-(\boxed{\alpha+\beta})a_{n+1}+\boxed{\alpha\beta}\,a_n=0$$

$$a_{n+2}-\boxed{\alpha}\,a_{n+1}=\boxed{\beta}\,a_{n+1}-\boxed{\alpha\beta}\,a_n$$

$$\therefore\ a_{n+2}-\alpha a_{n+1}=\beta(a_{n+1}-\alpha a_n) \cdots④$$

④ is also true for the equation $a_{n+2}-\beta a_{n+1}=\alpha(a_{n+1}-\beta a_n)$ when α and β switched.

Find the general term of each given sequence $\{a_n\}$ defined by the following conditions.

Ex. $a_1=0$, $a_2=2$, $a_{n+2}-5a_{n+1}+6a_n=0$

[Sol] Rearranging $a_{n+2}-5a_{n+1}+6a_n=0$,

$$\begin{cases} a_{n+2}-2a_{n+1}=\boxed{3}(a_{n+1}-2a_n) \cdots① \\ a_{n+2}-3a_{n+1}=\boxed{2}(a_{n+1}-3a_n) \cdots② \end{cases}$$

> Replacing a_{n+2}, a_{n+1}, a_n wi x^2, x, 1 respective
> $x^2-5x+6=0$
> $(x-2)(x-3)=$
> $\therefore\ x=2,\ 3$

From ①, the sequence $\{a_{n+1}-2a_n\}$ is the geometric sequence with 1st term $a_2-2a_1=2$ and common ratio $\boxed{3}$.

$$\therefore\ a_{n+1}-2a_n=2\cdot\boxed{3}^{n-1} \cdots③$$

From ②, the sequence $\{a_{n+1}-3a_n\}$ is the geometric sequence with 1' term $a_2-3a_1=2$ and common ratio $\boxed{2}$.

$$\therefore\ a_{n+1}-3a_n=2\cdot\boxed{2}^{n-1}=\boxed{2^n} \cdots④$$

From ③ and ④, $a_n=\boxed{2\cdot3^{n-1}-2^n}$ ← ③−④

N47b

(1) $a_1=1,\ a_2=5,\ a_{n+2}-7a_{n+1}+12a_n=0$

[Sol] Rearranging $a_{n+2}-7a_{n+1}+12a_n=0$,

$$\begin{cases} a_{n+2}-3a_{n+1}=4(a_{n+1}-3a_n) & \cdots ① \\ a_{n+2}-4a_{n+1}=3(a_{n+1}-4a_n) & \cdots ② \end{cases}$$

Replacing a_{n+2}, a_{n+1}, a_n with x^2, x, 1 respectively.
$x^2-7x+12=0$
$(x-3)(x-4)=0$
$\therefore\ x=3,\ 4$

From ①, the sequence $\{a_{n+1}-3a_n\}$ is the geometric sequence with 1st term $a_2-3a_1=2$ and common ratio 4.

$\therefore\ a_{n+1}-3a_n=2\cdot 4^{n-1}\ \cdots ③$

From ②, the sequence $\{a_{n+1}-4a_n\}$ is the geometric sequence with 1st term $a_2-4a_1=1$ and common ratio 3.

$\therefore\ a_{n+1}-4a_n=3^{n-1}\quad \cdots ④$

From ③ and ④, $a_n=2\cdot 4^{n-1}-3^{n-1}\quad [=2^{2n-1}-3^{n-1}]$

(2) $a_1=0,\ a_2=1,\ 5a_{n+1}=3a_{n+2}+2a_n$

[Sol] Since $5a_{n+1}=3a_{n+2}+2a_n$, $3a_{n+2}-5a_{n+1}+2a_n=0\ \cdots ①$

Rearranging ①,

$$\begin{cases} a_{n+2}-\dfrac{2}{3}a_{n+1}=a_{n+1}-\dfrac{2}{3}a_n & \cdots ② \\ a_{n+2}-a_{n+1}=\dfrac{2}{3}(a_{n+1}-a_n) & \cdots ③ \end{cases}$$

Replacing a_{n+2}, a_{n+1}, a_n with x^2, x, 1 respectively.
$3x^2-5x+2=0$
$(3x-2)(x-1)=0$
$\therefore\ x=\dfrac{2}{3},\ 1$

From ②, the sequence $\left\{a_{n+1}-\dfrac{2}{3}a_n\right\}$ is the geometric sequence with 1st term $a_2-\dfrac{2}{3}a_1=1$ and common ratio 1.

$\therefore\ a_{n+1}-\dfrac{2}{3}a_n=1\ \cdots ④$

From ③, the sequence $\{a_{n+1}-a_n\}$ is the geometric sequence with 1st term $a_2-a_1=1$ and common ratio $\dfrac{2}{3}$.

$\therefore\ a_{n+1}-a_n=\left(\dfrac{2}{3}\right)^{n-1}\ \cdots ⑤$

From ④ and ⑤, $a_n=3\left[1-\left(\dfrac{2}{3}\right)^{n-1}\right]$

Note Summary

The recurrence relation $a_{n+2}+pa_{n+1}+qa_n=0$ can be rearranged into $a_{n+2}-\alpha a_{n+1}=\beta(a_{n+1}-\alpha a_n)$ by using two solutions α and β of quadratic equation $x^2+px+q=0$. The sequence $\{a_{n+1}-\alpha a_n\}$ is a geometric sequence. Therefore, the general term of the sequence $\{a_n\}$ can be determined.

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Find the general terms of the following sequences $\{a_n\}$ and $\{b_n\}$ defined by the following conditions.

Ex.

$$\begin{cases} a_1=2,\ b_1=1 \\ a_{n+1}=3a_n+b_n \quad \cdots ① \\ b_{n+1}=a_n+3b_n \quad \cdots ② \end{cases}$$

[Sol] From ①, $b_n=-3a_n+a_{n+1}$ $\cdots ③$

From ③, $b_{n+1}=-3\boxed{a_{n+1}}+\boxed{a_{n+2}}$ $\cdots ④$ ← In ③, replacing n with $n+1$

Substituting ③ and ④ into ②,

$$\boxed{-3a_{n+1}+a_{n+2}}=a_n+3(\boxed{-3a_n+a_{n+1}})$$

$a_{n+2}-6a_{n+1}+8a_n=0$ $\cdots ⑤$

Rearranging ⑤,

$$\begin{cases} a_{n+2}-2a_{n+1}=\boxed{4}(a_{n+1}-2a_n) \quad \cdots ⑥ \\ a_{n+2}-4a_{n+1}=\boxed{2}(a_{n+1}-4a_n) \quad \cdots ⑦ \end{cases}$$

Replacing a_{n+2}, a_{n+1}, a_n with x^2, x, 1 respectively
$x^2-6x+8=0$
$(x-2)(x-4)=0$
$\therefore x=2,\ 4$

From ⑥, the sequence $\{a_{n+1}-2a_n\}$ is the geometric sequence with first term $a_2-2a_1=(3a_1+b_1)-2a_1=3$ and common ratio $\boxed{4}$.

From ①, $a_2=3a_1 \ldots$

$\therefore a_{n+1}-2a_n=3\cdot\boxed{4}^{n-1}$ $\cdots ⑧$

From ⑦, the sequence $\{a_{n+1}-4a_n\}$ is the geometric sequence with first term $a_2-4a_1=(3a_1+b_1)-4a_1=-1$ and common ratio $\boxed{2}$.

$\therefore a_{n+1}-4a_n=-\boxed{2}^{n-1}$ $\cdots ⑨$

From ⑧ and ⑨, $a_n=\frac{1}{2}(3\cdot\boxed{4}^{n-1}+\boxed{2}^{n-1})$ $\cdots ⑩$ ← From ⑧−⑨, $2a_n=3\cdot4^{n-1}\ldots$

From ③ and ⑩, $b_n=-3\cdot\frac{1}{2}(3\cdot\boxed{4}^{n-1}+\boxed{2}^{n-1})+\frac{1}{2}(3\cdot\boxed{4}^{n}+\boxed{2}^{n})$

$$=\frac{1}{2}(\boxed{3\cdot4^{n-1}-2^{n-1}})$$

Substituting $a_n=\frac{1}{2}(3\cdot4^{n-1}+\ldots$, $a_{n+1}=\frac{1}{2}(3\cdot4^{n}+2^{n})$ into $b_n=-3a_n+a_{n+1}$

48b

$$\begin{cases} a_1=1,\ b_1=3 \\ a_{n+1}=3a_n+b_n & \cdots① \\ b_{n+1}=2a_n+4b_n & \cdots② \end{cases}$$

ol] From ①, $b_n=-3a_n+a_{n+1}$ $\cdots③$

From ③, $b_{n+1}=-3a_{n+1}+a_{n+2}$ $\cdots④$

Substituting ③ and ④ into ②,

$$-3a_{n+1}+a_{n+2}=2a_n+4(-3a_n+a_{n+1})$$

$$a_{n+2}-7a_{n+1}+10a_n=0 \quad \cdots⑤$$

Rearranging ⑤,

$$\begin{cases} a_{n+2}-2a_{n+1}=5(a_{n+1}-2a_n) & \cdots⑥ \\ a_{n+2}-5a_{n+1}=2(a_{n+1}-5a_n) & \cdots⑦ \end{cases}$$

From ⑥, the sequence $\{a_{n+1}-2a_n\}$ is the geometric sequence with 1st term $a_2-2a_1=(3a_1+b_1)-2a_1=4$ and common ratio 5.

$$\therefore\ a_{n+1}-2a_n=4\cdot5^{n-1} \quad \cdots⑧$$

From ⑦, the sequence $\{a_{n+1}-5a_n\}$ is the geometric sequence with 1st term $a_2-5a_1=(3a_1+b_1)-5a_1=1$ and common ratio 2.

$$\therefore\ a_{n+1}-5a_n=2^{n-1} \quad \cdots⑨$$

From ⑧ and ⑨, $a_n=\frac{1}{3}(4\cdot5^{n-1}-2^{n-1})$ $\cdots⑩$

From ③ and ⑩, $b_n=-3\cdot\frac{1}{3}(4\cdot5^{n-1}-2^{n-1})+\frac{1}{3}(4\cdot5^n-2^n)$

$$=\frac{1}{3}(8\cdot5^{n-1}+2^{n-1})$$

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1. The sequence $\{a_n\}$ is defined by the following conditions.

$$a_1=\frac{1}{2},\ a_2=\frac{1}{3},\ a_{n+2}=\frac{a_n a_{n+1}}{2a_n-a_{n+1}+2a_n a_{n+1}}$$

(1) Let $b_n=\frac{1}{a_n}$. Express b_{n+2} in terms of b_{n+1} and b_n.

[Sol] Since $a_{n+2}=\frac{a_n a_{n+1}}{2a_n-a_{n+1}+2a_n a_{n+1}}$,

$$\frac{1}{a_{n+2}}=\frac{2a_n-a_{n+1}+2a_n a_{n+1}}{a_n a_{n+1}}=\frac{2}{a_{n+1}}-\frac{1}{a_n}+2$$

Let $b_n=\frac{1}{a_n}$. $b_{n+2}=2b_{n+1}-b_n+2$

(2) Let $b_{n+1}-b_n=c_n$. Express c_n in terms of n.

[Sol] Let $b_{n+1}-b_n=c_n$.

$$\begin{aligned}c_{n+1}&=b_{n+2}-b_{n+1}\\&=(2b_{n+1}-b_n+2)-b_{n+1}\\&=b_{n+1}-b_n+2\\&=c_n+2 \quad \cdots①\end{aligned}$$

From (1), $b_{n+2}=2b_{n+1}-b_n+2$

$$c_1=b_2-b_1=\frac{1}{a_2}-\frac{1}{a_1}=1 \quad \cdots②$$

$\{c_n\}$ is the arithmetic sequence with 1st term $c_1=1$ and common difference 2.

From ① and ②, $c_n=1+(n-1)\cdot 2=2n-1$

(3) Express a_n in terms of n.

[Sol] From (2), $b_{n+1}-b_n=2n-1$

Since the general term of the sequence of differences of $\{b_n\}$ is $2n-1$

when $n \geqq 2$,

$$\begin{aligned}b_n=b_1+\sum_{k=1}^{n-1}(2k-1)&=2+2\cdot\frac{1}{2}(n-1)n-(n-1)\\&=n^2-2n+3 \quad \cdots③\end{aligned}$$

Since $b_1=2$, ③ is also true when $n=1$.

$\therefore\ b_n=n^2-2n+3$

$\therefore\ a_n=\frac{1}{b_n}=\frac{1}{n^2-2n+3}$

149b

Given the sequence $\{a_n\}$, let S_n be the sum of the first n terms. Then,

$$S_1=1,\ S_2=-13,\ S_{n+2}=-13S_{n+1}-36S_n$$

are true. Express S_n and a_n in terms of n.

Sol] Since $S_{n+2}=-13S_{n+1}-36S_n$, $S_{n+2}+13S_{n+1}+36S_n=0$ ⋯①

Rearranging ①,

$$\begin{cases} S_{n+2}+9S_{n+1}=-4(S_{n+1}+9S_n) & \cdots ② \\ S_{n+2}+4S_{n+1}=-9(S_{n+1}+4S_n) & \cdots ③ \end{cases}$$

Replacing S_{n+2}, S_{n+1}, S_n with x^2, x, 1 respectively,
$x^2+13x+36=0$
$(x+9)(x+4)=0$
$\therefore\ x=-9,\ -4$

From ②, the sequence $\{S_{n+1}+9S_n\}$ is the geometric sequence with 1st term $S_2+9S_1=-4$ and common ratio -4.

$$\therefore\ S_{n+1}+9S_n=-4(-4)^{n-1}=(-4)^n \quad \cdots ④$$

From ③, the sequence $\{S_{n+1}+4S_n\}$ is the geometric sequence with 1st term $S_2+4S_1=-9$ and common ratio -9.

$$\therefore\ S_{n+1}+4S_n=-9(-9)^{n-1}=(-9)^n \quad \cdots ⑤$$

From ④ and ⑤, $S_n=\frac{1}{5}[(-4)^n-(-9)^n]$

The 1st term a_1 is $a_1=S_1=1$.

When $n \geqq 2$,

Sum of a Sequence and General Term (N34)

$$\begin{aligned} a_n&=S_n-S_{n-1} \\ &=\frac{1}{5}[(-4)^n-(-9)^n]-\frac{1}{5}[(-4)^{n-1}-(-9)^{n-1}] \\ &=\frac{1}{5}[(-4-1)(-4)^{n-1}+(9+1)(-9)^{n-1}] \\ &=-(-4)^{n-1}+2(-9)^{n-1} \quad \cdots ⑥ \end{aligned}$$

Since $a_1=1$, ⑥ is also true when $n=1$.

$$\therefore\ a_n=-(-4)^{n-1}+2(-9)^{n-1}$$

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Find the general term of each given sequence $\{a_n\}$ defined by the following conditions.

(1) $a_1=6,\ a_{n+1}=a_n+4n$

➡ N

[Sol] $a_{n+1}-a_n=4n$

Since the general term of the sequence of differences of $\{a_n\}$ is $4n$, when $n \geqq 2$,

$$a_n=a_1+\sum_{k=1}^{n-1}4k=6+4\cdot\frac{1}{2}(n-1)n$$

$$=2(n^2-n+3) \quad \cdots①$$

Since $a_1=6$, ① is also true when $n=1$.

$\therefore\ a_n=2(n^2-n+3)$

(2) $a_1=2,\ a_{n+1}=2a_n+5$

➡ N

[Sol] Since $a_{n+1}=2a_n+5$, $a_{n+1}+5=2(a_n+5)$

Let $b_n=a_n+5$.

$$\begin{cases} b_{n+1}=2b_n & \cdots① \\ b_1=a_1+5=7 & \cdots② \end{cases}$$

From ① and ②, $b_n=7\cdot 2^{n-1}$

$\therefore\ a_n=b_n-5=7\cdot 2^{n-1}-5$

(3) $a_1=\dfrac{1}{3},\ a_{n+1}=\dfrac{a_n}{4a_n-1}$

➡ N4

[Sol] Since $a_{n+1}=\dfrac{a_n}{4a_n-1}$, $\dfrac{1}{a_{n+1}}=\dfrac{4a_n-1}{a_n}=-\dfrac{1}{a_n}+4$

Let $b_n=\dfrac{1}{a_n}$. Since $b_{n+1}=-b_n+4$,

$$\begin{cases} b_{n+1}-2=-(b_n-2) & \cdots① \\ b_1-2=\dfrac{1}{a_1}-2=1 & \cdots② \end{cases}$$

From ① and ②, $b_n-2=(-1)^{n-1}$

$$b_n=(-1)^{n-1}+2$$

$$\therefore\ a_n=\frac{1}{b_n}=\frac{1}{(-1)^{n-1}+2}$$

(4) $a_1=2,\ a_2=1,\ a_{n+2}=a_{n+1}+6a_n$

➡ N47

[Sol] Since $a_{n+2}=a_{n+1}+6a_n$, $a_{n+2}-a_{n+1}-6a_n=0$ $\cdots①$

Rearranging ①,

$$\begin{cases} a_{n+2}+2a_{n+1}=3(a_{n+1}+2a_n) & \cdots② \\ a_{n+2}-3a_{n+1}=-2(a_{n+1}-3a_n) & \cdots③ \end{cases}$$

From ②, the sequence $\{a_{n+1}+2a_n\}$ is the geometric sequence with 1st term $a_2+2a_1=5$ and common ratio 3.

$$\therefore\ a_{n+1}+2a_n=5\cdot 3^{n-1}\ \cdots④$$

From ③, the sequence $\{a_{n+1}-3a_n\}$ is the geometric sequence with 1st term $a_2-3a_1=-5$ and common ratio -2.

$$\therefore\ a_{n+1}-3a_n=-5(-2)^{n-1}\ \cdots⑤$$

From ④ and ⑤, $a_n=3^{n-1}+(-2)^{n-1}$

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Ex. Given that n is a natural number, prove the following equality.

$2+4+6+\cdots+2n=n(n+1)$ …①

[Sol] (i) When $n=1$,

LHS $=\boxed{2}$, RHS $=1\cdot(1+1)=\boxed{2}$

Therefore, ① is true when $n=1$.

(ii) If ① is true when $n=k$,

$2+4+6+\cdots+2k=\boxed{k(k+1)}$ …②

When $n=k+1$, rearranging the LHS of ① by using ②,

$2+4+6+\cdots+2k+2(k+1)$

$=\boxed{k(k+1)}+2(k+1)$ From ②

$=(k+1)(\boxed{k+2})$

$=(k+1)[(\boxed{k+1})+1]$ ← The RHS of ① when $n=k+1$

Therefore, ① is also true when $n=k+1$.

From (i) and (ii), ① is true for all natural numbers n.

wers: 2, 2, $k(k+1)$, $k(k+1)$, $k+2$, $k+1$

The method of proof shown above is called ***mathematical induction***.

Mathematical Induction

To prove that a proposition P is true for all natural numbers n by mathemat induction, the following two statements have to be proved.

(i) P is true when $n=1$.

(ii) If P is true when $n=k$, then P is also true when $n=k+1$.

A statement that can be either true or false is called a ***proposition***.

5 | b

iven that n is a natural number, prove the following equality by mathematical iduction.

$1+3+5+\cdots+(2n-1)=n^2$ ⋯①

] (i) When $n=1$,

LHS $=1$, RHS $=1^2=1$

Therefore, ① is true when $n=1$.

(ii) If ① is true when $n=k$,

$1+3+5+\cdots+(2k-1)=k^2$ ⋯②

When $n=k+1$, rearranging the LHS of ① by using ②,

$1+3+5+\cdots+(2k-1)+[2(k+1)-1]$

$=k^2+2k+1$

$=(k+1)^2$

Therefore, ① is also true when $n=k+1$.

From (i) and (ii), ① is true for all natural numbers n.

Given that n is a natural number, prove the following equality by mathematical induction.

$4+16+28+\cdots+(12n-8)=2n(3n-1)$ ⋯①

l] (i) When $n=1$,

LHS $=4$, RHS $=2\cdot1\cdot(3\cdot1-1)=4$

Therefore, ① is true when $n=1$.

(ii) If ① is true when $n=k$,

$4+16+28+\cdots+(12k-8)=2k(3k-1)$ ⋯②

When $n=k+1$, rearranging the LHS of ① by using ②,

$4+16+28+\cdots+(12k-8)+[12(k+1)-8]$

$=2k(3k-1)+12k+4$

$=6k^2+10k+4$

$=2(k+1)(3k+2)$

$=2(k+1)[3(k+1)-1]$

Therefore, ① is also true when $n=k+1$.

From (i) and (ii), ① is true for all natural numbers n.

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Mathematical Induction

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1. Given that n is a natural number, prove the following equality by mathemati
induction.

$$1\cdot2+2\cdot3+3\cdot4+\cdots+n(n+1)=\frac{1}{3}n(n+1)(n+2) \quad \cdots①$$

[Sol] (i) When $n=1$,

$$\text{LHS}=1\cdot2=2,\ \text{RHS}=\frac{1}{3}\cdot1\cdot(1+1)(1+2)=2$$

Therefore, ① is true when $n=1$.

(ii) If ① is true when $n=k$,

$$1\cdot2+2\cdot3+3\cdot4+\cdots+k(k+1)=\frac{1}{3}k(k+1)(k+2) \quad \cdots②$$

When $n=k+1$, rearranging the LHS of ① by using ②,

$$1\cdot2+2\cdot3+3\cdot4+\cdots+k(k+1)+(k+1)[(k+1)+1]$$
$$=\frac{1}{3}k(k+1)(k+2)+(k+1)(k+2)$$
$$=\frac{1}{3}(k+1)(k+2)(k+3)$$
$$=\frac{1}{3}(k+1)[(k+1)+1][(k+1)+2]$$

Therefore, ① is also true when $n=k+1$.

From (i) and (ii), ① is true for all natural numbers n.

2b

Given that n is a natural number, prove the following equality by mathematical induction.

$$10+10^2+10^3+\cdots+10^n=\frac{10}{9}(10^n-1) \quad \cdots ①$$

[Sol] (i) When $n=1$,

LHS$=10$, RHS$=\frac{10}{9}(10^1-1)=10$

Therefore, ① is true when $n=1$.

(ii) If ① is true when $n=k$,

$$10+10^2+10^3+\cdots+10^k=\frac{10}{9}(10^k-1) \quad \cdots ②$$

When $n=k+1$, rearranging the LHS of ① by using ②,

$$10+10^2+10^3+\cdots+10^k+10^{k+1}$$

$$=\frac{10}{9}(10^k-1)+10^{k+1}$$

$$=\frac{10}{9}(10^k-1+9\cdot 10^k)$$

$\boxed{10^k+9\cdot 10^k=(1+9)\cdot 10^k=10^{k+1}}$

$$=\frac{10}{9}(10^{k+1}-1)$$

Therefore, ① is also true when $n=k+1$.

From (i) and (ii), ① is true for all natural numbers n.

Mathematical Induction

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1. Given that n is a natural number, prove the following equality by mathemati induction.

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1} \quad \cdots ①$$

[Sol] (i) When $n=1$,

$$\text{LHS}=\frac{1}{1\cdot 2}=\frac{1}{2},\ \text{RHS}=\frac{1}{1+1}=\frac{1}{2}$$

Therefore, ① is true when $n=1$.

(ii) If ① is true when $n=k$,

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{k(k+1)}=\frac{k}{k+1} \quad \cdots ②$$

When $n=k+1$, rearranging the LHS of ① by using ②,

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{k(k+1)}+\frac{1}{(k+1)\{(k+1)+1\}}$$

$$=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}$$

$$=\frac{k(k+2)+1}{(k+1)(k+2)}$$

$$=\frac{(k+1)^2}{(k+1)(k+2)}$$

$$=\frac{k+1}{k+2}$$

$$=\frac{k+1}{(k+1)+1}$$

Therefore, ① is also true when $n=k+1$.

From (i) and (ii), ① is true for all natural numbers n.

iven that n is a natural number, prove the following equality by mathematical iduction.

$$1+\frac{3}{2^2}+\frac{4}{2^3}+\cdots+\frac{n+1}{2^n}=3-\frac{n+3}{2^n} \quad \cdots①$$

] (i) When $n=1$,

$$\text{LHS}=1,\ \text{RHS}=3-\frac{1+3}{2^1}=1$$

Therefore, ① is true when $n=1$.

(ii) If ① is true when $n=k$,

$$1+\frac{3}{2^2}+\frac{4}{2^3}+\cdots+\frac{k+1}{2^k}=3-\frac{k+3}{2^k} \quad \cdots②$$

When $n=k+1$, rearranging the LHS of ① by using ②,

$$1+\frac{3}{2^2}+\frac{4}{2^3}+\cdots+\frac{k+1}{2^k}+\frac{(k+1)+1}{2^{k+1}}$$

$$=3-\frac{k+3}{2^k}+\frac{k+2}{2^{k+1}}$$

$$=3-\frac{2(k+3)-(k+2)}{2^{k+1}}$$

$$=3-\frac{k+4}{2^{k+1}}$$

$$=3-\frac{(k+1)+3}{2^{k+1}}$$

Therefore, ① is also true when $n=k+1$.

From (i) and (ii), ① is true for all natural numbers n.

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Ex. Given that n is a natural number, prove the following statement by mathematical induction.

n^3+2n is a multiple of 3.

[Sol] Let the statement "n^3+2n is a multiple of 3" be ①.

(i) When $n=1$,

$$n^3+2n=\boxed{1}^3+2\cdot\boxed{1}=\boxed{3}$$

Therefore, ① is true when $n=1$.

(ii) If ① is true when $n=k$,

using an integer m,

$$k^3+2k=\boxed{3}\,m \quad \cdots②$$

← k^3+2k is a multiple of 3.

When $n=k+1$, rearranging by using ②,

$$\begin{aligned}&(k+1)^3+2(k+1)\\&=(k^3+3k^2+3k+1)+(2k+2)\\&=(k^3+2k)+\boxed{3}(k^2+k+1)\\&=\boxed{3}\,m+\boxed{3}(k^2+k+1)\\&=\boxed{3}(m+k^2+k+1)\end{aligned}$$

From ②

Since $m+k^2+k+1$ is an integer,

$(k+1)^3+2(k+1)$ is a multiple of 3.

Therefore, ① is also true when $n=k+1$.

From (i) and (ii), ① is true for all natural numbers n.

N54b

1. Given that n is a natural number, prove the following statement by mathematical induction.

$2n^3-3n^2+n+6$ is a multiple of 6.

[Sol] Let the statement "$2n^3-3n^2+n+6$ is a multiple of 6" be ①.

(i) When $n=1$,

$$2n^3-3n^2+n+6=2\cdot1^3-3\cdot1^2+1+6=6$$

Therefore, ① is true when $n=1$.

(ii) If ① is true when $n=k$,

using an integer m,

$$2k^3-3k^2+k+6=6m \quad \cdots ②$$

When $n=k+1$, rearranging by using ②,

$$2(k+1)^3-3(k+1)^2+(k+1)+6$$
$$=2k^3+6k^2+6k+2-3k^2-6k-3+k+1+6$$
$$=(2k^3-3k^2+k+6)+6k^2$$
$$=6m+6k^2$$
$$=6(m+k^2)$$

Since $m+k^2$ is an integer,

$2(k+1)^3-3(k+1)^2+(k+1)+6$ is a multiple of 6.

Therefore, ① is also true when $n=k+1$.

From (i) and (ii), ① is true for all natural numbers n.

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1. Given that n is a natural number, prove the following statement by mathematic
   induction.

   $3^{2n}-1$ is a multiple of 8.

[Sol] Let the statement "$3^{2n}-1$ is a multiple of 8" be ①.

(i) When $n=1$,

$$3^{2n}-1=3^2-1=8$$

Therefore, ① is true when $n=1$.

(ii) If ① is true when $n=k$,

using an integer m,

$$3^{2k}-1=8m \quad \cdots ②$$

When $n=k+1$, rearranging by using ②,

$$\begin{aligned} &3^{2(k+1)}-1 \\ &=9\cdot 3^{2k}-1 \\ &=9(8m+1)-1 \\ &=9\cdot 8m+8 \\ &=8(9m+1) \end{aligned}$$

$3^{2(k+1)}=3^{2k+2}=3^2\cdot 3^{2k}$

From ②,
$3^{2k}=8m+1$

Since $9m+1$ is an integer,

$3^{2(k+1)}-1$ is a multiple of 8.

Therefore, ① is also true when $n=k+1$.

From (i) and (ii), ① is true for all natural numbers n.

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Given that n is a natural number, prove the following statement by mathematical induction.

$11^{n+1}+12^{2n-1}$ is a multiple of 19.

[Sol] Let the statement "$11^{n+1}+12^{2n-1}$ is a multiple of 19" be ①.

(i) When $n=1$,

$$11^{n+1}+12^{2n-1}=11^2+12^1=133=19\cdot 7$$

Therefore, ① is true when $n=1$.

(ii) If ① is true when $n=k$,

using an integer m,

$$11^{k+1}+12^{2k-1}=19m \quad \cdots ②$$

When $n=k+1$, rearranging by using ②,

$$11^{(k+1)+1}+12^{2(k+1)-1}$$

$$=11\cdot 11^{k+1}+12^{2k+1}$$

From ②, $11^{k+1}=19m-12^{2k-1}$

$$=11(19m-12^{2k-1})+12^2\cdot 12^{2k-1}$$

$$=11\cdot 19m+133\cdot 12^{2k-1}$$

$$=19(11m+7\cdot 12^{2k-1})$$

Since $11m+7\cdot 12^{2k-1}$ is an integer,

$11^{(k+1)+1}+12^{2(k+1)-1}$ is a multiple of 19.

Therefore, ① is also true when $n=k+1$.

From (i) and (ii), ① is true for all natural numbers n.

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Ex. Given that n is a natural number greater than or equal to 3, prove the following inequality by mathematical induction.

$3^n > 8n$ ···①

[Sol] (i) When $n=3$,

$\text{LHS}=3^3=27$

$\text{RHS}=8\cdot\boxed{3}=\boxed{24}$

Therefore, LHS > RHS

Thus, ① is true when $n=3$.

(ii) For $k \geqq 3$, if ① is true when $n=k$,

$3^k > 8k$ ···②

When $n=k+1$, considering the difference of both sides of ①, then from ②,

Proof of LHS − RHS > 0 (J194)

$3^{k+1}-8(\boxed{k+1})=3\cdot 3^k-8(\boxed{k+1})$

From ②

$>3\cdot\boxed{8k}-8(\boxed{k+1})$

$=8(\boxed{2k-1})>0$

Since $k \geqq 3$, $2k-1>0$

$\therefore\ 3^{k+1}>8(\boxed{k+1})$

Therefore, ① is also true when $n=k+1$.

From (i) and (ii), ① is true for all natural numbers n greater than or equal to 3.

Ans. 3, 24, $k+1$, $k+1$, $8k$, $k+1$, $2k-1$, $k+1$

As shown in **Ex.**, if the statement is proved when $n=3$, then from (i) and (ii), the statement is true for all natural numbers n greater than or equal to 3.

N56b

Given that n is a natural number greater than or equal to 3, prove the following inequality by mathematical induction.

$2^n > 2n+1 \quad \cdots ①$

Sol] (i) When $n=3$,

$\text{LHS}=2^3=8$

$\text{RHS}=2\cdot 3+1=7$

Therefore, LHS > RHS

Thus, ① is true when $n=3$.

(ii) For $k \geqslant 3$, if ① is true when $n=k$,

$2^k > 2k+1 \quad \cdots ②$

When $n=k+1$, considering the difference of both sides of ①, then from ②,

$$\begin{aligned} 2^{k+1}-[2(k+1)+1] &= 2\cdot 2^k-(2k+3) \\ &> 2(2k+1)-(2k+3) \\ &= 2k-1 > 0 \end{aligned}$$

$\therefore 2^{k+1} > 2(k+1)+1$

Therefore, ① is also true when $n=k+1$.

From (i) and (ii), ① is true for all natural numbers n greater than or equal to 3.

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Find the general term of each given sequence $\{a_n\}$ defined by the following conditions.

Ex. $a_1=2,\ a_{n+1}=2-\dfrac{1}{a_n}$ …①

$a_1=2$
$a_2=2-\dfrac{1}{a_1}$
$a_3=2-\dfrac{1}{a_2}$
⋮

[Sol] From the conditions,

$$a_1=2,\ a_2=\frac{3}{2},\ a_3=\frac{4}{3},\ a_4=\boxed{\frac{5}{4}},\ \cdots$$

Therefore, the general term of $\{a_n\}$ is assumed to be

$$a_n=\frac{n+1}{\boxed{n}}\ \cdots②$$

Prove that the assumption ② is true by mathematical induction.

(i) When $n=1$, in ②,

$$a_1=\frac{1+1}{\boxed{1}}=\boxed{2}$$

Therefore, ② is true when $n=1$. ← It coincides with condition $a_1=2$.

(ii) If ② is true when $n=k$,

$$a_k=\frac{k+1}{\boxed{k}}$$

When $n=k+1$, substituting $n=k$ into ①,

$$a_{k+1}=2-\frac{1}{a_k}$$

$a_k=\dfrac{k+1}{k}$

$$=2-\frac{\boxed{k}}{k+1}$$

$$=\frac{\boxed{k+2}}{k+1}=\frac{(\boxed{k+1})+1}{k+1}$$

Therefore, ② is also true when $n=k+1$.

From (i) and (ii), ② is true for all natural numbers n.

$$\therefore\ a_n=\frac{n+1}{\boxed{n}}$$

N57b

(1) $a_1=2$, $a_{n+1}=\dfrac{a_n}{1+a_n}$ ···①

[Sol] From the conditions,

$$a_1=2,\ a_2=\frac{2}{3},\ a_3=\frac{2}{5},\ a_4=\frac{2}{7},\ \cdots$$

Therefore, the general term of $\{a_n\}$ is assumed to be

$$a_n=\frac{2}{2n-1}\ \cdots②$$

Prove that the assumption ② is true by mathematical induction.

(i) When $n=1$, in ②,

$$a_1=\frac{2}{2\cdot1-1}=2$$

Therefore, ② is true when $n=1$.

(ii) If ② is true when $n=k$,

$$a_k=\frac{2}{2k-1}$$

When $n=k+1$, substituting $n=k$ into ①,

$$a_{k+1}=\frac{a_k}{1+a_k}$$

$$=\frac{\frac{2}{2k-1}}{1+\frac{2}{2k-1}}$$

$$=\frac{2}{2k+1}=\frac{2}{2(k+1)-1}$$

Therefore, ② is also true when $n=k+1$.

From (i) and (ii), ② is true for all natural numbers n.

$$\therefore\ a_n=\frac{2}{2n-1}$$

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Ex. Given that n is a natural number, compare the values of 2^n and $3n+1$.
(Fill in the gray boxes ☐ with either $>$ or $<$.)

[Sol] The table below shows the values of 2^n and $3n+1$ when $n=1, 2, 3, 4, 5$.

n	1	2	3	4	5
2^n	2	4	8	16	32
$3n+1$	4	7	10	13	16

From the table,
when $n \geqq 4$, the following inequality is assumed to be true:

$$2^n \boxed{>} 3n+1 \quad \cdots ①$$

Prove that the assumption ① is true by mathematical induction.

(i) ① is true when $n=4$.

(ii) For $k \geqq 4$, if ① is true when $n=k$,

$$2^k \boxed{>} 3k+1 \quad \cdots ②$$

When $n=k+1$, considering the difference of both sides of then from ②,

$$2^{k+1}-[3(\boxed{k+1})+1]=2\cdot 2^k-(\boxed{3k+4})$$

$$>2(\boxed{3k+1})-(\boxed{3k+4})$$

$$=\boxed{3k-2}>0$$

$$\therefore\ 2^{k+1}>3(\boxed{k+1})+1$$

Therefore, ① is also true when $n=k+1$.

From (i) and (ii), ① is true for all natural numbers n when $n \geqq 4$.

Thus, when $n=1, 2, 3,$ $2^n \boxed{<} 3n+1$ and

when $n \geqq 4,$ $2^n \boxed{>} 3n+1$

N58b

1. Given that n is a natural number, compare the values of 3^n and $5n+1$.

[Sol] The table below shows the values of 3^n and $5n+1$ when $n=1, 2, 3, 4, 5$.

n	1	2	3	4	5
3^n	3	9	27	81	243
$5n+1$	6	11	16	21	26

From the table,

when $n \geqslant 3$, the following inequality is assumed to be true:

$3^n > 5n+1$ ···①

Prove that the assumption ① is true by mathematical induction.

(i) ① is true when $n=3$.

(ii) For $k \geqslant 3$, if ① is true when $n=k$,

$3^k > 5k+1$ ···②

When $n=k+1$, considering the difference of both sides of ①, then from ②,

$$\begin{aligned} 3^{k+1}-[5(k+1)+1] &= 3\cdot 3^k-(5k+6) \\ &> 3(5k+1)-(5k+6) \\ &= 10k-3 > 0 \end{aligned}$$

$\therefore\ 3^{k+1} > 5(k+1)+1$

Therefore, ① is also true when $n=k+1$.

From (i) and (ii), ① is true for all natural numbers n when $n \geqslant 3$.

Thus, when $n=1, 2$, $3^n < 5n+1$ and

when $n \geqslant 3$, $3^n > 5n+1$

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Mathematical Induction

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1. Given that n is a natural number, prove the following equality by mathemati induction.

$$1\cdot n+2\cdot(n-1)+3\cdot(n-2)+\cdots\cdots+(n-1)\cdot 2+n\cdot 1=\frac{1}{6}n(n+1)(n$$

[Sol] Let $1\cdot n+2\cdot(n-1)+3\cdot(n-2)+\cdots\cdots+(n-1)\cdot 2+n\cdot 1=\frac{1}{6}n(n+1)(n+2)$ ···

(i) When $n=1$,

$$\text{LHS}=1\cdot 1=1,\ \text{RHS}=\frac{1}{6}\cdot 1\cdot(1+1)(1+2)=1$$

Therefore, ① is true when $n=1$.

(ii) If ① is true when $n=k$,

$$1\cdot k+2\cdot(k-1)+3\cdot(k-2)+\cdots\cdots+(k-1)\cdot 2+k\cdot 1=\frac{1}{6}k(k+1)(k+2)$$

When $n=k+1$, rearranging the LHS of ① by using ②,

$$1\cdot(k+1)+2\cdot[(k+1)-1]+3\cdot[(k+1)-2]+\cdots\cdots+[(k+1)-1]\cdot 2+(k+$$

$$=[1\cdot k+2\cdot(k-1)+3\cdot(k-2)+\cdots\cdots+k\cdot 1]+\boxed{1+2+3+\cdots\cdots+k+(k+1)}$$

$$=\frac{1}{6}k(k+1)(k+2)+\frac{1}{2}(k+1)(k+2)$$

$$\boxed{\sum_{k=1}^{n}k=\frac{1}{2}n(n+1)}$$

$$=\frac{1}{6}(k+1)(k+2)(k+3)$$

$$=\frac{1}{6}(k+1)[(k+1)+1][(k+1)+2]$$

Therefore, ① is also true when $n=k+1$.

From (i) and (ii), ① is true for all natural numbers n.

$$\begin{array}{rllllll} & 1\cdot(k+1) & +2\cdot k & +3\cdot(k-1) & +\cdots\cdots & +k\cdot 2 & +(k+ \\ -) & 1\cdot k & +2\cdot(k-1) & +3\cdot(k-2) & +\cdots\cdots & +k\cdot 1 & \\ \hline & 1 & +2 & +3 & +\cdots\cdots & +k & +(k+ \end{array}$$

en that n is a natural number greater than or equal to 2, prove the following
quality by mathematical induction.

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}>\frac{2n}{n+1} \quad \cdots ①$$

(i) When $n=2$,

$$\text{LHS}=\frac{1}{1}+\frac{1}{2}=\frac{3}{2}$$

$$\text{RHS}=\frac{2\cdot 2}{2+1}=\frac{4}{3}$$

Therefore, LHS > RHS

Thus, ① is true when $n=2$.

(ii) For $k \geqq 2$, if ① is true when $n=k$,

$$\boxed{\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{k}}>\boxed{\frac{2k}{k+1}} \quad \cdots ②$$

When $n=k+1$, considering the difference of both sides of ①, then from ②,

$$\left(\boxed{\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{k}}+\frac{1}{k+1}\right)-\frac{2(k+1)}{(k+1)+1}$$

From ②

$$>\left(\boxed{\frac{2k}{k+1}}+\frac{1}{k+1}\right)-\frac{2(k+1)}{k+2}$$

$$=\frac{2k+1}{k+1}-\frac{2(k+1)}{k+2}$$

$$=\frac{(2k+1)(k+2)-2(k+1)^2}{(k+1)(k+2)}$$

$$=\frac{k}{(k+1)(k+2)}>0 \longleftarrow \text{Since } k \geqq 2$$

$$\therefore \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{k}+\frac{1}{k+1}>\frac{2(k+1)}{(k+1)+1}$$

Therefore, ① is also true when $n=k+1$.

rom (i) and (ii), ① is true for all natural numbers n greater than or
qual to 2.

Mathematical Induction

Name

Date / /

Time : to :

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1. Given that n is a natural number, prove the following equality by mathemati[...]

➡ N[...]

induction.

$$1^2+2^2+3^2+\cdots+n^2=\frac{1}{6}n(n+1)(2n+1)\quad\cdots①$$

[Sol] (i) When $n=1$,

$$\text{LHS}=1^2=1.\ \text{RHS}=\frac{1}{6}\cdot1\cdot(1+1)(2\cdot1+1)=1$$

Therefore, ① is true when $n=1$.

(ii) If ① is true when $n=k$,

$$1^2+2^2+3^2+\cdots+k^2=\frac{1}{6}k(k+1)(2k+1)\quad\cdots②$$

When $n=k+1$, rearranging the LHS of ① by using ②,

$$\begin{aligned}
&1^2+2^2+3^2+\cdots+k^2+(k+1)^2\\
&=\frac{1}{6}k(k+1)(2k+1)+(k+1)^2\\
&=\frac{1}{6}(k+1)[k(2k+1)+6(k+1)]\\
&=\frac{1}{6}(k+1)(k+2)(2k+3)\\
&=\frac{1}{6}(k+1)[(k+1)+1][2(k+1)+1]
\end{aligned}$$

Therefore, ① is also true when $n=k+1$.

From (i) and (ii), ① is true for all natural numbers n.

N60b

2. Given that n is a natural number greater than or equal to 2, prove the following inequality by mathematical induction.

$3^n > 4n \quad \cdots①$

➡ N56

[Sol] (i) When $n=2$,

$$\text{LHS}=3^2=9$$

$$\text{RHS}=4\cdot2=8$$

Therefore, LHS > RHS

Thus, ① is true when $n=2$.

(ii) For $k \geqslant 2$, if ① is true when $n=k$,

$3^k > 4k \quad \cdots②$

When $n=k+1$, considering the difference of both sides of ①, then from ②,

$$\begin{aligned} 3^{k+1}-4(k+1) &= 3\cdot3^k-4(k+1) \\ &> 3\cdot4k-4(k+1) \\ &= 4(2k-1) > 0 \end{aligned}$$

$\therefore\ 3^{k+1} > 4(k+1)$

Therefore, ① is also true when $n=k+1$.

From (i) and (ii), ① is true for all natural numbers n greater than or equal to 2.

Infinite Sequences

Name

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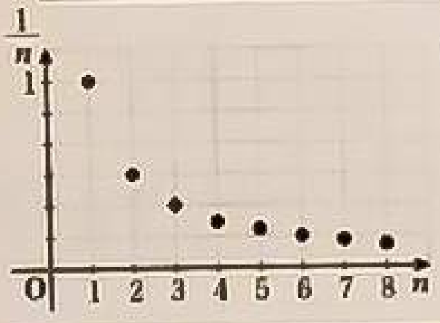
A sequence with infinite terms a_1, a_2, a_3, ..., a_n, ... is called an *infi*
sequence, and it is expressed as $\{a_n\}$.

Given the sequence 1, $\frac{1}{2}$, $\frac{1}{3}$, ..., $\frac{1}{n}$, ... fill in the following table and p
the points representing the values of $\frac{1}{n}$ when $n=4$, 5, 6, 7, 8 on the coordinate plane below. Then, consider how the value of $\frac{1}{n}$ changes as n increases infinitely.

[Sol] When $n=1$, 2, 3, 4, 5, 10, 100, the values of $\frac{1}{n}$ are as shown in table below.

n	1	2	3	4	5	···	10	···	100	···
$\frac{1}{n}$	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	···	$\frac{1}{10}$	···	$\frac{1}{100}$	···



Therefore, the value of $\frac{1}{n}$ decreases and approaches $\boxed{0}$ as n increases infinitely.

Given the sequence $\{a_n\}$, if a_n approaches a constant value α as n approach
infinity, then $\{a_n\}$ is said to ***converge*** to α, where it is expressed as follows:

$$\lim_{n\to\infty} a_n = \alpha \quad \text{or} \quad a_n \to \alpha \text{ as } n \to \infty$$

The value α is called the ***limit value*** of $\{a_n\}$. In other words, the ***limit*** of
is α. If the value of all terms of the sequence is constant c, then the limit val
also considered as c and expressed as follows: $\lim_{n\to\infty} c = c$

The symbol ∞ is read as "*infinity*" and represents an unbounded quantity that is greate
every real number.

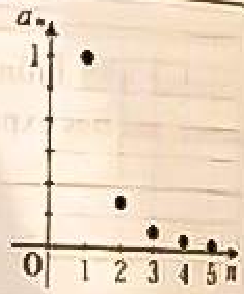
N61b

For each given sequence $\{a_n\}$, plot the points representing the values of a_n when $n=1,\ 2,\ 3,\ 4,\ 5$ on the coordinate plane, and then find the limit value.

Ex.

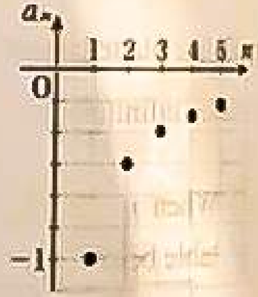
$1,\ \frac{1}{4},\ \frac{1}{9},\ \cdots,\ \frac{1}{n^2},\ \cdots$

[Sol] $\lim_{n\to\infty}\frac{1}{n^2}=0$



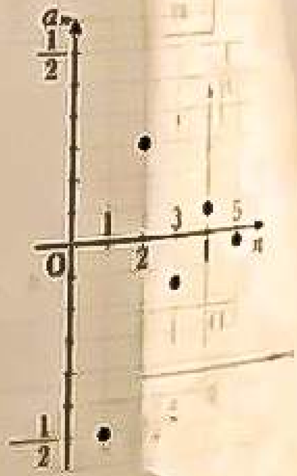
(1) $-1,\ -\frac{1}{2},\ -\frac{1}{3},\ \cdots,\ -\frac{1}{n},\ \cdots$

[Sol] $\lim_{n\to\infty}\left(-\frac{1}{n}\right)=0$



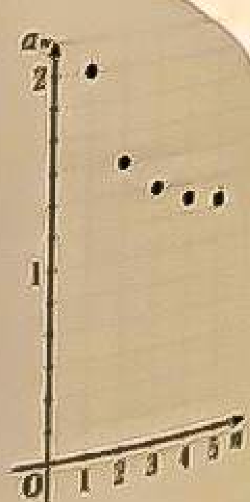
(2) $-\frac{1}{2},\ \frac{1}{4},\ -\frac{1}{8},\ \cdots,\ \left(-\frac{1}{2}\right)^n,\ \cdots$

[Sol] $\lim_{n\to\infty}\left(-\frac{1}{2}\right)^n=0$



(3) $2,\ \frac{3}{2},\ \frac{4}{3},\ \cdots,\ 1+\frac{1}{n},\ \cdots$

[Sol] $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)=1$



N62a

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Infinite Sequences

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0	1	2	3	4

When the sequence $\{a_n\}$ does not converge, $\{a_n\}$ is said to ***diverge***.

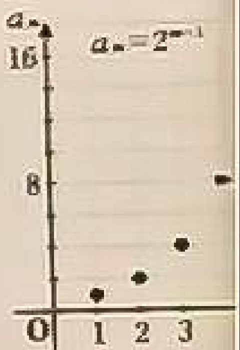
When $\{a_n\}$ ***diverges to positive infinity***, it is said that the limit of $\{a_n\}$ is pos
infinity and is expressed as follows: $\lim_{n\to\infty} a_n = \infty$ or $a_n \to \infty$ as $n \to \infty$

When $\{a_n\}$ ***diverges to negative infinity***, it is said that the limit of $\{a_n\}$ is neg
infinity and is expressed as follows: $\lim_{n\to\infty} a_n = -\infty$ or $a_n \to -\infty$ as $n \to$

When a divergent sequence does not diverge to positive or negative infinity
sequence is said to ***oscillate***.

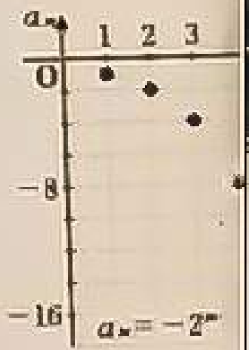
1. For each given sequence $\{a_n\}$, plot the points representing the values of a_n w
$n=4, 5$ on the coordinate plane, and then find the limit. (When it oscill
write "Oscillates.")

(1) $1, 2, 4, \dots, 2^{n-1}, \dots$



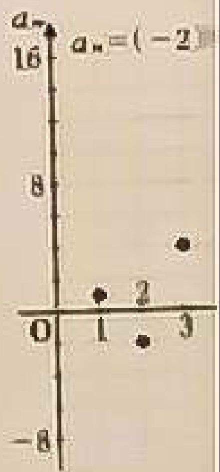
[Sol] $\lim_{n\to\infty} 2^{n-1} = \infty$

(2) $-1, -2, -4, \dots, -2^{n-1}, \dots$



[Sol] $\lim_{n\to\infty}(-2^{n-1}) = -\infty$

(3) $1, -2, 4, \dots, (-2)^{n-1}, \dots$



[Sol] Oscillates

N62b

Limit of a Sequence

Converges	$\lim_{n\to\infty} a_n = \alpha$	(converges to a constant value α)	…①
Diverges	$\lim_{n\to\infty} a_n = \infty$	(diverges to positive infinity)	…②
	$\lim_{n\to\infty} a_n = -\infty$	(diverges to negative infinity)	…③
	Oscillates	(no limit)	…④

2. Classify each given limit by indicating the appropriate number ① to ④.
If the limit is classified as ①, state the limit value.

Ex. $\lim_{n\to\infty} \frac{2}{n}$ (①, limit value 0) $\lim_{n\to\infty} (-3)^n$ (④)

(1) $\lim_{n\to\infty} \frac{1}{n^2+1}$ (**①, limit value 0**)

(2) $\lim_{n\to\infty} \left(2+\frac{1}{n}\right)$ (**①, limit value 2**)

(3) $\lim_{n\to\infty} (2n+5)$ (②)

(4) $\lim_{n\to\infty} (-2n+5)$ (③)

(5) $\lim_{n\to\infty} \cos\frac{\pi}{n}$ (**①, limit value 1**)

(6) $\lim_{n\to\infty} (-5)^n$ (④)

(7) $\lim_{n\to\infty} -4^n$ (③)

(8) $\lim_{n\to\infty} \frac{(-1)^n}{n}$ (①, limit value 0)

(9) $\lim_{n\to\infty} \left(n-\frac{1}{n}\right)$ (②)

(10) $\lim_{n\to\infty} \sin\frac{n\pi}{2}$ (④)

Substituting $n=10, 100, \cdots\cdots$
$\cos\frac{\pi}{10}, \cos\frac{\pi}{100}, \cdots\cdots$
$\lim_{n\to\infty} \cos\frac{\pi}{n}$ approaches $\cos 0$.

Substituting $n=1, 2, 3, 4, 5, \cdots$
$\sin\frac{\pi}{2}, \sin\pi, \sin\frac{3\pi}{2}, \sin 2\pi, \sin\frac{5\pi}{2}$
$\therefore 1, 0, -1, 0, 1 \cdots$

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Infinite Sequences

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	1	2	3	4

Properties of Limits of Sequences

When the sequences $\{a_n\}$ and $\{b_n\}$ converge, where $\lim_{n\to\infty} a_n=\alpha$ and $\lim_{n\to\infty} b_n=\beta$

$\lim_{n\to\infty} ka_n=k\alpha$ (k is a constant)

$\lim_{n\to\infty}(a_n+b_n)=\alpha+\beta,$ $\lim_{n\to\infty}(a_n-b_n)=\alpha-\beta$

$\lim_{n\to\infty} a_nb_n=\alpha\beta$

$\lim_{n\to\infty}\dfrac{a_n}{b_n}=\dfrac{\alpha}{\beta}$ $(\beta\neq 0)$

Find the following limits.

Ex.

$$\lim_{n\to\infty}\frac{2n+1}{3n^2+n}=\lim_{n\to\infty}\frac{\frac{2}{n}+\frac{1}{n^2}}{3+\frac{1}{n}}=0 \leftarrow$$

Dividing the numerator and the denominator by the term with the highest power in the denominator (n^2 in this case)

(1) $$\lim_{n\to\infty}\frac{2n+3}{4n^2+5n}=\lim_{n\to\infty}\frac{\frac{2}{n}+\frac{3}{n^2}}{4+\frac{5}{n}}=0$$

(2) $$\lim_{n\to\infty}\frac{8-5n}{4+n}=\lim_{n\to\infty}\frac{\frac{8}{n}-5}{\frac{4}{n}+1}=-5$$

(3) $$\lim_{n\to\infty}\frac{n^2+5n+4}{3-2n^2}=\lim_{n\to\infty}\frac{1+\frac{5}{n}+\frac{4}{n^2}}{\frac{3}{n^2}-2}=-\frac{1}{2}$$

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4) $\lim\limits_{n\to\infty}\dfrac{2n^2-3n}{n+1}=\lim\limits_{n\to\infty}\dfrac{\boxed{2n-3}}{1+\dfrac{1}{n}}=\infty$

5) $\lim\limits_{n\to\infty}\dfrac{-2n^3-5n}{n^2+2n+1}=\lim\limits_{n\to\infty}\dfrac{-2n-\dfrac{5}{n}}{1+\dfrac{2}{n}+\dfrac{1}{n^2}}=-\infty$

6) $\lim\limits_{n\to\infty}\dfrac{-2n^2(n+2)}{2n^3-5}=\lim\limits_{n\to\infty}\dfrac{-2n^3-4n^2}{2n^3-5}=\lim\limits_{n\to\infty}\dfrac{-2-\dfrac{4}{n}}{2-\dfrac{5}{n^3}}=-1$

) $\lim\limits_{n\to\infty}\dfrac{5+n+n^5}{n(4-3n^2)}=\lim\limits_{n\to\infty}\dfrac{5+n+n^5}{4n-3n^3}=\lim\limits_{n\to\infty}\dfrac{\dfrac{5}{n^3}+\dfrac{1}{n^2}+n^2}{\dfrac{4}{n^2}-3}=-\infty$

$\lim\limits_{n\to\infty}\left(\dfrac{5}{n^3}+\dfrac{1}{n^2}+n^2\right)=\infty$

$\lim\limits_{n\to\infty}\left(\dfrac{4}{n^2}-3\right)=-3$

) $\lim\limits_{n\to\infty}\dfrac{\sqrt{n^2+1}}{2n}=\lim\limits_{n\to\infty}\dfrac{\sqrt{1+\boxed{\dfrac{1}{n^2}}}}{2}=\dfrac{1}{2}$

$\dfrac{\sqrt{n^2+1}}{n}=\sqrt{\dfrac{n^2+1}{n^2}}=\sqrt{\dfrac{n^2}{n^2}+\dfrac{1}{n^2}}$

$\lim\limits_{n\to\infty}\dfrac{\sqrt{4n^2+3n}}{6n}=\lim\limits_{n\to\infty}\dfrac{\sqrt{4+\dfrac{3}{n}}}{6}=\dfrac{1}{3}$

) $\lim\limits_{n\to\infty}\dfrac{2n^2+4n+3}{\sqrt{3n^2+1}}=\lim\limits_{n\to\infty}\dfrac{2n+4+\dfrac{3}{n}}{\sqrt{3+\dfrac{1}{n^2}}}=\infty$

Ex., sinc[illegible] $=\infty$ and $\lim\limits_{n\to\infty}(2n+1)=\infty$, the limit becomes an inde[illegible]

$\frac{\infty}{\infty}$. T[illegible] cases, the expression has to be rearranged.

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Infinite Sequences

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Find the following limits.

Ex.

$$\lim_{n\to\infty}\log_2\frac{16n+1}{2n}=\lim_{n\to\infty}\log_2\frac{16+\frac{1}{n}}{2}$$
$$=\log_2 8$$
$$=3$$

$\boxed{\log_a a^m = m}$

(1) $$\lim_{n\to\infty}\log_3\frac{9n+1}{3n}=\lim_{n\to\infty}\log_3\frac{9+\frac{1}{n}}{3}$$
$$=\log_3 3$$
$$=1$$

$\boxed{\log_a a = 1}$

(2) $$\lim_{n\to\infty}\left[\log_3(2n^2-1)-\log_3(6n^2+2)\right]=\lim_{n\to\infty}\log_3\frac{2n^2-1}{6n^2+2}$$

$\log_a M - \log$
$= \log_a \frac{M}{N}$

$$=\lim_{n\to\infty}\log_3\frac{2-\frac{1}{n^2}}{6+\frac{2}{n^2}}$$
$$=\log_3\frac{1}{3}$$
$$=-1$$

(3) $$\lim_{n\to\infty}\left[\log_2\sqrt{4n^4+3n}-\log_2(2n^2+n)\right]=\lim_{n\to\infty}\log_2\frac{\sqrt{4n^4+3n}}{2n^2+n}$$
$$=\lim_{n\to\infty}\log_2\frac{\sqrt{4+\frac{3}{n^3}}}{2+\frac{1}{n}}$$
$$=\log_2 1$$
$$=0$$

$\boxed{\log_a 1 = 0}$

$\log_a a = 1,\quad \log_a 1 = 0,\quad \log_a\frac{M}{N}=\log_a M-\log_a N$

N64b

Ex.

$$\lim_{n\to\infty}\frac{1+2+3+\cdots+n}{3n^2}=\lim_{n\to\infty}\frac{\frac{1}{2}n(n+1)}{3n^2} \leftarrow \boxed{1+2+3+\cdots+n=\sum_{k=1}^{n}k}$$

$$=\lim_{n\to\infty}\frac{n+1}{6n}$$

$$=\lim_{n\to\infty}\frac{1+\frac{1}{n}}{6}$$

$$=\frac{1}{6}$$

(4) $$\lim_{n\to\infty}\frac{1^2+2^2+3^2+\cdots+n^2}{n^3}=\lim_{n\to\infty}\frac{\frac{1}{6}n(n+1)(2n+1)}{n^3} \leftarrow \boxed{1^2+2^2+3^2+\cdots+n^2=\sum_{k=1}^{n}k^2}$$

$$=\lim_{n\to\infty}\frac{2n^2+3n+1}{6n^2}$$

$$=\lim_{n\to\infty}\frac{2+\frac{3}{n}+\frac{1}{n^2}}{6}$$

$$=\frac{1}{3}$$

(5) $$\lim_{n\to\infty}\frac{2^3+4^3+6^3+\cdots+(2n)^3}{n^4}=\lim_{n\to\infty}\frac{8\left[\frac{1}{2}n(n+1)\right]^2}{n^4} \leftarrow \boxed{\begin{aligned}&2^3+4^3+6^3+\cdots+(2n)^3\\&=\sum_{k=1}^{n}(2k)^3\\&=8\sum_{k=1}^{n}k^3\end{aligned}}$$

$$=\lim_{n\to\infty}\frac{2n^2+4n+2}{n^2}$$

$$=\lim_{n\to\infty}\left(2+\frac{4}{n}+\frac{2}{n^2}\right)$$

$$=2$$

$\sum_{k=1}^{n}k^2=$

Infinite Sequences

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Find the following limits.

Ex. $\lim\limits_{n\to\infty}(n^2-3n)=\lim\limits_{n\to\infty}n^2\left(1-\dfrac{3}{n}\right)=\infty$ ← Taking out the term with the highest power

(1) $\lim\limits_{n\to\infty}(n^2-10n)=\lim\limits_{n\to\infty}n^2\left(1-\dfrac{10}{n}\right)=\infty$

(2) $\lim\limits_{n\to\infty}(2n^3-3n^2+4)=\lim\limits_{n\to\infty}n^3\left(2-\dfrac{3}{n}+\dfrac{4}{n^3}\right)=\infty$

(3) $\lim\limits_{n\to\infty}(4n^2-2n^3)=\lim\limits_{n\to\infty}n^3\left(\dfrac{4}{n}-2\right)=-\infty$ ← $\lim\limits_{n\to\infty}n^3=\infty$, $\lim\limits_{n\to\infty}\left(\dfrac{4}{n}-2\right)=-2$

(4) $\lim\limits_{n\to\infty}(n-\sqrt{n})=\lim\limits_{n\to\infty}n\left(1-\dfrac{1}{\sqrt{n}}\right)=\infty$

(5) $\lim\limits_{n\to\infty}(\sqrt{n^3}-n^2)=\lim\limits_{n\to\infty}n^2\left(\dfrac{1}{\sqrt{n}}-1\right)=-\infty$

In **Ex.**, since $\lim\limits_{n\to\infty}n^2=\infty$ and $\lim\limits_{n\to\infty}3n=\infty$, the limit becomes an indeterminate form $\infty-$
Therefore, in such cases, the expression has to be rearranged.

Ex.

$$\lim_{n\to\infty}(\sqrt{n^2+3n}-n)=\lim_{n\to\infty}\frac{(\sqrt{n^2+3n}-n)(\sqrt{n^2+3n}+n)}{\sqrt{n^2+3n}+n}$$

$$=\lim_{n\to\infty}\frac{3n}{\sqrt{n^2+3n}+n}$$

$$=\lim_{n\to\infty}\frac{3}{\sqrt{1+\frac{3}{n}}+1}$$

$$=\frac{3}{2}$$

Considering $\sqrt{n^2+3n}-n=\frac{\sqrt{n^2+3n}-n}{1}$, then multiplying the numerator and the denominator by $\sqrt{n^2+3n}+n$

(6) $$\lim_{n\to\infty}(\sqrt{n^2+n+1}-n)=\lim_{n\to\infty}\frac{(\sqrt{n^2+n+1}-n)(\sqrt{n^2+n+1}+n)}{\sqrt{n^2+n+1}+n}$$

$$=\lim_{n\to\infty}\frac{n+1}{\sqrt{n^2+n+1}+n}$$

$$=\lim_{n\to\infty}\frac{1+\frac{1}{n}}{\sqrt{1+\frac{1}{n}+\frac{1}{n^2}}+1}$$

$$=\frac{1}{2}$$

(7) $$\lim_{n\to\infty}(\sqrt{2n}+\sqrt{n})(\sqrt{n+1}-\sqrt{n})$$

$$=\lim_{n\to\infty}\frac{(\sqrt{2n}+\sqrt{n})(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{\sqrt{n+1}+\sqrt{n}}$$

$$=\lim_{n\to\infty}\frac{\sqrt{2n}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$$

Dividing the numerator and the denominator by $\sqrt{n}$

$$=\lim_{n\to\infty}\frac{\sqrt{2}+1}{\sqrt{1+\frac{1}{n}}+1}$$

$$=\frac{\sqrt{2}+1}{2}$$

Infinite Sequences

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Find the following limits.

Ex.

$$\lim_{n\to\infty}\frac{1}{\sqrt{n^2+5n}-n}=\lim_{n\to\infty}\frac{\sqrt{n^2+5n}+n}{(\sqrt{n^2+5n}-n)(\sqrt{n^2+5n}+n)}$$

Multiplying the numerator and the denominator by $\sqrt{n^2+5n}+$

$$=\lim_{n\to\infty}\frac{\sqrt{n^2+5n}+n}{5n}$$

$$=\lim_{n\to\infty}\frac{\sqrt{1+\frac{5}{n}}+1}{5}$$

$$=\frac{2}{5}$$

(1) $$\lim_{n\to\infty}\frac{3}{\sqrt{4n^2+2n}-2n}=\lim_{n\to\infty}\frac{3(\sqrt{4n^2+2n}+2n)}{(\sqrt{4n^2+2n}-2n)(\sqrt{4n^2+2n}+2n)}$$

$$=\lim_{n\to\infty}\frac{3(\sqrt{4n^2+2n}+2n)}{2n}$$

$$=\lim_{n\to\infty}\frac{3\left(\sqrt{4+\frac{2}{n}}+2\right)}{2}$$

$$=6$$

(2) $$\lim_{n\to\infty}\frac{5}{\sqrt{n^2+2}-\sqrt{n}}=\lim_{n\to\infty}\frac{5(\sqrt{n^2+2}+\sqrt{n})}{(\sqrt{n^2+2}-\sqrt{n})(\sqrt{n^2+2}+\sqrt{n})}$$

$$=\lim_{n\to\infty}\frac{5(\sqrt{n^2+2}+\sqrt{n})}{n^2-n+2}$$

$$=\lim_{n\to\infty}\frac{5\left(\sqrt{\frac{1}{n^2}+\frac{2}{n^4}}+\sqrt{\frac{1}{n^3}}\right)}{1-\frac{1}{n}+\frac{2}{n^2}}$$

$$=0$$

(3) $\lim\limits_{n\to\infty}\dfrac{\sqrt{n+3}-\sqrt{n}}{\sqrt{n+1}-\sqrt{n}}$

$$=\lim_{n\to\infty}\frac{(\sqrt{n+3}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})}$$

Multiplying the numerator and the denominator by $\sqrt{n+1}+\sqrt{n}$

$$=\lim_{n\to\infty}(\sqrt{n+3}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})$$

$$=\lim_{n\to\infty}\frac{(\sqrt{n+3}-\sqrt{n})(\sqrt{n+3}+\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{\sqrt{n+3}+\sqrt{n}}$$

Considering $\sqrt{n+3}-\sqrt{n}=\dfrac{\sqrt{n+3}-\sqrt{n}}{1}$, then multiplying the numerator and the denominator by $\sqrt{n+3}+\sqrt{n}$

$$=\lim_{n\to\infty}\frac{3(\sqrt{n+1}+\sqrt{n})}{\sqrt{n+3}+\sqrt{n}}$$

$$=\lim_{n\to\infty}\frac{3\left(\sqrt{1+\frac{1}{n}}+1\right)}{\sqrt{1+\frac{3}{n}}+1}$$

$$=3$$

Alternative Solution

$$\lim_{n\to\infty}\frac{\sqrt{n+3}-\sqrt{n}}{\sqrt{n+1}-\sqrt{n}}$$

$$=\lim_{n\to\infty}\frac{(\sqrt{n+3}-\sqrt{n})(\sqrt{n+3}+\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})(\sqrt{n+3}+\sqrt{n})}$$

$$=\lim_{n\to\infty}\frac{3(\sqrt{n+1}+\sqrt{n})}{\sqrt{n+3}+\sqrt{n}}=\lim_{n\to\infty}\frac{3\left(\sqrt{1+\frac{1}{n}}+1\right)}{\sqrt{1+\frac{3}{n}}+1}$$

$$=3$$

(4) $\lim\limits_{n\to\infty}\dfrac{2(\sqrt{n+5}-\sqrt{n+3})}{\sqrt{n+2}-\sqrt{n}}$

$$=\lim_{n\to\infty}\frac{2(\sqrt{n+5}-\sqrt{n+3})(\sqrt{n+2}+\sqrt{n})}{(\sqrt{n+2}-\sqrt{n})(\sqrt{n+2}+\sqrt{n})}$$

$$=\lim_{n\to\infty}(\sqrt{n+5}-\sqrt{n+3})(\sqrt{n+2}+\sqrt{n})$$

$$=\lim_{n\to\infty}\frac{(\sqrt{n+5}-\sqrt{n+3})(\sqrt{n+5}+\sqrt{n+3})(\sqrt{n+2}+\sqrt{n})}{\sqrt{n+5}+\sqrt{n+3}}$$

$$=\lim_{n\to\infty}\frac{2(\sqrt{n+2}+\sqrt{n})}{\sqrt{n+5}+\sqrt{n+3}}$$

$$=\lim_{n\to\infty}\frac{2\left(\sqrt{1+\frac{2}{n}}+1\right)}{\sqrt{1+\frac{5}{n}}+\sqrt{1+\frac{3}{n}}}$$

Alternative Solution

$$\lim_{n\to\infty}\frac{2(\sqrt{n+5}-\sqrt{n+3})}{\sqrt{n+2}-\sqrt{n}}$$

$$=\lim_{n\to\infty}\frac{2(\sqrt{n+5}-\sqrt{n+3})(\sqrt{n+5}+\sqrt{n+3})(\sqrt{n+2}+\sqrt{n})}{(\sqrt{n+2}-\sqrt{n})(\sqrt{n+2}+\sqrt{n})(\sqrt{n+5}+\sqrt{n+3})}$$

$$=\lim_{n\to\infty}\frac{2(\sqrt{n+2}+\sqrt{n})}{\sqrt{n+5}+\sqrt{n+3}}=\lim_{n\to\infty}\frac{2\left(\sqrt{1+\frac{2}{n}}+1\right)}{\sqrt{1+\frac{5}{n}}+\sqrt{1+\frac{3}{n}}}$$

$$=2$$

KUMON

Infinite Sequences

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Limits of Sequences and Their Relationships

1. For all n, when $a_n \leqq b_n$,

   if $\lim\limits_{n\to\infty} a_n = \alpha$ and $\lim\limits_{n\to\infty} b_n = \beta$, then $\alpha \leqq \beta$

   if $\lim\limits_{n\to\infty} a_n = \infty$, then $\lim\limits_{n\to\infty} b_n = \infty$

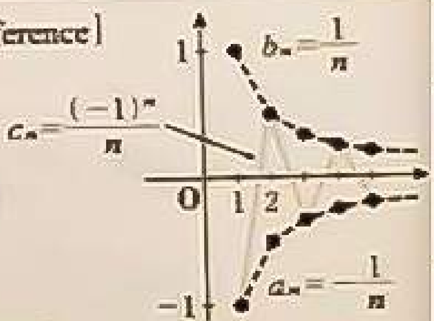
2. For all n, when $a_n \leqq c_n \leqq b_n$,

   if $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = \alpha$, then $\lim\limits_{n\to\infty} c_n = \alpha$

Find the following limits.

Ex. $\lim\limits_{n\to\infty} \dfrac{(-1)^n}{n}$

[Reference]



[Sol] Since $-1 \leqq (-1)^n \leqq 1$,

$n > 0$

$$-\frac{1}{n} \leqq \frac{(-1)^n}{n} \leqq \frac{1}{n}$$

Then, since $\lim\limits_{n\to\infty}\left(-\dfrac{1}{n}\right) = 0$ and $\lim\limits_{n\to\infty} \dfrac{1}{n} = 0$,

$$\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$$

(1) $\lim\limits_{n\to\infty} \dfrac{(-1)^n}{\sqrt{n}}$

[Sol] Since $-1 \leqq (-1)^n \leqq 1$,

$$-\frac{1}{\sqrt{n}} \leqq \frac{(-1)^n}{\sqrt{n}} \leqq \frac{1}{\sqrt{n}}$$

Then, since $\lim\limits_{n\to\infty}\left(-\dfrac{1}{\sqrt{n}}\right) = 0$ and $\lim\limits_{n\to\infty} \dfrac{1}{\sqrt{n}} = 0$,

$$\lim_{n\to\infty} \frac{(-1)^n}{\sqrt{n}} = 0$$

As for Limits of Sequences and Their Relationships, statement 1 is also true when $a_n < b$
statement 2 is also true when $a_n \leqq c_n < b_n$, $a_n < c_n \leqq b_n$, $a_n < c_n < b_n$.

N67b

(2) $\lim_{n\to\infty}\frac{1}{n}\sin\frac{n\pi}{2}$

[Sol] Since $-1\leqslant\sin\frac{n\pi}{2}\leqslant 1$, ← $-1\leqslant\sin\theta\leqslant 1$ (M131)

$$-\frac{1}{n}\leqslant\frac{1}{n}\sin\frac{n\pi}{2}\leqslant\frac{1}{n}$$

Then, since $\lim_{n\to\infty}\left(-\frac{1}{n}\right)=0$ and $\lim_{n\to\infty}\frac{1}{n}=0$,

$$\lim_{n\to\infty}\frac{1}{n}\sin\frac{n\pi}{2}=0$$

(3) $\lim_{n\to\infty}\frac{n+1}{n^2}\cos n\theta$ (θ is a constant)

[Sol] Since $-1\leqslant\cos n\theta\leqslant 1$,

$$-\frac{n+1}{n^2}\leqslant\frac{n+1}{n^2}\cos n\theta\leqslant\frac{n+1}{n^2}$$

Then, since $\lim_{n\to\infty}\left(-\frac{n+1}{n^2}\right)=\lim_{n\to\infty}\left(-\frac{1}{n}-\frac{1}{n^2}\right)=0$ and

$$\lim_{n\to\infty}\frac{n+1}{n^2}=\lim_{n\to\infty}\left(\frac{1}{n}+\frac{1}{n^2}\right)=0,$$

$$\lim_{n\to\infty}\frac{n+1}{n^2}\cos n\theta=0$$

(4) $\lim_{n\to\infty}\frac{2^n}{n}$ $\left(\text{Using inequality } 2^n>\frac{n^2}{2}\right)$

[Sol] Since $2^n>\frac{n^2}{2}$,

$$\frac{2^n}{n}>\boxed{\frac{n}{2}}$$

Then, since $\lim_{n\to\infty}\frac{n}{2}=\infty$,

$$\lim_{n\to\infty}\frac{2^n}{n}=\infty$$

N68a

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Infinite Sequences

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Find the following limits.

Ex. $\displaystyle\lim_{n\to\infty}\frac{1}{2n-1}(n+\sin n\theta)$

[Sol] $\displaystyle\lim_{n\to\infty}\frac{1}{2n-1}(n+\sin n\theta)$

$$=\lim_{n\to\infty}\left(\frac{n}{2n-1}+\frac{1}{2n-1}\sin n\theta\right)$$

Then, $\displaystyle\lim_{n\to\infty}\frac{n}{2n-1}=\lim_{n\to\infty}\frac{1}{2-\frac{1}{n}}=\boxed{\frac{1}{2}}$ …① ← $\displaystyle\lim_{n\to\infty}\frac{n}{2n-1}$ converges.

Also, since $-1\leqq\sin n\theta\leqq 1$,

$$\boxed{-\frac{1}{2n-1}}\leqq\frac{1}{2n-1}\sin n\theta\leqq\boxed{\frac{1}{2n-1}}$$ ← $2n-1>0$

Then, since $\displaystyle\lim_{n\to\infty}\left(\boxed{-\frac{1}{2n-1}}\right)=\boxed{0}$ and $\displaystyle\lim_{n\to\infty}\boxed{\frac{1}{2n-1}}=\boxed{0}$

$\displaystyle\lim_{n\to\infty}\frac{1}{2n-1}\sin n\theta=\boxed{0}$ …② ← $\displaystyle\lim_{n\to\infty}\frac{1}{2n-1}\sin n\theta$ converges.

From ① and ②,

$\displaystyle\lim_{n\to\infty}\frac{1}{2n-1}(n+\sin n\theta)=\boxed{\frac{1}{2}}$ ← From ① and ②, $\displaystyle\lim_{n\to\infty}\left(\frac{n}{2n-1}+\frac{1}{2n-1}\sin n\theta\right)=\lim_{n\to\infty}\frac{n}{2n-1}+\lim_{n\to\infty}\frac{1}{2n-1}\sin n\theta$

Ex. $\frac{1}{2}$, $-\frac{1}{2n-1}$, $\frac{1}{2n-1}$, $-\frac{1}{2n-1}$, 0, $\frac{1}{2n-1}$, 0, 0, $\frac{1}{2}$

N68b

(1) $\lim_{n\to\infty}\frac{1}{5n-2}\left(n+\cos\frac{n\pi}{6}\right)$

[Sol] $\lim_{n\to\infty}\frac{1}{5n-2}\left(n+\cos\frac{n\pi}{6}\right)$

$=\lim_{n\to\infty}\left(\frac{n}{5n-2}+\frac{1}{5n-2}\cos\frac{n\pi}{6}\right)$

Then, $\lim_{n\to\infty}\frac{n}{5n-2}=\lim_{n\to\infty}\frac{1}{5-\frac{2}{n}}=\frac{1}{5}$ …①

Also, since $-1\leqslant\cos\frac{n\pi}{6}\leqslant 1$,

$-\frac{1}{5n-2}\leqslant\frac{1}{5n-2}\cos\frac{n\pi}{6}\leqslant\frac{1}{5n-2}$

Then, since $\lim_{n\to\infty}\left(-\frac{1}{5n-2}\right)=0$ and $\lim_{n\to\infty}\frac{1}{5n-2}=0$,

$\lim_{n\to\infty}\frac{1}{5n-2}\cos\frac{n\pi}{6}=0$ …②

From ① and ②,

$\lim_{n\to\infty}\frac{1}{5n-2}\left(n+\cos\frac{n\pi}{6}\right)=\frac{1}{5}$

Infinite Sequences

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1. Given the sequence $\{a_n\}$: $a_1=5$, $a_2=12$, $a_3=21$, $a_4=32$, ……, when t sequence of differences $\{b_n\}$ of this sequence becomes an arithmetic sequen solve the following questions.

(1) Find a_n.

[Sol] $\{b_n\}$ is 7, 9, 11, ……

$\therefore\ b_n=7+(n-1)\cdot 2=2n+5$

When $n \geqq 2$,

$$a_n=a_1+\sum_{k=1}^{n-1}(2k+5)$$

$$=5+2\cdot\frac{1}{2}(n-1)n+5(n-1)$$

$$=n^2+4n \quad \cdots ①$$

Since $a_1=5$, ① is also true when $n=1$.

$\therefore\ a_n=n^2+4n$

$\{a_n\}$: 5 12 21 32……
$\{b_n\}$: 7 9 11……

$\{b_n\}$ is the arithmetic sequence wit 1st term 7 and common difference

When $n \geqq 2$,
$a_n=a_1+\sum_{k=1}^{n-1} b_k$
(N32)

(2) Find $\lim_{n\to\infty}(\sqrt{a_n}-n)$ and $\lim_{n\to\infty}\left(\sqrt{\frac{a_n}{n}}-\sqrt{n}\right)$.

[Sol] From (1),

$$\lim_{n\to\infty}(\sqrt{a_n}-n)=\lim_{n\to\infty}(\sqrt{n^2+4n}-n)=\lim_{n\to\infty}\frac{(\sqrt{n^2+4n}-n)(\sqrt{n^2+4n}+}{\sqrt{n^2+4n}+n}$$

$$=\lim_{n\to\infty}\frac{4n}{\sqrt{n^2+4n}+n}$$

$$=\lim_{n\to\infty}\frac{4}{\sqrt{1+\frac{4}{n}}+1}$$

$$=2$$

$$\lim_{n\to\infty}\left(\sqrt{\frac{a_n}{n}}-\sqrt{n}\right)=\lim_{n\to\infty}\left(\sqrt{\frac{n^2+4n}{n}}-\sqrt{n}\right)$$

$$=\lim_{n\to\infty}(\sqrt{n+4}-\sqrt{n})$$

$$=\lim_{n\to\infty}\frac{(\sqrt{n+4}-\sqrt{n})(\sqrt{n+4}+\sqrt{n})}{\sqrt{n+4}+\sqrt{n}}$$

$$=\lim_{n\to\infty}\frac{4}{\sqrt{n+4}+\sqrt{n}}$$

$$=0$$

N69b

2. Given that n is a positive integer and $x \geqslant 0$, solve the following questions.

(1) Using inequality $(1+x)^n \geqslant 1+nx+\frac{1}{2}n(n-1)x^2$, prove that $1+\sqrt{\frac{2}{n}} > n^{\frac{1}{n}}$ is true.

[Sol] In $(1+x)^n \geqslant 1+nx+\frac{1}{2}n(n-1)x^2$, let $x=\sqrt{\frac{2}{n}}$.

$$\left(1+\sqrt{\frac{2}{n}}\right)^n \geqslant 1+n\sqrt{\frac{2}{n}}+\frac{1}{2}n(n-1)\cdot\frac{2}{n}$$

$$=1+\sqrt{2n}+n-1$$

$$=n+\sqrt{2n}>n$$ ← $\sqrt{2n}>0$

$$\therefore \left(1+\sqrt{\frac{2}{n}}\right)^n > n$$

$1+\sqrt{\frac{2}{n}}>0$, $n>0$; therefore,

When $a>0$, $b>0$ and $p>0$, $a^p<b^p \Leftrightarrow a<b$ (K188)

$$1+\sqrt{\frac{2}{n}} > n^{\frac{1}{n}}$$

(2) Find the value of $\lim_{n\to\infty} n^{\frac{1}{n}}$.

[Sol] Since $n \geqslant 1$, $n^{\frac{1}{n}} \geqslant 1^{\frac{1}{n}}=1$ ···① ← For all positive integers n, $1^{\frac{1}{n}}=1$

From ① and (1),

$$1 \leqslant n^{\frac{1}{n}} < 1+\sqrt{\frac{2}{n}}$$

Then, since $\lim_{n\to\infty} 1=1$ and $\lim_{n\to\infty}\left(1+\sqrt{\frac{2}{n}}\right)=1$, ← $\lim_{n\to\infty} c=c$ (c is a constant)

$$\lim_{n\to\infty} n^{\frac{1}{n}}=1$$

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Find the following limits.

(1) $\lim_{n\to\infty}\frac{5-2n^2}{3n^2+4}=\lim_{n\to\infty}\frac{\frac{5}{n^2}-2}{3+\frac{4}{n^2}}=-\frac{2}{3}$

(2) $\lim_{n\to\infty}\frac{2n^2+3n+1}{12n^3-18n}=\lim_{n\to\infty}\frac{\frac{2}{n}+\frac{3}{n^2}+\frac{1}{n^3}}{12-\frac{18}{n^2}}=0$

(3) $\lim_{n\to\infty}\frac{n^2+6}{\sqrt{5n^2+4}}=\lim_{n\to\infty}\frac{n+\frac{6}{n}}{\sqrt{5+\frac{4}{n^2}}}=\infty$

(4) $\lim_{n\to\infty}(n^3-6n^2)=\lim_{n\to\infty}n^3\left(1-\frac{6}{n}\right)=\infty$

N70b

(5) $\displaystyle\lim_{n\to\infty}(\sqrt{n^2-9n}-n)=\lim_{n\to\infty}\frac{(\sqrt{n^2-9n}-n)(\sqrt{n^2-9n}+n)}{\sqrt{n^2-9n}+n}$ ➡ N65

$$=\lim_{n\to\infty}\frac{-9n}{\sqrt{n^2-9n}+n}$$

$$=\lim_{n\to\infty}\frac{-9}{\sqrt{1-\frac{9}{n}}+1}$$

$$=-\frac{9}{2}$$

(6) $\displaystyle\lim_{n\to\infty}\frac{4}{\sqrt{n^2+3n}-n}=\lim_{n\to\infty}\frac{4(\sqrt{n^2+3n}+n)}{(\sqrt{n^2+3n}-n)(\sqrt{n^2+3n}+n)}$ ➡ N66

$$=\lim_{n\to\infty}\frac{4(\sqrt{n^2+3n}+n)}{3n}$$

$$=\lim_{n\to\infty}\frac{4\left(\sqrt{1+\frac{3}{n}}+1\right)}{3}$$

$$=\frac{8}{3}$$

➡ N67

(7) $\displaystyle\lim_{n\to\infty}\frac{1}{n}\cos\frac{n\pi}{3}$

[Sol] Since $-1\leqq\cos\frac{n\pi}{3}\leqq 1$,

$$-\frac{1}{n}\leqq\frac{1}{n}\cos\frac{n\pi}{3}\leqq\frac{1}{n}$$

Then, since $\displaystyle\lim_{n\to\infty}\left(-\frac{1}{n}\right)=0$ and $\displaystyle\lim_{n\to\infty}\frac{1}{n}=0$,

$$\lim_{n\to\infty}\frac{1}{n}\cos\frac{n\pi}{3}=0$$

Infinite Geometric Sequences

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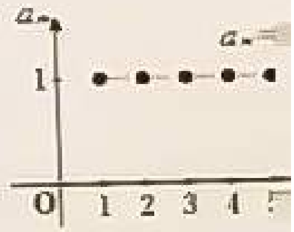
A sequence $a,\ ar,\ ar^2,\ \ldots,\ ar^{n-1},\ \ldots$ is called an ***infinite geometric sequ***
with 1st term a and common ratio r.

Find the limit for each given infinite geometric sequence. (When it oscill
write "Oscillates.")

[1] $2,\ 4,\ 8,\ 16,\ \ldots,\ 2^n,\ \ldots$

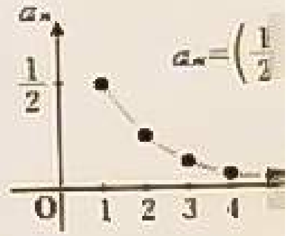
[Sol] $\lim_{n\to\infty} 2^n = \boxed{\infty}$

[2] $1,\ 1,\ 1,\ 1,\ \ldots,\ 1^n,\ \ldots$



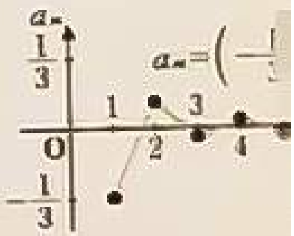
[Sol] $\lim_{n\to\infty} 1^n = \boxed{1}$

[3] $\frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \frac{1}{16},\ \ldots,\ \left(\frac{1}{2}\right)^n,\ \ldots$



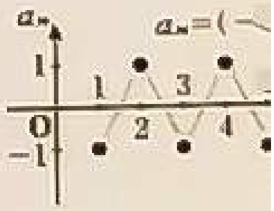
[Sol] $\lim_{n\to\infty}\left(\frac{1}{2}\right)^n = \boxed{0}$

[4] $-\frac{1}{3},\ \frac{1}{9},\ -\frac{1}{27},\ \frac{1}{81},\ \ldots,\ \left(-\frac{1}{3}\right)^n,\ \ldots$



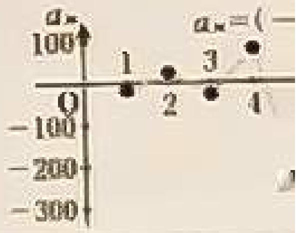
[Sol] $\lim_{n\to\infty}\left(-\frac{1}{3}\right)^n = \boxed{0}$

[5] $-1,\ 1,\ -1,\ 1,\ \ldots,\ (-1)^n,\ \ldots$



[Sol] **Oscillates**

[6] $-3,\ 9,\ -27,\ 81,\ \ldots,\ (-3)^n,\ \ldots$



[Sol] **Oscillates**

∞, 1, 0, 0, Oscillates, Oscillates

N71b

Limit of an Infinite Geometric Sequence $\{r^n\}$

When $r>1$, $\lim_{n\to\infty} r^n=\infty$ — Diverges

When $r=1$, $\lim_{n\to\infty} r^n=1$

When $|r|<1$, $\lim_{n\to\infty} r^n=0$ — Converges

When $r\le-1$, Oscillates (no limit) — Diverges

Classify each given limit by indicating the appropriate number ① to ④.

Converges ---①

Diverges to positive infinity ---②

Diverges to negative infinity ---③

Oscillates (no limit) ---④

If the limit is classified as ①, state the limit value.

Ex.

$\lim_{n\to\infty}\left(-\frac{1}{5}\right)^n$ ← $\left|-\frac{1}{5}\right|<1$ (①, limit value 0)

(1) $\lim_{n\to\infty}\left(-\frac{2}{3}\right)^n$

(①, limit value 0)

(2) $\lim_{n\to\infty}\left(\frac{5}{4}\right)^n$

(②)

(3) $\lim_{n\to\infty}\left[-\left(\frac{5}{3}\right)^n\right]$

(③)

$\lim_{n\to\infty}\left(-\frac{4}{3}\right)^n$

(④)

(5) $\lim_{n\to\infty}\left(\frac{\sqrt{5}}{2}\right)^n$

(②)

Since $\sqrt{5}>2$, $\frac{\sqrt{5}}{2}>1$

(6) $\lim_{n\to\infty}\left[1+\left(\frac{\sqrt{3}}{2}\right)^n\right]$

(①, limit value 1)

(7) $\lim_{n\to\infty}\left[-\left(\frac{3}{2}\right)^{n+1}\right]$

(③)

$-\left(\frac{3}{2}\right)^{n+1}=-\frac{3}{2}\left(\frac{3}{2}\right)^n$

(8) $\lim_{n\to\infty}\left[\left(\frac{1}{2}\right)^n+(-\sqrt{2})^n\right]$

(④)

$\lim_{n\to\infty}\left(\frac{1}{2}\right)^n=0$ and $\lim_{n\to\infty}(-\sqrt{2})^n$ oscillates

N72a

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Find the following limits.

Ex.

$$\lim_{n\to\infty}\frac{1+3^n}{3^n+4^n}=\lim_{n\to\infty}\frac{\left(\frac{1}{4}\right)^n+\left(\frac{3}{4}\right)^n}{\left(\frac{3}{4}\right)^n+1}=0$$

Dividing the numerator and the denominator by the term with the largest absolute value of the base in the denominator (4^n in this case

(1) $$\lim_{n\to\infty}\frac{1+2^n}{2^n+3^n}=\lim_{n\to\infty}\frac{\left(\frac{1}{3}\right)^n+\left(\frac{2}{3}\right)^n}{\left(\frac{2}{3}\right)^n+1}=0$$

(2) $$\lim_{n\to\infty}\frac{(-2)^n+2\cdot 3^n}{3^n+1}=\lim_{n\to\infty}\frac{\left(-\frac{2}{3}\right)^n+2}{1+\left(\frac{1}{3}\right)^n}=2$$

(3) $$\lim_{n\to\infty}\frac{3^{n+1}}{4^n+2^n}=\lim_{n\to\infty}\frac{3\left(\frac{3}{4}\right)^n}{1+\left(\frac{1}{2}\right)^n}=0$$

(4) $$\lim_{n\to\infty}\frac{5^n-3^{n-1}}{5^n+3^n}=\lim_{n\to\infty}\frac{1-\frac{1}{3}\left(\frac{3}{5}\right)^n}{1+\left(\frac{3}{5}\right)^n}=1$$

(5) $$\lim_{n\to\infty}\frac{3^{n-1}+4^{n+1}}{3^n-4^n}=\lim_{n\to\infty}\frac{\frac{1}{3}\left(\frac{3}{4}\right)^n+4}{\left(\frac{3}{4}\right)^n-1}=-4$$

(6) $\displaystyle\lim_{n\to\infty}\frac{4^n}{2^n+3^n}=\lim_{n\to\infty}\frac{\left(\frac{4}{3}\right)^n}{\left(\frac{2}{3}\right)^n+1}=\infty$

(7) $\displaystyle\lim_{n\to\infty}\frac{-5^n+3^{n+1}}{3^n+4^n}=\lim_{n\to\infty}\frac{-\left(\frac{5}{4}\right)^n+3\left(\frac{3}{4}\right)^n}{\left(\frac{3}{4}\right)^n+1}=-\infty$

(8) $\displaystyle\lim_{n\to\infty}\frac{1+(-3)^n}{2^n-(-3)^n}=\lim_{n\to\infty}\frac{\left(-\frac{1}{3}\right)^n+1}{\left(-\frac{2}{3}\right)^n-1}=-1$

$\displaystyle\lim_{n\to\infty}\frac{4^n+(-8)^n}{(-6)^n+5^n}=\lim_{n\to\infty}\frac{\left(-\frac{2}{3}\right)^n+\left(\frac{4}{3}\right)^n}{1+\left(-\frac{5}{6}\right)^n}=\infty$

$\displaystyle\lim_{n\to\infty}\frac{9^{n+1}+3^n}{3^{2n}-(-5)^n}=\lim_{n\to\infty}\frac{9^{n+1}+3^n}{9^n-(-5)^n}=\lim_{n\to\infty}\frac{9+\left(\frac{1}{3}\right)^n}{1-\left(-\frac{5}{9}\right)^n}=9$

$\displaystyle\lim_{n\to\infty}\frac{7+4^{2n}}{6^n-2}=\lim_{n\to\infty}\frac{7+16^n}{6^n-2}=\lim_{n\to\infty}\frac{7\left(\frac{1}{6}\right)^n+\left(\frac{8}{3}\right)^n}{1-2\left(\frac{1}{6}\right)^n}=\infty$

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Find the following limits.

Ex. $\lim_{n\to\infty}(3^n-2^n)=\lim_{n\to\infty}3^n\left[1-\left(\frac{2}{3}\right)^n\right]=\infty$ ← Taking out the term with the largest absolute value of the bas

(1) $\lim_{n\to\infty}(4^n-3^n)=\lim_{n\to\infty}4^n\left[1-\left(\frac{3}{4}\right)^n\right]=\infty$

(2) $\lim_{n\to\infty}(3^n-5^n)=\lim_{n\to\infty}5^n\left[\left(\frac{3}{5}\right)^n-1\right]=-\infty$

(3) $\lim_{n\to\infty}\{4^n+(-2)^n\}=\lim_{n\to\infty}4^n\left[1+\left(-\frac{1}{2}\right)^n\right]=\infty$

(4) $\lim_{n\to\infty}(2^{3n}-3^{2n})=\lim_{n\to\infty}(8^n-9^n)=\lim_{n\to\infty}9^n\left[\left(\frac{8}{9}\right)^n-1\right]=-\infty$

(5) $\lim_{n\to\infty}\frac{5^n-6^n}{2^{2n}}=\lim_{n\to\infty}\frac{6^n\left[\left(\frac{5}{6}\right)^n-1\right]}{4^n}=\lim_{n\to\infty}\left(\frac{3}{2}\right)^n\left[\left(\frac{5}{6}\right)^n-1\right]=-\infty$

[Alternative Solution $\lim_{n\to\infty}\frac{5^n-6^n}{2^{2n}}=\lim_{n\to\infty}\left[\left(\frac{5}{4}\right)^n-\left(\frac{3}{2}\right)^n\right]=\lim_{n\to\infty}\left(\frac{3}{2}\right)^n\left[\left(\frac{5}{6}\right)^n-1\right]=-\infty$

(6) $\lim_{n\to\infty}\frac{4^n-3^{n+1}}{2^n}=\lim_{n\to\infty}\frac{4^n\left[1-3\left(\frac{3}{4}\right)^n\right]}{2^n}=\lim_{n\to\infty}2^n\left[1-3\left(\frac{3}{4}\right)^n\right]=\infty$

[Alternative Solution
$\lim_{n\to\infty}\frac{4^n-3^{n+1}}{2^n}=\lim_{n\to\infty}\left[2^n-3\left(\frac{3}{2}\right)^n\right]=\lim_{n\to\infty}2^n\left[1-3\left(\frac{3}{4}\right)^n\right]=\infty$]

N73b

$$\lim_{n\to\infty}\frac{2\cdot 2^2\cdot\cdots\cdot 2^n}{3^{[illegible]}}=\lim_{n\to\infty}\frac{2^{[illegible]}}{3^{[illegible]}}$$

[illegible]

$$=\lim_{n\to\infty}\frac{2^{\frac{1}{2}[illegible]}}{9^{\frac{1}{2}[illegible]}}$$

$$=\lim_{n\to\infty}\left(\frac{2}{9}\right)^{\frac{1}{2}[illegible]}$$

$$=0$$

$$\lim_{n\to\infty}\frac{2\cdot 2^2\cdot\cdots\cdot 2^n}{8^{[illegible]}}=\lim_{n\to\infty}\frac{2^{[illegible]}}{8^{[illegible]}}$$

$$=\lim_{n\to\infty}\frac{2^{\frac{1}{2}[illegible]}}{64^{\frac{1}{2}[illegible]}}$$

$$=\lim_{n\to\infty}\left(\frac{1}{32}\right)^{\frac{1}{2}[illegible]}$$

$$=0$$

$$\lim_{n\to\infty}\frac{2^{[illegible]}}{16\cdot 16^2\cdot\cdots\cdot 16^n}=\lim_{n\to\infty}\frac{2^{[illegible]}}{16^{1+2+\cdots+n}}$$

$$=\lim_{n\to\infty}\frac{16^{\frac{1}{2}[illegible]}}{16^{\frac{1}{2}[illegible]}}$$

$$=\lim_{n\to\infty}1$$

$$=1$$

Alternative Solution

$$\lim_{n\to\infty}\frac{2^{[illegible]}}{16\cdot 16^2\cdot\cdots\cdot 16^n}=\lim_{n\to\infty}\frac{2^{[illegible]}}{16^{[illegible]}}=\lim_{n\to\infty}\frac{2^{[illegible]}}{16^{\frac{1}{2}[illegible]}}=\lim_{n\to\infty}\frac{2^{[illegible]}}{2^{[illegible]}}=\lim_{n\to\infty}1=1$$

Infinite Geometric Sequences

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Find the following limits.

Ex.

$$\lim_{n\to\infty}\frac{r^n}{1+r^n} \quad (r\neq -1)$$

[Sol] (i) When $|r|<1$,

$$\lim_{n\to\infty}\frac{r^n}{1+r^n}=\frac{0}{1+0}=0 \quad \leftarrow \boxed{\lim_{n\to\infty} r^n=0}$$

(ii) When $r=1$,

$$\lim_{n\to\infty}\frac{r^n}{1+r^n}=\frac{1}{1+1}=\frac{1}{2} \quad \leftarrow \boxed{\lim_{n\to\infty} r^n=1}$$

(iii) When $|r|>1$,

$$\lim_{n\to\infty}\frac{r^n}{1+r^n}=\lim_{n\to\infty}\frac{1}{\left(\frac{1}{r}\right)^n+1}=\frac{1}{0+1}=1 \quad \leftarrow \boxed{\lim_{n\to\infty}\left(\frac{1}{r}\right)^n=}$$

(1) $\lim_{n\to\infty}\frac{1-r^n}{1+r^n} \quad (r\neq -1)$

[Sol] (i) When $|r|<1$,

$$\lim_{n\to\infty}\frac{1-r^n}{1+r^n}=\frac{1-0}{1+0}=1$$

(ii) When $r=1$,

$$\lim_{n\to\infty}\frac{1-r^n}{1+r^n}=\frac{1-1}{1+1}=0$$

(iii) When $|r|>1$,

$$\lim_{n\to\infty}\frac{1-r^n}{1+r^n}=\lim_{n\to\infty}\frac{\left(\frac{1}{r}\right)^n-1}{\left(\frac{1}{r}\right)^n+1}=\frac{0-1}{0+1}=-1$$

In **Ex.** above, when $r=-1$, the denominator becomes 0 if n is an odd number. Th[e]
the condition $r\neq -1$ is added.

(2) $\lim_{n\to\infty}\frac{r^{n+1}-1}{r^n+1}$ $(r\neq -1)$

[Sol] (i) **When $|r|<1$,**

$$\lim_{n\to\infty}\frac{r^{n+1}-1}{r^n+1}=\frac{0-1}{0+1}=-1$$

(ii) **When $r=1$,**

$$\lim_{n\to\infty}\frac{r^{n+1}-1}{r^n+1}=\frac{1-1}{1+1}=0$$

(iii) **When $|r|>1$,**

$$\lim_{n\to\infty}\frac{r^{n+1}-1}{r^n+1}=\lim_{n\to\infty}\frac{r-\left(\frac{1}{r}\right)^n}{1+\left(\frac{1}{r}\right)^n}=\frac{r-0}{1+0}=r$$

(3) $\lim_{n\to\infty}\frac{r^{2n+1}-1}{r^{2n}+1}$

(Solving for four different cases: $|r|<1$, $r=1$, $r=-1$ and $|r|>1$)

[Sol] (i) **When $|r|<1$,**

$$\lim_{n\to\infty}\frac{r^{2n+1}-1}{r^{2n}+1}=\frac{0-1}{0+1}=-1$$

(ii) **When $r=1$,**

$$\lim_{n\to\infty}\frac{r^{2n+1}-1}{r^{2n}+1}=\frac{1-1}{1+1}=0$$

(iii) **When $r=-1$,**

$$\lim_{n\to\infty}\frac{r^{2n+1}-1}{r^{2n}+1}=\frac{-1-1}{1+1}=-1 \leftarrow$$

When m is a natural number,
$(-1)^{2m}=1$
$(-1)^{2m+1}=-1$

(iv) **When $|r|>1$,**

$$\lim_{n\to\infty}\frac{r^{2n+1}-1}{r^{2n}+1}=\lim_{n\to\infty}\frac{r-\left(\frac{1}{r}\right)^{2n}}{1+\left(\frac{1}{r}\right)^{2n}}=\frac{r-0}{1+0}=r$$

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Find the following limits.

Ex. $\displaystyle\lim_{n\to\infty}\frac{r^n+2^n}{r^{n+1}+4^n}\quad(r\neq-4)$

[Sol] (i) When $|r|<4$,

$$\lim_{n\to\infty}\frac{r^n+2^n}{r^{n+1}+4^n}=\lim_{n\to\infty}\frac{\left(\frac{r}{4}\right)^n+\left(\frac{1}{2}\right)^n}{r\left(\frac{r}{4}\right)^n+1}=\frac{0+0}{r\cdot0+1}=0$$

(ii) When $r=4$,

$$\lim_{n\to\infty}\frac{r^n+2^n}{r^{n+1}+4^n}=\lim_{n\to\infty}\frac{4^n+2^n}{4^{n+1}+4^n}=\lim_{n\to\infty}\frac{4^n\left[1+\left(\frac{1}{2}\right)^n\right]}{5\cdot4^n}=\frac{1+0}{5}=$$

(iii) When $|r|>4$,

$$\lim_{n\to\infty}\frac{r^n+2^n}{r^{n+1}+4^n}=\lim_{n\to\infty}\frac{1+\left(\frac{2}{r}\right)^n}{r+\left(\frac{4}{r}\right)^n}=\frac{1+0}{r+0}=\frac{1}{r}$$

(1) $\displaystyle\lim_{n\to\infty}\frac{r^{n+1}+3^n}{r^n+6^n}\quad(r\neq-6)$

[Sol] (i) When $|r|<6$,

$$\lim_{n\to\infty}\frac{r^{n+1}+3^n}{r^n+6^n}=\lim_{n\to\infty}\frac{r\left(\frac{r}{6}\right)^n+\left(\frac{1}{2}\right)^n}{\left(\frac{r}{6}\right)^n+1}=\frac{r\cdot0+0}{0+1}=0$$

(ii) When $r=6$,

$$\lim_{n\to\infty}\frac{r^{n+1}+3^n}{r^n+6^n}=\lim_{n\to\infty}\frac{6^{n+1}+3^n}{6^n+6^n}=\lim_{n\to\infty}\frac{6^n\left[6+\left(\frac{1}{2}\right)^n\right]}{2\cdot6^n}=\frac{6+0}{2}=$$

(iii) When $|r|>6$,

$$\lim_{n\to\infty}\frac{r^{n+1}+3^n}{r^n+6^n}=\lim_{n\to\infty}\frac{r+\left(\frac{3}{r}\right)^n}{1+\left(\frac{6}{r}\right)^n}=\frac{r+0}{1+0}=r$$

N75b

(2) $\lim_{n\to\infty}\dfrac{r^{2n+1}-5^{2n}}{r^{2n}+5^{2n}}$

[Sol] (i) **When $|r|<5$,**

$$\lim_{n\to\infty}\frac{r^{2n+1}-5^{2n}}{r^{2n}+5^{2n}}=\lim_{n\to\infty}\frac{r\left(\dfrac{r}{5}\right)^{2n}-1}{\left(\dfrac{r}{5}\right)^{2n}+1}=\frac{r\cdot 0-1}{0+1}=-1$$

(ii) **When $r=5$,**

$$\lim_{n\to\infty}\frac{r^{2n+1}-5^{2n}}{r^{2n}+5^{2n}}=\lim_{n\to\infty}\frac{5^{2n+1}-5^{2n}}{5^{2n}+5^{2n}}=\lim_{n\to\infty}\frac{4\cdot 5^{2n}}{2\cdot 5^{2n}}=2$$

(iii) **When $r=-5$,**

$$\lim_{n\to\infty}\frac{r^{2n+1}-5^{2n}}{r^{2n}+5^{2n}}=\lim_{n\to\infty}\frac{(-5)^{2n+1}-5^{2n}}{(-5)^{2n}+5^{2n}}=\lim_{n\to\infty}\frac{-6\cdot 5^{2n}}{2\cdot 5^{2n}}=-3$$

$\boxed{(-5)^{2n}=5^{2n}}$

(iv) **When $|r|>5$,**

$$\lim_{n\to\infty}\frac{r^{2n+1}-5^{2n}}{r^{2n}+5^{2n}}=\lim_{n\to\infty}\frac{r-\left(\dfrac{5}{r}\right)^{2n}}{1+\left(\dfrac{5}{r}\right)^{2n}}=\frac{r-0}{1+0}=r$$

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EX. Find the range of values of x for which the sequence $\{(x-1)^n\}$ converges. Then, state the limit values.

[Sol] Since the common ratio is $x-1$, $-1 < x-1 \leqslant 1$

$\therefore\ 0 < x \leqslant 2$

The sequence $\{r^n\}$ converges when $-1 < r \leqslant 1$.

Also, the limit values are,

when $0 < x < 2$, $\lim\limits_{n\to\infty}(x-1)^n = 0$ and

When $|r| < 1$, $\lim\limits_{n\to\infty} r^n = 0$

when $x = 2$, $\lim\limits_{n\to\infty}(x-1)^n = 1$

When $r = 1$, $\lim\limits_{n\to\infty} r^n = 1$

1. Find the range of values of x for which the sequence $\{(x+2)^n\}$ converg
Then, state the limit values.

[Sol] Since the common ratio is $x+2$, $-1 < x+2 \leqslant 1$

$\therefore\ -3 < x \leqslant -1$

Also, the limit values are,

when $-3 < x < -1$, $\lim\limits_{n\to\infty}(x+2)^n = 0$ and

when $x = -1$, $\lim\limits_{n\to\infty}(x+2)^n = 1$

2. Find the range of values of x for which the sequence $\left\{\left(\dfrac{x+1}{3}\right)^n\right\}$ conver

Then, state the limit values.

[Sol] Since the common ratio is $\dfrac{x+1}{3}$, $-1 < \dfrac{x+1}{3} \leqslant 1$

$\therefore\ -4 < x \leqslant 2$

Also, the limit values are,

when $-4 < x < 2$, $\lim\limits_{n\to\infty}\left(\dfrac{x+1}{3}\right)^n = 0$ and

when $x = 2$, $\lim\limits_{n\to\infty}\left(\dfrac{x+1}{3}\right)^n = 1$

N76b

3. Find the range of values of x for which the sequence $\{(x^2-x-1)^n\}$ converges. Then, state the limit values.

[Sol] Since the common ratio is x^2-x-1, $-1<x^2-x-1\leqq 1$

So, $-1<x^2-x-1$ ···① and also $x^2-x-1\leqq 1$ ···②

From ①, $x^2-x>\boxed{0}$; therefore, $x(x-1)>\boxed{0}$

$\therefore x<\boxed{0}$, $\boxed{1}<x$ ···③

From ②, $x^2-x-2\leqq\boxed{0}$; therefore, $(x+1)(x-2)\leqq\boxed{0}$

$\therefore \boxed{-1}\leqq x\leqq\boxed{2}$ ···④

From ③ and ④, the range of values of x is

$\boxed{-1}\leqq x<\boxed{0}$, $\boxed{1}<x\leqq\boxed{2}$

Also, the limit values are,

when $-1<x<0$, $1<x<2$, $\lim_{n\to\infty}(x^2-x-1)^n=0$ and

when $x=-1$, 2, $\lim_{n\to\infty}(x^2-x-1)^n=1$

4. Find the range of values of x for which the sequence $\{(x^2-3x+1)^n\}$ converges. Then, state the limit values.

[Sol] Since the common ratio is x^2-3x+1, $-1<x^2-3x+1\leqq 1$

So, $-1<x^2-3x+1$ ···① and also $x^2-3x+1\leqq 1$ ···②

From ①, $x^2-3x+2>0$; therefore, $(x-1)(x-2)>0$

$\therefore x<1$, $2<x$ ···③

From ②, $x^2-3x\leqq 0$; therefore, $x(x-3)\leqq 0$

$\therefore 0\leqq x\leqq 3$ ···④

From ③ and ④, the range of values of x is

$0\leqq x<1$, $2<x\leqq 3$

Also, the limit values are,

when $0<x<1$, $2<x<3$, $\lim_{n\to\infty}(x^2-3x+1)^n=0$ and

when $x=0$, 3, $\lim_{n\to\infty}(x^2-3x+1)^n=1$

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1. Find the range of values of x for which the sequence $\left\{\left(\dfrac{x^2+2x-5}{x^2-x+2}\right)^n\right\}$ converges. Then, state the limit values.

[Sol] Since the common ratio is $\dfrac{x^2+2x-5}{x^2-x+2}$, $-1<\dfrac{x^2+2x-5}{x^2-x+2}\leqq 1$

Since $x^2-x+2>0$, rearranging the inequality, ← $x^2-x+2=\left(x-\dfrac{1}{2}\right)^2+\dfrac{7}{4}>$

$$-(x^2-x+2)<x^2+2x-5\leqq x^2-x+2$$

So,

$-(x^2-x+2)<x^2+2x-5$ …① and also $x^2+2x-5\leqq x^2-x+2$ …②

From ①, $2x^2+x-3>0$; therefore, $(2x+3)(x-1)>0$

$$\therefore\ x<-\frac{3}{2},\ 1<x \quad \cdots③$$

From ②, $3x\leqq 7$

$$\therefore\ x\leqq\frac{7}{3} \quad \cdots④$$

From ③ and ④, the range of values of x is

$$x<-\frac{3}{2},\ 1<x\leqq\frac{7}{3}$$

Also, the limit values are,

when $x<-\dfrac{3}{2}$, $1<x<\dfrac{7}{3}$, $\displaystyle\lim_{n\to\infty}\left(\frac{x^2+2x-5}{x^2-x+2}\right)^n=0$ and

when $x=\dfrac{7}{3}$, $\displaystyle\lim_{n\to\infty}\left(\frac{x^2+2x-5}{x^2-x+2}\right)^n=1$

177b

Given $a_n=(4\sin^2\theta+2\cos\theta-3)^n$, find the range of values of θ for which the sequence $\{a_n\}$ converges. $(0\leq\theta\leq\pi)$

ol] Since the common ratio is $4\sin^2\theta+2\cos\theta-3$, $-1<4\sin^2\theta+2\cos\theta-3\leq 1$

$$4\sin^2\theta+2\cos\theta-3=4(1-\cos^2\theta)+2\cos\theta-3 \leftarrow \boxed{\sin^2\theta+\cos^2\theta=1}$$
$$=-4\cos^2\theta+2\cos\theta+1$$

Therefore, rearranging the inequality,

$$-1<-4\cos^2\theta+2\cos\theta+1\leq 1$$

So,

$-1<-4\cos^2\theta+2\cos\theta+1$ ···① and also $-4\cos^2\theta+2\cos\theta+1\leq 1$ ···②

From ①, $2\cos^2\theta-\cos\theta-1<0$; therefore, $(2\cos\theta+1)(\cos\theta-1)<0$

$$\therefore\ -\frac{1}{2}<\cos\theta<1 \quad \cdots③$$

From ②, $2\cos^2\theta-\cos\theta\geq 0$; therefore, $\cos\theta(2\cos\theta-1)\geq 0$

$$\therefore\ \cos\theta\leq 0,\ \frac{1}{2}\leq\cos\theta \quad \cdots④$$

From ③ and ④, $-\frac{1}{2}<\cos\theta\leq 0,\ \frac{1}{2}\leq\cos\theta<1$

Since $0\leq\theta\leq\pi$,

when $\cos\theta=-\frac{1}{2}$, $\theta=\frac{2}{3}\pi$

when $\cos\theta=0$, $\theta=\frac{\pi}{2}$

when $\cos\theta=\frac{1}{2}$, $\theta=\frac{\pi}{3}$

when $\cos\theta=1$, $\theta=0$

Therefore, the range of values of θ is

$$0<\theta\leq\frac{\pi}{3},\ \frac{\pi}{2}\leq\theta<\frac{2}{3}\pi \leftarrow$$

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Find the limits of each given sequence $\{a_n\}$ defined by the following condition

Ex. $a_1=1,\ a_{n+1}=\frac{2}{3}a_n-1 \quad (n=1,\ 2,\ 3,\ \ldots)$

[Sol] Since $a_{n+1}=\frac{2}{3}a_n-1,\ a_{n+1}+3=\frac{2}{3}(a_n+3)$

Replacing both a_{n+1} and a_n with x, $x=\frac{2}{3}x-1$ $\therefore\ x=-3$ (N44)

Let $b_n=a_n+3$.

$$\begin{cases} b_{n+1}=\frac{2}{3}b_n & \cdots① \\ b_1=a_1+3=4 & \cdots② \end{cases}$$

$\{b_n\}$ is the geometric sequence with 1st term 4 and common ratio $\frac{2}{3}$

From ① and ②, $b_n=4\left(\frac{2}{3}\right)^{n-1}$

$\therefore\ a_n=b_n-3=4\left(\frac{2}{3}\right)^{n-1}-3$

$\therefore\ \lim\limits_{n\to\infty}a_n=\lim\limits_{n\to\infty}\left[4\left(\frac{2}{3}\right)^{n-1}-3\right]=-3$

(1) $a_1=3,\ a_{n+1}=\frac{1}{2}a_n+3 \quad (n=1,\ 2,\ 3,\ \ldots)$

[Sol] Since $a_{n+1}=\frac{1}{2}a_n+3,\ a_{n+1}-6=\frac{1}{2}(a_n-6)$

Replacing both a_{n+1} and a_n with x, $x=\frac{1}{2}x$ ... $\therefore\ x=6$

Let $b_n=a_n-6$.

$$\begin{cases} b_{n+1}=\frac{1}{2}b_n & \cdots① \\ b_1=a_1-6=-3 & \cdots② \end{cases}$$

From ① and ②, $b_n=-3\left(\frac{1}{2}\right)^{n-1}$

$\therefore\ a_n=b_n+6=-3\left(\frac{1}{2}\right)^{n-1}+6$

$\therefore\ \lim\limits_{n\to\infty}a_n=\lim\limits_{n\to\infty}\left[-3\left(\frac{1}{2}\right)^{n-1}+6\right]=6$

78b

$a_1=2$, $a_{n+1}=\dfrac{a_n}{a_n+3}$ $(n=1,2,3,\dots)$

Since $a_{n+1}=\dfrac{a_n}{a_n+3}$, $\dfrac{1}{a_{n+1}}=\dfrac{\boxed{a_n}+3}{\boxed{a_n}}=\dfrac{3}{\boxed{a_n}}+\boxed{1}$

Let $b_n=\dfrac{1}{a_n}$.

Since $b_{n+1}=3b_n+1$, $b_{n+1}+\dfrac{1}{2}=3\left(b_n+\dfrac{1}{2}\right)$ ←

Let $c_n=b_n+\dfrac{1}{2}$.

$$\begin{cases} c_{n+1}=3c_n & \text{—①} \\ c_1=b_1+\dfrac{1}{2}=\dfrac{1}{a_1}+\dfrac{1}{2}=1 & \text{—②} \end{cases}$$

From ① and ②, $c_n=3^{n-1}$

$\therefore\ b_n=c_n-\dfrac{1}{2}=3^{n-1}-\dfrac{1}{2}$

$\therefore\ a_n=\dfrac{1}{b_n}=\dfrac{1}{3^{n-1}-\dfrac{1}{2}}=\dfrac{2}{2\cdot 3^{n-1}-1}$

$\therefore\ \lim\limits_{n\to\infty}a_n=\lim\limits_{n\to\infty}\dfrac{2}{2\cdot 3^{n-1}-1}=0$

N79a

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Infinite Geometric Sequences

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1. Find the range of values of θ which satisfies $\lim_{n\to\infty}\dfrac{\tan^n\theta+2}{2\tan^n\theta+2}=\dfrac{1}{2}$.

$\left(0 \leqq \theta < \dfrac{\pi}{2}\right)$

[Sol] (i) When $0 \leqq \tan\theta < 1$, i.e. $0 \leqq \theta < \dfrac{\pi}{4}$,

$$\lim_{n\to\infty}\tan^n\theta=0$$

$$\therefore \lim_{n\to\infty}\frac{\tan^n\theta+2}{2\tan^n\theta+2}=\frac{0+2}{2\cdot 0+2}=1$$

(ii) When $\tan\theta=1$, i.e. $\theta=\dfrac{\pi}{4}$,

$$\lim_{n\to\infty}\tan^n\theta=1$$

$$\therefore \lim_{n\to\infty}\frac{\tan^n\theta+2}{2\tan^n\theta+2}=\frac{1+2}{2\cdot 1+2}=\frac{3}{4}$$

(iii) When $\tan\theta > 1$, i.e. $\dfrac{\pi}{4} < \theta < \dfrac{\pi}{2}$,

$$\lim_{n\to\infty}\frac{1}{\tan^n\theta}=0$$

$$\therefore \lim_{n\to\infty}\frac{\tan^n\theta+2}{2\tan^n\theta+2}=\lim_{n\to\infty}\frac{1+\dfrac{2}{\tan^n\theta}}{2+\dfrac{2}{\tan^n\theta}}=\frac{1+2\cdot 0}{2+2\cdot 0}=\frac{1}{2}$$

From (i) ~ (iii), the range of values of θ is

$$\frac{\pi}{4} < \theta < \frac{\pi}{2}$$

N79b

Given the sequence $\{a_n\}$ which is defined by $a_1=0$, $a_2=1$ and $a_{n+2}=\frac{1}{4}(a_{n+1}+3a_n)$ $(n=1, 2, 3, \dots)$, solve the following questions.

(1) Let $b_n=a_{n+1}-a_n$ $(n=1, 2, 3, \dots)$. Express b_n in terms of n.

[Sol] Since $a_{n+2}=\frac{1}{4}(a_{n+1}+3a_n)$,

$$a_{n+2}-\frac{1}{4}a_{n+1}-\frac{3}{4}a_n=0 \quad \cdots①$$

Rearranging ①,

$$a_{n+2}-a_{n+1}=-\frac{3}{4}(a_{n+1}-a_n)$$

Replacing a_{n+2}, a_{n+1}, a_n with x^2, x, 1 respectively,
$x^2-\frac{1}{4}x-\frac{3}{4}=0$
$4x^2-x-3=0$
$(4x+3)(x-1)=0$
$\therefore x=-\frac{3}{4}, 1$
(N47)

Let $b_n=a_{n+1}-a_n$.

$$\begin{cases} b_{n+1}=-\dfrac{3}{4}b_n & \cdots② \\ b_1=a_2-a_1=1 & \cdots③ \end{cases}$$

$\{b_n\}$ is the geometric sequence with 1st term $a_2-a_1=1$ and common ratio $-\frac{3}{4}$.

From ② and ③, $b_n=\left(-\frac{3}{4}\right)^{n-1}$

(2) Express a_n in terms of n.

[Sol] From (1), since the general term of the sequence of differences of $\{a_n\}$ is $\left(-\frac{3}{4}\right)^{n-1}$,

when $n\geqq 2$,

When $n\geqq 2$, $a_n=a_1+\sum_{k=1}^{n-1}b_k$ (N32)

$$a_n=a_1+\sum_{k=1}^{n-1}\left(-\frac{3}{4}\right)^{k-1}=0+\frac{1-\left(-\frac{3}{4}\right)^{n-1}}{1-\left(-\frac{3}{4}\right)}$$

$$=\frac{4}{7}\left[1-\left(-\frac{3}{4}\right)^{n-1}\right] \quad \cdots④$$

Since $a_1=0$, ④ is also true when $n=1$.

$$\therefore a_n=\frac{4}{7}\left[1-\left(-\frac{3}{4}\right)^{n-1}\right]$$

(3) Find the limit value $\lim_{n\to\infty}a_n$.

[Sol] From (2), $\lim_{n\to\infty}a_n=\lim_{n\to\infty}\frac{4}{7}\left[1-\left(-\frac{3}{4}\right)^{n-1}\right]=\frac{4}{7}$

N80a

KUMON

Infinite Geometric Sequences

Name

Date / /

Time : to :

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1. Find the following limits.

(1) $\lim_{n\to\infty}\frac{3^n+2^n}{2^{2n}-3^n}=\lim_{n\to\infty}\frac{3^n+2^n}{4^n-3^n}=\lim_{n\to\infty}\frac{\left(\frac{3}{4}\right)^n+\left(\frac{1}{2}\right)^n}{1-\left(\frac{3}{4}\right)^n}=0$

➡ N72

(2) $\lim_{n\to\infty}\frac{4^n+2^n}{3^n}=\lim_{n\to\infty}\frac{4^n\left[1+\left(\frac{1}{2}\right)^n\right]}{3^n}=\lim_{n\to\infty}\left(\frac{4}{3}\right)^n\left[1+\left(\frac{1}{2}\right)^n\right]=\infty$

➡ N7

[Alternative Solution

$\lim_{n\to\infty}\frac{4^n+2^n}{3^n}=\lim_{n\to\infty}\left[\left(\frac{4}{3}\right)^n+\left(\frac{2}{3}\right)^n\right]=\lim_{n\to\infty}\left(\frac{4}{3}\right)^n\left[1+\left(\frac{1}{2}\right)^n\right]=\infty$]

(3) $\lim_{n\to\infty}\frac{r^{2n-1}}{r^{2n}+1}$

➡ N

[Sol] (i) When $|r|<1$,

$$\lim_{n\to\infty}\frac{r^{2n-1}}{r^{2n}+1}=\frac{0}{0+1}=0$$

(ii) When $r=1$,

$$\lim_{n\to\infty}\frac{r^{2n-1}}{r^{2n}+1}=\frac{1}{1+1}=\frac{1}{2}$$

(iii) When $r=-1$,

$$\lim_{n\to\infty}\frac{r^{2n-1}}{r^{2n}+1}=\frac{-1}{1+1}=-\frac{1}{2}$$

(iv) When $|r|>1$,

$$\lim_{n\to\infty}\frac{r^{2n-1}}{r^{2n}+1}=\lim_{n\to\infty}\frac{\frac{1}{r}}{1+\left(\frac{1}{r}\right)^{2n}}=\frac{\frac{1}{r}}{1+0}=\frac{1}{r}$$

Find the range of values of x for which the sequence $\left\{\left(\frac{x+3}{4}\right)^n\right\}$ converges. Then, state the limit values.

➡ N76

[Sol] Since the common ratio is $\frac{x+3}{4}$, $-1<\frac{x+3}{4}\leqq 1$

$\therefore\ -7<x\leqq 1$

Also, the limit values are,

when $-7<x<1$, $\lim_{n\to\infty}\left(\frac{x+3}{4}\right)^n=0$ and

when $x=1$, $\lim_{n\to\infty}\left(\frac{x+3}{4}\right)^n=1$

Find the limit of the sequence $\{a_n\}$ defined by the following conditions.

➡ N78

$a_1=1,\ a_{n+1}=3-\frac{1}{2}a_n\quad(n=1,\ 2,\ 3,\ \dots)$

[Sol] Since $a_{n+1}=3-\frac{1}{2}a_n$, $a_{n+1}-2=-\frac{1}{2}(a_n-2)$

Let $b_n=a_n-2$

$$\begin{cases} b_{n+1}=-\frac{1}{2}b_n & \cdots① \\ b_1=a_1-2=-1 & \cdots② \end{cases}$$

From ① and ②, $b_n=-\left(-\frac{1}{2}\right)^{n-1}$

$\therefore\ a_n=b_n+2=-\left(-\frac{1}{2}\right)^{n-1}+2$

$\therefore\ \lim_{n\to\infty}a_n=\lim_{n\to\infty}\left[-\left(-\frac{1}{2}\right)^{n-1}+2\right]=2$

N81a

KUMON

Infinite Geometric Series

Name

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Time : to :

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Given an infinite sequence $\{a_n\}$, the expression $a_1+a_2+a_3+\cdots+a_n+\cdots$ …(
is called the ***infinite series***, where a_1 and a_n are called the 1st ***term*** and the n^{th} ***term*** respectively.

Also, given an infinite sequence $\{a_n\}$, let S_n be the sum of the first n terms.
When the infinite sequence $\{S_n\}$ converges, the infinite series ① is said to conver
When the infinite sequence $\{S_n\}$ diverges, the infinite series ① is said to diver
Likewise, $a+ar+ar^2+\cdots+ar^{n-1}+\cdots$ which is the infinite series that is deriv
from the infinite geometric sequence with 1st term a and common ratio r is call
the ***infinite geometric series*** with 1st term a and common ratio r.

Determine whether the following infinite geometric series converges or diverg

$a+ar+ar^2+\cdots+ar^{n-1}+\cdots$

[Sol] When $a=0$, all terms become 0; therefore, the infinite geometric se converges to 0.

When $a\neq 0$, let S_n be the sum of the first n terms.

When $r\neq 1$, $S_n=\dfrac{a(1-r^n)}{1-r}$ …①

When $r=1$, $S_n=na$

(i) When $|r|<1$, since $\lim\limits_{n\to\infty} r^n=0$, $\lim\limits_{n\to\infty} S_n=\lim\limits_{n\to\infty}\dfrac{a(1-r^n)}{1-r}=\boxed{\dfrac{a}{1-r}}$

Therefore, the infinite geometric series converges and the sum is $\boxed{\dfrac{a}{1-r}}$.

(ii) When $r=1$,
since $S_n=na$ and $a\neq 0$, the sequence $\{S_n\}$ diverges.
Therefore, the infinite geometric series diverges.

(iii) When $r\leqq -1$ and $r>1$,
since the sequence $\{r^n\}$ diverges,
from ①, the sequence $\{S_n\}$ also $\boxed{\text{diverges}}$.
Therefore, the infinite geometric series $\boxed{\text{diverges}}$.

When $r <$
$\{r^n\}$ oscilla
(no limit).
When $r >$
$\lim r^n =$ ∞
Therefore
either case
diverges. (

$\dfrac{a}{1-r}$, $\dfrac{a}{1-r}$, diverges, diverges

N81b

Convergence and Divergence of an Infinite Geometric Series

Given an infinite geometric series $a+ar+ar^2+\cdots+ar^{n-1}+\cdots$, the following is true.

When $a\neq 0$,

if $|r|<1$, then the series converges and the sum is $\frac{a}{1-r}$,

if $|r|\geqslant 1$, then the series diverges.

When $a=0$, the series converges and the sum is 0.

Determine whether each given infinite geometric series converges or diverges. If it converges, find the sum S.

Ex. $2+1+\frac{1}{2}+\frac{1}{4}+\cdots$

[Sol] The 1st term is 2 and the common ratio is $\frac{1}{2}$. Since $\left|\frac{1}{2}\right|<1$, the series converges.

$$\therefore\ S=\frac{2}{1-\frac{1}{2}}=4$$

(1) $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots$

[Sol] The 1st term is 1 and the common ratio is $\frac{1}{3}$. Since $\left|\frac{1}{3}\right|<1$, the series converges.

$$\therefore\ S=\frac{1}{1-\frac{1}{3}}=\frac{3}{2}$$

(2) $3-\frac{3}{10}+\frac{3}{100}-\frac{3}{1000}+\cdots$

[Sol] The 1st term is 3 and the common ratio is $-\frac{1}{10}$. Since $\left|-\frac{1}{10}\right|<1$, the series converges.

$$\therefore\ S=\frac{3}{1-\left(-\frac{1}{10}\right)}=\frac{30}{11}$$

(3) $-1+2-4+8-\cdots$

[Sol] The 1st term is -1 and the common ratio is -2. Since $|-2|>1$, the series diverges.

N82a

KUMON

Infinite Geometric Series

Name

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Determine whether each given infinite geometric series converges or diverges. If it converges, find the sum S.

(1) $1-\frac{\sqrt{2}}{2}+\frac{1}{2}-\frac{\sqrt{2}}{4}+\cdots$

[Sol] The 1st term is 1 and the common ratio is $-\frac{\sqrt{2}}{2}$. Since $\left|-\frac{\sqrt{2}}{2}\right|<1$, the series **converges.**

$$\therefore S=\frac{1}{1-\left(-\frac{\sqrt{2}}{2}\right)}$$
$$=\frac{2}{2+\sqrt{2}}$$
$$=\frac{2(2-\sqrt{2})}{(2+\sqrt{2})(2-\sqrt{2})}$$
$$=2-\sqrt{2}$$

Multiplying the numerator and the denominator by $2-\sqrt{2}$

(2) $6-2\sqrt{3}+2-\frac{2\sqrt{3}}{3}+\cdots$

[Sol] The 1st term is 6 and the common ratio is $\frac{-2\sqrt{3}}{6}=-\frac{\sqrt{3}}{3}$.
Since $\left|-\frac{\sqrt{3}}{3}\right|<1$, the series **converges.**

$$\therefore S=\frac{6}{1-\left(-\frac{\sqrt{3}}{3}\right)}$$
$$=\frac{18}{3+\sqrt{3}}$$
$$=\frac{18(3-\sqrt{3})}{(3+\sqrt{3})(3-\sqrt{3})}$$
$$=3(3-\sqrt{3})$$

32b

$\sqrt{2}+(2-\sqrt{2})+(3\sqrt{2}-4)+(10-7\sqrt{2})+\cdots$

[Sol] The 1st term is $\sqrt{2}$ and the common ratio is $\frac{2-\sqrt{2}}{\sqrt{2}}=\sqrt{2}-1$.
Since $|\sqrt{2}-1|<1$, the series converges.

$$\therefore S=\frac{\sqrt{2}}{1-(\sqrt{2}-1)}$$
$$=\frac{\sqrt{2}}{2-\sqrt{2}}$$
$$=\frac{\sqrt{2}(2+\sqrt{2})}{(2-\sqrt{2})(2+\sqrt{2})}$$
$$=\sqrt{2}+1$$

(4) $(\sqrt{3}-1)-2(2-\sqrt{3})+2(3\sqrt{3}-5)-4(7-4\sqrt{3})+\cdots$

[Sol] The 1st term is $\sqrt{3}-1$ and the common ratio is

$$\frac{-2(2-\sqrt{3})}{\sqrt{3}-1}=\frac{-2(2-\sqrt{3})(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)}=-\sqrt{3}+1.$$

Since $|-\sqrt{3}+1|<1$, the series converges.

$$\therefore S=\frac{\sqrt{3}-1}{1-(-\sqrt{3}+1)}$$
$$=\frac{\sqrt{3}-1}{\sqrt{3}}$$
$$=\frac{3-\sqrt{3}}{3}$$

N83a

KUMON

Infinite Geometric Series

Name

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Find the range of values of a real number x for which each given infinite geometric series converges. Then, find the sum S.

Ex. $1+2x+4x^2+\cdots$

[Sol] Since the common ratio is $2x$, $-1<2x<1$

$\therefore\ -\frac{1}{2}<x<\frac{1}{2}$

Also, $S=\frac{1}{1-2x}$ ← $\boxed{S=\frac{a}{1-r}}$

Let a be the 1st term and r be the common ratio. If $|r|<1$ when $a\neq 0$, then the infinite geometric series converges.

(1) $3+x+\frac{x^2}{3}+\cdots$

[Sol] Since the common ratio is $\frac{x}{3}$, $-1<\frac{x}{3}<1$

$\therefore\ -3<x<3$

Also, $S=\frac{3}{1-\frac{x}{3}}=\frac{9}{3-x}$

(2) $1+(2-x^2)+(2-x^2)^2+\cdots$

[Sol] Since the common ratio is $2-x^2$, $-1<2-x^2<1$

So, $-1<2-x^2$ …① and also $2-x^2<1$ …②

From ①, $x^2<3$; therefore, $-\sqrt{3}<x<\sqrt{3}$ …③

From ②, $x^2>1$; therefore, $x<-1,\ 1<x$ …④

From ③ and ④, the range of values of x is

$-\sqrt{3}<x<-1,\ 1<x<\sqrt{3}$

Also, $S=\frac{1}{1-(2-x^2)}=\frac{1}{x^2-1}$

3b

$x+x(1-x^2)+x(1-x^2)^2+\cdots$

(i) When $x\neq0$, since the common ratio is $1-x^2$, $-1<1-x^2<1$

So, $-1<1-x^2$ …① and also $1-x^2<1$ …②

From ①, $x^2<\boxed{2}$; therefore, $\boxed{-\sqrt{2}}<x<\boxed{\sqrt{2}}$ …③

From ②, $x^2>\boxed{0}$; therefore, $x<\boxed{0}$, $\boxed{0}<x$ …④

From ③ and ④, $\boxed{-\sqrt{2}}<x<\boxed{0}$, $\boxed{0}<x<\boxed{\sqrt{2}}$

Also, $S=\dfrac{\boxed{x}}{1-(1-x^2)}=\boxed{\dfrac{1}{x}}$

(ii) When $x=0$, all terms become 0; therefore, the series converges.

$\therefore\ S=\boxed{0}$

From (i) and (ii), the range of values of x is $\boxed{-\sqrt{2}}<x<\boxed{\sqrt{2}}$.

Also, when $\boxed{-\sqrt{2}}<x<\boxed{0}$, $\boxed{0}<x<\boxed{\sqrt{2}}$, $S=\boxed{\dfrac{1}{x}}$ and

when $x=\boxed{0}$, $S=\boxed{0}$

$x+x(1-x)^2+x(1-x)^4+\cdots$

(i) When $x\neq0$, since the common ratio is $(1-x)^2$, $-1<(1-x)^2<1$

So, $-1<(1-x)^2$ …① and also $(1-x)^2<1$ …②

From ①, $(1-x)^2+1>0$; therefore, it is true for all real numbers x.

From ②, $x(x-2)<0$; therefore, $0<x<2$ …③

From ① and ③, $0<x<2$

Also, $S=\dfrac{x}{1-(1-x)^2}=-\dfrac{1}{x-2}$

(ii) When $x=0$, all terms become 0; therefore, the series converges.

$\therefore\ S=0$

From (i) and (ii), the range of values of x is $0\leqq x<2$

Also, when $0<x<2$, $S=-\dfrac{1}{x-2}$ and

when $x=0$, $S=0$

N84a

KUMON

Infinite Geometric Series

Name

Date / /

Time : to :

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Ex. Given the infinite geometric series whose sum is 9 and 2^{nd} term is -4, find the 1^{st} term a and the common ratio r.

[Sol] Since the sum is 9, $a \neq 0$, $-1 < r < 1$ ← Since the sum is 9, the infinite geometric series converges.

$$\begin{cases} \dfrac{a}{1-r} = 9 & \cdots ① \\ ar = -4 & \cdots ② \end{cases}$$

From ① and ②,

From ①, $a = 9(1-r)$ Substituting this into ②

$$9(1-r) \cdot r = -4$$
$$9r^2 - 9r - 4 = 0$$
$$(3r+1)(3r-4) = 0$$
$$r = -\frac{1}{3}, \frac{4}{3}$$

Since $-1 < r < 1$, $r = -\frac{1}{3}$

From ②, $a = 12$

1. Given the infinite geometric series whose sum is $\frac{8}{3}$ and 2^{nd} term is -2, fin the 1^{st} term a and the common ratio r.

[Sol] Since the sum is $\frac{8}{3}$, $a \neq 0$, $-1 < r < 1$

$$\begin{cases} \dfrac{a}{1-r} = \dfrac{8}{3} & \cdots ① \\ ar = -2 & \cdots ② \end{cases}$$

From ① and ②,

$$\frac{8}{3}(1-r) \cdot r = -2$$
$$4r^2 - 4r - 3 = 0$$
$$(2r+1)(2r-3) = 0$$
$$r = -\frac{1}{2}, \frac{3}{2}$$

Since $-1 < r < 1$, $r = -\frac{1}{2}$

From ②, $a = 4$

2. Given the infinite geometric series whose sum is 3, squaring each term of the series gives another infinite geometric series whose sum is 6. Find the 1st term a and the common ratio r of the original infinite geometric series.

[Sol] Since the sum is 3, $a \neq 0$, $-1 < r < 1$

$$\begin{cases} \dfrac{a}{1-r} = 3 & \cdots① \\ \dfrac{a^2}{1-r^2} = 6 & \cdots② \end{cases}$$

1st term: a^2, common ratio: r^2, sum: 6

From ① and ②,

From ①, $a = 3(1-r)$
Substituting this into ②

$$\frac{[3(1-r)]^2}{1-r^2} = 6$$

$$\frac{9(1-r)^2}{(1+r)(1-r)} = 6$$

$$3(1-r) = 2(1+r)$$

$$\therefore\ r = \frac{1}{5}$$

From ①, $a = \dfrac{12}{5}$

Alternative Solution

Since the sum is 3, $a \neq 0$, $-1 < r < 1$

$$\begin{cases} \dfrac{a}{1-r} = 3 & \cdots① \\ \dfrac{a^2}{1-r^2} = 6 & \cdots② \end{cases}$$

From ①, $a = 3(1-r)$ $\cdots$③

From ②, $a^2 = 6(1-r^2)$ $\cdots$④

From ③ and ④,

$$[3(1-r)]^2 = 6(1-r^2)$$

$$5r^2 - 6r + 1 = 0$$

$$(5r-1)(r-1) = 0$$

$$r = \frac{1}{5},\ 1$$

Since $-1 < r < 1$, $r = \dfrac{1}{5}$

From ③, $a = \dfrac{12}{5}$

3. Given the infinite geometric series whose sum is 2, cubing each term of the series gives another infinite geometric series whose sum is $\dfrac{32}{13}$. Find the 1st term a and the common ratio r of the original infinite geometric series.

[Sol] Since the sum is 2, $a \neq 0$, $-1 < r < 1$

$$\begin{cases} \dfrac{a}{1-r} = 2 & \cdots① \\ \dfrac{a^3}{1-r^3} = \dfrac{32}{13} & \cdots② \end{cases}$$

$a^3 - b^3 = (a-b)(a^2+ab+b^2)$

From ① and ②,

$$\frac{[2(1-r)]^3}{1-r^3} = \frac{32}{13}$$

$$\frac{8(1-r)^3}{(1-r)(1+r+r^2)} = \frac{32}{13}$$

$$13(1-r)^2 = 4(1+r+r^2)$$

$$3r^2 - 10r + 3 = 0$$

$$(3r-1)(r-3) = 0$$

$$r = \frac{1}{3},\ 3$$

Since $-1 < r < 1$, $r = \dfrac{1}{3}$

From ①, $a = \dfrac{4}{3}$

Alternative Solution

Since the sum is 2, $a \neq 0$, $-1 < r < 1$

$$\begin{cases} \dfrac{a}{1-r} = 2 & \cdots① \\ \dfrac{a^3}{1-r^3} = \dfrac{32}{13} & \cdots② \end{cases}$$

From ①, $a = 2(1-r)$ $\cdots$③

From ②, $a^3 = \dfrac{32}{13}(1-r^3)$ $\cdots$④

From ③ and ④,

$$[2(1-r)]^3 = \frac{32}{13}(1-r^3)$$

$$3r^3 - 13r^2 + 13r - 3 = 0$$

$$(r-1)(3r^2 - 10r + 3) = 0$$

$$(r-1)(3r-1)(r-3) = 0$$

$$r = \frac{1}{3},\ 1,\ 3$$

Since $-1 < r < 1$, $r = \dfrac{1}{3}$

From ③, $a = \dfrac{4}{3}$

N85a

KUMON

Infinite Geometric Series

Name

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Time : to :

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A decimal which contains a digit or block of digits that repeats without end in its decimal part is called a ***repeating decimal***. The repeating decimal is expressed by placing "." over the first and the last repeating digits as shown below:

$0.33333\ldots=0.\dot{3}$, $0.454545\ldots=0.\dot{4}\dot{5}$, $0.123123123\ldots=0.\dot{1}2\dot{3}$

Also, the repeating decimal can be expressed by a fraction using the sum of a[n] infinite geometric series.

Convert the following repeating decimals into fractions.

Ex. $0.\dot{6}$

[Sol] $0.\dot{6}=0.6+0.06+0.006+\cdots$

$0.\dot{6}$ is the infinite geometric series with 1st term 0.6 and common ratio 0.1. Since $|0.1|<1$, it converges.

$$\therefore\ 0.\dot{6}=\frac{0.6}{1-0.1}=\frac{2}{3}$$

(1) $0.\dot{1}\dot{2}$

[Sol] $0.\dot{1}\dot{2}=0.12+0.0012+0.000012+\cdots$

$0.\dot{1}\dot{2}$ is the infinite geometric series with 1st term 0.12 and common ratio 0
. Since $|0.01|<1$, it converges.

$$\therefore\ 0.\dot{1}\dot{2}=\frac{0.12}{1-0.01}=\frac{4}{33}$$

(2) $0.\dot{3}0\dot{6}$

[Sol] $0.\dot{3}0\dot{6}=0.306+0.000306+0.000000306+\cdots$

$0.\dot{3}0\dot{6}$ is the infinite geometric series with 1st term 0.306 and common r
0.001. Since $|0.001|<1$, it converges.

$$\therefore\ 0.\dot{3}0\dot{6}=\frac{0.306}{1-0.001}=\frac{34}{111}$$

N85b

(3) $0.3\dot{2}\dot{4}$

[Sol] $0.3\dot{2}\dot{4}=0.3+0.024+0.00024+0.0000024+\cdots$

The 2nd and subsequent terms of the RHS is the infinite geometric series with 1st term 0.024 and common ratio 0.01.

Since $|0.01|<1$, it converges.

$$\therefore\ 0.3\dot{2}\dot{4}=\boxed{0.3}+\frac{\boxed{0.024}}{1-\boxed{0.01}}=\frac{3}{10}+\frac{4}{165}=\frac{107}{330}$$

(4) $3.5\dot{2}$

[Sol] $3.5\dot{2}=3.5+0.02+0.002+0.0002+\cdots$

The 2nd and subsequent terms of the RHS is the infinite geometric series with 1st term 0.02 and common ratio 0.1.

Since $|0.1|<1$, it converges.

$$\therefore\ 3.5\dot{2}=3.5+\frac{0.02}{1-0.1}=\frac{7}{2}+\frac{1}{45}=\frac{317}{90}$$

(5) $5.\dot{4}7\dot{7}$

[Sol] $5.\dot{4}7\dot{7}=5+0.477+0.000477+0.000000477+\cdots$

The 2nd and subsequent terms of the RHS is the infinite geometric series with 1st term 0.477 and common ratio 0.001.

Since $|0.001|<1$, it converges.

$$\therefore\ 5.\dot{4}7\dot{7}=5+\frac{0.477}{1-0.001}=5+\frac{53}{111}=\frac{608}{111}$$

N86a

KUMON

Infinite Geometric Series

Name

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Ex. On a number line, starting from origin O, point P moves 1 unit in the positive direction and then moves $\frac{1}{2}$ units in the negative direction. Then, it moves $\frac{1}{2^2}$ units in the positive direction and then moves $\frac{1}{2^3}$ units in the negative direction. When this motion continues infinitely, find the coordinate of the point which point P approaches.

[Sol] The coordinates of point P are as follows.

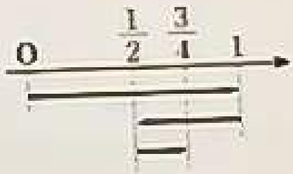
$$1$$

$$1-\frac{1}{2}$$

$$1-\frac{1}{2}+\frac{1}{2^2}$$

$$1-\frac{1}{2}+\frac{1}{2^2}-\frac{1}{2^3}$$

$$\vdots$$



Let x be the coordinate of the point which point P approaches.

$$x=1-\frac{1}{2}+\frac{1}{2^2}-\frac{1}{2^3}+\cdots$$

x is the infinite geometric series with 1st term 1 and common rati

$\boxed{-\frac{1}{2}}$. Since $\left|\boxed{-\frac{1}{2}}\right|<1$, it converges.

$$\therefore\ x=\frac{1}{1-\left(\boxed{-\frac{1}{2}}\right)}=\boxed{\frac{2}{3}}$$

Thus, the coordinate of the point which point P approaches is $\boxed{\frac{2}{3}}$.

N86b

1. On a number line, starting from origin O, point P moves 1 unit in the posi
direction and then moves $\frac{1}{3}$ units in the negative direction. Then, it mov
$\frac{1}{3^2}$ units in the positive direction and then moves $\frac{1}{3^3}$ units in the nega
direction. When this motion continues infinitely, find the coordinate of th
point which point P approaches.

[Sol] The coordinates of point P are as follows.

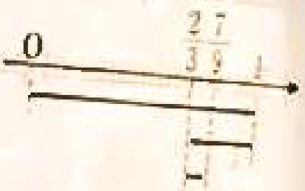
$$1$$

$$1-\frac{1}{3}$$

$$1-\frac{1}{3}+\frac{1}{3^2}$$

$$1-\frac{1}{3}+\frac{1}{3^2}-\frac{1}{3^3}$$

$$\vdots$$



Let x be the coordinate of the point which point P approaches.

$$x=1-\frac{1}{3}+\frac{1}{3^2}-\frac{1}{3^3}+\cdots$$

x is the infinite geometric series with 1st term 1 and common ratio $-\frac{1}{3}$.

Since $\left|-\frac{1}{3}\right|<1$, it converges.

$$\therefore\ x=\frac{1}{1-\left(-\frac{1}{3}\right)}=\frac{3}{4}$$

Thus, the coordinate of the point which point P approaches is $\frac{3}{4}$

N87a

KUMON

Infinite Geometric Series

Name ______________________

Date ______ / ______ / ______

Time ______ : ______ to ______ : ______

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Ex. On a coordinate plane, starting from origin O, point P moves 1 unit in the positive direction to the x-axis and then moves $\frac{1}{2}$ units in the positive direction to the y-axis. Then, it moves $\frac{1}{2^2}$ units in the negative direction to the x-axis and then moves $\frac{1}{2^3}$ units in the negative direction to the y-axis. When this motion continues infinitely, find the coordinates of the point which point P approaches.

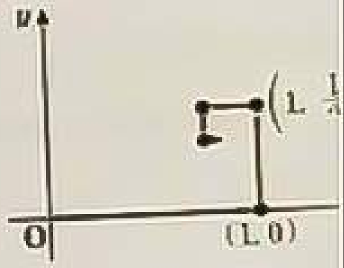
[Sol] Let (x, y) be the coordinates of the point which point P approaches.

The x-coordinate of point P is

$$x=1-\frac{1}{2^2}+\frac{1}{2^4}-\frac{1}{2^6}+\cdots$$

x is the infinite geometric series with 1st term 1 and common ratio $\boxed{-\frac{1}{4}}$. Since $\left|\boxed{-\frac{1}{4}}\right|<1$, it converges.

$$\therefore\ x=\frac{1}{1-\left(\boxed{-\frac{1}{4}}\right)}=\boxed{\frac{4}{5}}$$



The y-coordinate of point P is

$$y=\frac{1}{2}-\frac{1}{2^3}+\frac{1}{2^5}-\frac{1}{2^7}+\cdots$$

y is the infinite geometric series with 1st term $\frac{1}{2}$ and common ratio $\boxed{-\frac{1}{4}}$. Since $\left|\boxed{-\frac{1}{4}}\right|<1$, it converges.

$$\therefore\ y=\frac{\frac{1}{2}}{1-\left(\boxed{-\frac{1}{4}}\right)}=\boxed{\frac{2}{5}}$$

Thus, the coordinates of the point which point P approaches are $\left(\boxed{\frac{4}{5}}, \boxed{\frac{2}{5}}\right)$.

N87b

1. On a coordinate plane, starting from origin O, point P moves 1 unit in the positive direction to the x-axis and then moves $\frac{2}{3}$ units in the positive direction to the y-axis. Then, it moves $\left(\frac{2}{3}\right)^2$ units in the positive direction to the x-axis and then moves $\left(\frac{2}{3}\right)^3$ units in the positive direction to the y-axis. When this motion continues infinitely, find the coordinates of the point which point P approaches.

[Sol] Let (x, y) be the coordinates of the point which point P approaches.

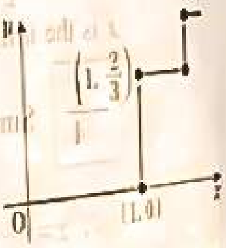
The x-coordinate of point P is

$$x=1+\left(\frac{2}{3}\right)^2+\left(\frac{2}{3}\right)^4+\left(\frac{2}{3}\right)^6+\cdots$$

x is the infinite geometric series with 1st term 1 and common ratio $\frac{4}{9}$.

Since $\left|\frac{4}{9}\right|<1$, it converges.

$$\therefore\ x=\frac{1}{1-\frac{4}{9}}=\frac{9}{5}$$



The y-coordinate of point P is

$$y=\frac{2}{3}+\left(\frac{2}{3}\right)^3+\left(\frac{2}{3}\right)^5+\left(\frac{2}{3}\right)^7+\cdots$$

y is the infinite geometric series with 1st term $\frac{2}{3}$ and common ratio $\frac{4}{9}$.

Since $\left|\frac{4}{9}\right|<1$, it converges.

$$\therefore\ y=\frac{\frac{2}{3}}{1-\frac{4}{9}}=\frac{6}{5}$$

Thus, the coordinates of the point which point P approaches are $\left(\frac{9}{5}, \frac{6}{5}\right)$.

Infinite Geometric Series

Name

Date / /

Time : to :

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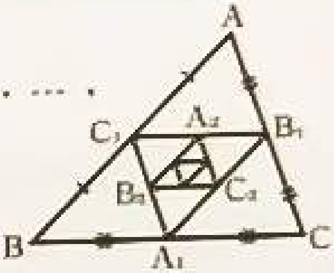
Ex. Given that $\triangle ABC$ has a perimeter of 3, form $\triangle A_1B_1C_1$ whose vertices are the midpoints of the sides of $\triangle ABC$ and then form $\triangle A_2B_2C_2$ whose vertices are the midpoints of the sides of $\triangle A_1B_1C_1$. When triangles are formed infinitely as

$\triangle A_1B_1C_1$, $\triangle A_2B_2C_2$, $\triangle A_3B_3C_3$, ..., $\triangle A_nB_nC_n$, ... ,

find the sum L of the perimeters.



[Sol] Let L_n be the perimeter of $\triangle A_nB_nC_n$.

$$\begin{cases} L_{n+1}=\dfrac{1}{2}L_n & \cdots① \\ L_1=\dfrac{1}{2}\cdot 3=\dfrac{3}{2} & \cdots② \end{cases}$$

From the Midpoint Theorem (※),

$A_{n+1}B_{n+1}=\frac{1}{2}A_nB_n$, $B_{n+1}C_{n+1}=\frac{1}{2}B_nC_n$, $C_{n+1}A_{n+1}=\frac{1}{2}C_nA_n$

Therefore,

$A_{n+1}B_{n+1}+B_{n+1}C_{n+1}+C_{n+1}A_{n+1}=\frac{1}{2}(A_nB_n+B_nC_n+C_nA_n)$

From ① and ②,

$$L=\frac{3}{2}+\frac{3}{2}\cdot\frac{1}{2}+\frac{3}{2}\left(\frac{1}{2}\right)^2+\frac{3}{2}\left(\frac{1}{2}\right)^3+\cdots$$

L is the infinite geometric series with 1st term $\frac{3}{2}$ and common ratio $\boxed{\frac{1}{2}}$. Since $\left|\boxed{\frac{1}{2}}\right|<1$, it converges.

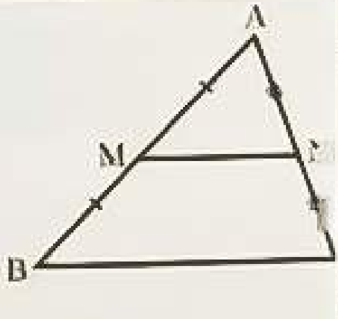
$$\therefore\ L=\frac{\frac{3}{2}}{1-\boxed{\frac{1}{2}}}=\boxed{3}$$

ANS: $\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{2}$, 3

※ If M and N are the midpoints of two sides AB and AC of $\triangle ABC$, the following relationships are true:

MN and BC are parallel, $MN=\frac{1}{2}BC$

This is called the ***Midpoint Theorem***.

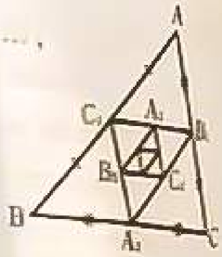


N88b

1. Given that $\triangle ABC$ has a perimeter of 10, form $\triangle A_1B_1C_1$ whose vertices are the midpoints of the sides of $\triangle ABC$ and then form $\triangle A_2B_2C_2$ whose vertices are the midpoints of the sides of $\triangle A_1B_1C_1$. When triangles are formed infinitely as

$\triangle A_1B_1C_1$, $\triangle A_2B_2C_2$, $\triangle A_3B_3C_3$, ..., $\triangle A_nB_nC_n$, ... ,

find the sum L of the perimeters.



[Sol] Let L_n be the perimeter of $\triangle A_nB_nC_n$.

$$\begin{cases} L_{n+1}=\dfrac{1}{2}L_n & \cdots ① \\ L_1=\dfrac{1}{2}\cdot 10=5 & \cdots ② \end{cases}$$

From ① and ②,

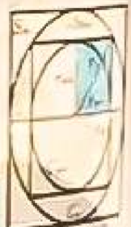
$$L=5+5\cdot\frac{1}{2}+5\left(\frac{1}{2}\right)^2+5\left(\frac{1}{2}\right)^3+\cdots$$

L is the infinite geometric series with 1st term 5 and common ratio $\frac{1}{2}$.

Since $\left|\frac{1}{2}\right|<1$, it converges.

$$\therefore\ L=\frac{5}{1-\frac{1}{2}}=10$$

2. Given square S_n and circle C_n ($n=1, 2, \dots$), C_n is inscribed in S_n and S_{n+1} is inscribed in C_n. Let r_n be the radius of C_n and a be the length of each side of S_1. Find the sum L of the circumferences.



[Sol] Let L_n be the circumference of C_n.

$$\begin{cases} L_{n+1}=2\pi r_{n+1}=2\pi\cdot\dfrac{\sqrt{2}}{2}r_n=\dfrac{\sqrt{2}}{2}L_n & \cdots ① \\ L_1=2\pi r_1=\pi a & \cdots ② \end{cases}$$

From ① and ②,

$$L=\pi a+\pi a\cdot\frac{\sqrt{2}}{2}+\pi a\left(\frac{\sqrt{2}}{2}\right)^2+\cdots$$

L is the infinite geometric series with 1st term πa and common ratio $\frac{\sqrt{2}}{2}$

Since $\left|\frac{\sqrt{2}}{2}\right|<1$, it converges.

$$\therefore\ L=\frac{\pi a}{1-\frac{\sqrt{2}}{2}}=\frac{2\pi a}{2-\sqrt{2}}=\frac{2\pi a(2+\sqrt{2})}{(2-\sqrt{2})(2+\sqrt{2})}=(2+\sqrt{2})\pi a$$

N89a

KUMON

Infinite Geometric Series

Name

Date / /

Time : to :

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1. Let n be a positive integer and r be a real number that satisfies $r>1$. Given that the sequence $\{a_n\}$ satisfies $a_1r+a_2r^2+\cdots+a_nr^n=(-1)^n$, solve the following questions.

(1) Express a_1, a_2, a_3 in terms of r.

[Sol] When $n=1$, $a_1r=-1$; therefore, $a_1=-\frac{1}{r}$

When $n=2$, $a_1r+a_2r^2=1$; therefore, $a_2=\frac{2}{r^2}$ ← $-1+a_2r^2=1$

When $n=3$, $a_1r+a_2r^2+a_3r^3=-1$; therefore, $a_3=-\frac{2}{r^3}$ ← $1+a_3r^3=-1$

(2) When $n \geqq 2$, express a_n in terms of r.

[Sol] When $n \geqq 2$,

$$\begin{cases} a_1r+a_2r^2+\cdots+a_{n-1}r^{n-1}=(-1)^{n-1} & \cdots ① \\ a_1r+a_2r^2+\cdots+a_{n-1}r^{n-1}+a_nr^n=(-1)^n & \cdots ② \end{cases}$$

From ① and ②,

$$a_nr^n=(-1)^n-(-1)^{n-1}$$

$$\therefore\ a_n=\frac{(-1)^n-(-1)^{n-1}}{r^n}=2\left(-\frac{1}{r}\right)^n$$

← $\frac{(-1)^n-(-1)^{n-1}}{r^n}=\frac{(-1)^n+(-1)^n}{r^n}=\frac{2(-1)^n}{r^n}$

(3) Find the sum of the infinite series $a_1+a_2+\cdots+a_n+\cdots$.

[Sol] From (1) and (2), $a_2+a_3+\cdots+a_n+\cdots$ is the infinite geometric series with 1st term $\frac{2}{r^2}$ and common ratio $-\frac{1}{r}$.

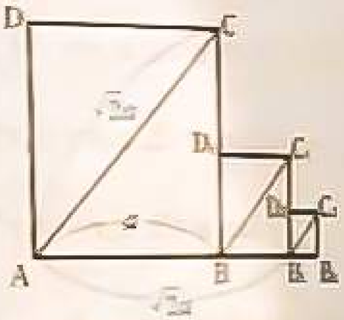
Since $\left|-\frac{1}{r}\right|<1$, it converges.

$$\therefore\ a_1+a_2+\cdots+a_n+\cdots$$
$$=a_1+(a_2+a_3+\cdots+a_n+\cdots)$$
$$=-\frac{1}{r}+\frac{\frac{2}{r^2}}{1-\left(-\frac{1}{r}\right)}$$
$$=-\frac{r-1}{r(r+1)}$$

From (1), $a_1=-\frac{1}{r}$

N89b

2. Given square ABCD with side length a, extend the line of side AB through point B in the opposite direction to point A. Then, place point B_1 on this extended line of side AB such that $AB_1 = AC$. Make a square $BB_1C_1D_1$ using BB_1 as a side and extend the line of side BB_1 from point B_1 in the opposite direction to point B. Then, place point B_2 on this extended line of side BB_1 such that $BB_2 = BC_1$. Given points B_1, B_2, B_3, … as described above, find the sum of the lengths of line segments AB, BB_1, B_1B_2, …



[Sol] Since $AB = a$,

$$\boxed{\begin{aligned}AB_1 &= AC\\ &= \sqrt{2}AB\end{aligned}}$$

$$\begin{aligned}BB_1 &= AB_1 - AB\\ &= \sqrt{2}AB - AB\\ &= \sqrt{2}a - a\\ &= (\sqrt{2}-1)a\end{aligned}$$

$$\begin{aligned}B_1B_2 &= BB_2 - BB_1\\ &= \sqrt{2}BB_1 - BB_1\\ &= (\sqrt{2}-1)\cdot(\sqrt{2}-1)a\\ &= (\sqrt{2}-1)^2 a\end{aligned}$$

$$\begin{aligned}B_2B_3 &= B_1B_3 - B_1B_2\\ &= \sqrt{2}B_1B_2 - B_1B_2\\ &= (\sqrt{2}-1)\cdot(\sqrt{2}-1)^2 a\\ &= (\sqrt{2}-1)^3 a\end{aligned}$$

$\vdots$

Let L be the sum of the lengths of line segments AB, BB_1, B_1B_2, …

$$L = a + (\sqrt{2}-1)a + (\sqrt{2}-1)^2 a + (\sqrt{2}-1)^3 a + \cdots$$

Therefore, L is the infinite geometric series with 1st term a and common ratio $\sqrt{2}-1$. Since $|\sqrt{2}-1| < 1$, it converges.

$$\therefore\ L = \frac{a}{1-(\sqrt{2}-1)} = \frac{a}{2-\sqrt{2}} = \frac{a(2+\sqrt{2})}{(2-\sqrt{2})(2+\sqrt{2})} = \frac{2+\sqrt{2}}{2}a$$

Thus, the sum of the lengths of line segments AB, BB_1, B_1B_2, … is $\dfrac{2+\sqrt{2}}{2}a$

KUMON

N90a

Infinite Geometric Series

Name

Date / /

Time : to :

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0	—	—	1	2~

1. Determine whether the following infinite geometric series converges or diverges. If it converges, find the sum S. ➡ N8

$$16+8\sqrt{2}+8+4\sqrt{2}+\cdots$$

[Sol] The 1st term is 16 and the common ratio is $\frac{8\sqrt{2}}{16}=\frac{\sqrt{2}}{2}$. Since $\left|\frac{\sqrt{2}}{2}\right|<1$, the series **converges**.

$$\therefore\ S=\frac{16}{1-\frac{\sqrt{2}}{2}}$$

$$=\frac{32}{2-\sqrt{2}}$$

$$=\frac{32(2+\sqrt{2})}{(2-\sqrt{2})(2+\sqrt{2})}$$

$$=\mathbf{16(2+\sqrt{2})}$$

2. Find the range of values of a real number x for which the following infinite geometric series converges. Then, find the sum S. ➡ N

$$2+2(x^2-3)+2(x^2-3)^2+\cdots$$

[Sol] Since the common ratio is x^2-3, $-1<x^2-3<1$

So, $-1<x^2-3$ ⋯① and also $x^2-3<1$ ⋯②

From ①, $x^2>2$; therefore, $x<-\sqrt{2},\ \sqrt{2}<x$ ⋯③

From ②, $x^2<4$; therefore, $-2<x<2$ ⋯④

From ③ and ④, the range of values of x is

$$-2<x<-\sqrt{2},\ \sqrt{2}<x<2$$

Also, $S=\dfrac{2}{1-(x^2-3)}=\dfrac{2}{4-x^2}$

N90b

3. Convert the following repeating decimal into a fraction.
$0.03\dot{1}\dot{8}$

[Sol] $0.03\dot{1}\dot{8} = 0.03 + 0.0018 + 0.000018 + 0.00000018 + \cdots$
The 2nd and subsequent terms of the RHS is the infinite geometric series with 1st term 0.0018 and common ratio 0.01.
Since $|0.01| < 1$, it converges.

$$\therefore\ 0.03\dot{1}\dot{8} = 0.03 + \frac{0.0018}{1-0.01} = \frac{3}{100} + \frac{1}{550} = \frac{7}{220}$$

4. On a number line, starting from origin O, point P moves 1 unit in the positive direction and then moves $\frac{1}{3^2}$ units in the negative direction. Then, it moves $\frac{1}{3^4}$ units in the positive direction and then moves $\frac{1}{3^6}$ units in the negative direction. When this motion continues infinitely, find the coordinate of the point which point P approaches.

[Sol] The coordinates of point P are as follows.

1

$1-\frac{1}{3^2}$

$1-\frac{1}{3^2}+\frac{1}{3^4}$

$1-\frac{1}{3^2}+\frac{1}{3^4}-\frac{1}{3^6}$

$\vdots$

Let x be the coordinate of the point which point P approaches.

$$x = 1 - \frac{1}{3^2} + \frac{1}{3^4} - \frac{1}{3^6} + \cdots$$

x is the infinite geometric series with 1st term 1 and common ratio $-\frac{1}{9}$.

Since $\left|-\frac{1}{9}\right| < 1$, it converges.

$$\therefore\ x = \frac{1}{1-\left(-\frac{1}{9}\right)} = \frac{9}{10}$$

The coordinate of the point which point P approaches is $\frac{9}{10}$

N91a

KUMON

Infinite Series

Name

Date / /

Time : to :

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Given the infinite series $a_1+a_2+a_3+\cdots+a_n+\cdots$ $\cdots$①, the sum of the first n terms

$S_n=a_1+a_2+a_3+\cdots+a_n$

is called the ***partial sum*** of the first n terms of the infinite series.

The infinite series ① can be written as $\sum_{n=1}^{\infty} a_n$.

Determine whether each given infinite series converges or diverges.
If it converges, find the sum.

Ex. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}}$

[Sol] $\displaystyle\frac{1}{\sqrt{n+1}+\sqrt{n}}=\frac{\sqrt{n+1}-\sqrt{n}}{(\sqrt{n+1}+\sqrt{n})(\sqrt{n+1}-\sqrt{n})}$

$=\sqrt{n+1}-\sqrt{n}$

Therefore, let S_n be the partial sum of the first n terms.

$S_n=(\sqrt{2}-\sqrt{1})+(\sqrt{3}-\sqrt{2})+(\sqrt{4}-\sqrt{3})+\cdots+(\sqrt{n+1}-\sqrt{n})$

$=-1+\sqrt{n+1}$

$\therefore \lim_{n\to\infty}S_n=\lim_{n\to\infty}(-1+\sqrt{n+1})=\infty$

Thus, the series diverges.

(1) $\displaystyle\sum_{n=1}^{\infty}\frac{3}{\sqrt{3n+2}+\sqrt{3n-1}}$

[Sol] $\displaystyle\frac{3}{\sqrt{3n+2}+\sqrt{3n-1}}=\frac{3(\sqrt{3n+2}-\sqrt{3n-1})}{(\sqrt{3n+2}+\sqrt{3n-1})(\sqrt{3n+2}-\sqrt{3n-1})}$

$=\sqrt{3n+2}-\sqrt{3n-1}$

Therefore, let S_n be the partial sum of the first n terms.

$S_n=(\sqrt{5}-\sqrt{2})+(\sqrt{8}-\sqrt{5})+(\sqrt{11}-\sqrt{8})+\cdots+(\sqrt{3n+2}-\sqrt{3n-$

$=-\sqrt{2}+\sqrt{3n+2}$

$\therefore \lim_{n\to\infty}S_n=\lim_{n\to\infty}(-\sqrt{2}+\sqrt{3n+2})=\infty$

Thus, the series diverges.

N91b

(2) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n+1}+\sqrt{2n-1}}$

[Sol] $\frac{1}{\sqrt{2n+1}+\sqrt{2n-1}}=\frac{\sqrt{2n+1}-\sqrt{2n-1}}{(\sqrt{2n+1}+\sqrt{2n-1})(\sqrt{2n+1}-\sqrt{2n-1})}$

$=\frac{1}{2}(\sqrt{2n+1}-\sqrt{2n-1})$

Therefore, let S_n be the partial sum of the first n terms.

$S_n=\frac{1}{2}(\sqrt{3}-\sqrt{1})+\frac{1}{2}(\sqrt{5}-\sqrt{3})+\frac{1}{2}(\sqrt{7}-\sqrt{5})+\cdots+\frac{1}{2}(\sqrt{2n+1}-\sqrt{2n-1})$

$=\frac{1}{2}(-1+\sqrt{2n+1})$

$\therefore \lim_{n\to\infty} S_n=\lim_{n\to\infty}\frac{1}{2}(-1+\sqrt{2n+1})=\infty$

Thus, the series diverges.

(3) $\sum_{n=1}^{\infty} \frac{1}{(n+1)\sqrt{n}+n\sqrt{n+1}}$

[Sol] $\frac{1}{(n+1)\sqrt{n}+n\sqrt{n+1}}=\frac{(n+1)\sqrt{n}-n\sqrt{n+1}}{[(n+1)\sqrt{n}+n\sqrt{n+1}][(n+1)\sqrt{n}-n\sqrt{n+1}]}$

$=\frac{(n+1)\sqrt{n}-n\sqrt{n+1}}{n(n+1)}$

$=\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}$

Therefore, let S_n be the partial sum of the first n terms.

$S_n=\left(\frac{1}{\sqrt{1}}-\frac{1}{\sqrt{2}}\right)+\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{3}}\right)+\left(\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}\right)+\cdots+\left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right)$

$=1-\frac{1}{\sqrt{n+1}}$

$\therefore \lim_{n\to\infty} S_n=\lim_{n\to\infty}\left(1-\frac{1}{\sqrt{n+1}}\right)=1$

Thus, the series converges and the sum is 1.

N92a

KUMON

Infinite Series

Name

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Determine whether each given infinite series converges or diverges.
If it converges, find the sum.

Ex. $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\dfrac{1}{3\cdot 4}+\cdots+\dfrac{1}{n(n+1)}+\cdots$

[Sol] Let $\dfrac{1}{n(n+1)}=\dfrac{a}{n}-\dfrac{b}{n+1}$.

Multiplying both sides by $n(n+1)$

$1=a(n+1)-bn$

$=(a-b)n+a$

Since $1=(a-b)n+a$, $\begin{cases} a-b=0 \\ a=1 \end{cases}$

$\therefore\ a=1,\ b=1$

Therefore, let S_n be the partial sum of the first n terms.

$$S_n=\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$$=1-\frac{1}{n+1}$$

$$\therefore\ \lim_{n\to\infty}S_n=\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)=1$$

Thus, the series converges and the sum is 1.

(1) $\dfrac{3}{1\cdot 4}+\dfrac{3}{4\cdot 7}+\dfrac{3}{7\cdot 10}+\cdots+\dfrac{3}{(3n-2)(3n+1)}+\cdots$

[Sol] Let $\dfrac{3}{(3n-2)(3n+1)}=\dfrac{a}{3n-2}-\dfrac{b}{3n+1}$.

$3=a(3n+1)-b(3n-2)$

$=3(a-b)n+(a+2b)$

$\therefore\ a=1,\ b=1$

Therefore, let S_n be the partial sum of the first n terms.

$$S_n=\left(\frac{1}{1}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{7}\right)+\left(\frac{1}{7}-\frac{1}{10}\right)+\cdots+\left(\frac{1}{3n-2}-\frac{1}{3n+1}\right)$$

$$=1-\frac{1}{3n+1}$$

$$\therefore\ \lim_{n\to\infty}S_n=\lim_{n\to\infty}\left(1-\frac{1}{3n+1}\right)=1$$

Thus, the series converges and the sum is 1.

92b

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\cdots+\frac{1}{(2n-1)(2n+1)}+\cdots$$

[Sol] Let $\dfrac{1}{(2n-1)(2n+1)}=\dfrac{a}{2n-1}-\dfrac{b}{2n+1}$

$$1=a(2n+1)-b(2n-1)$$
$$=2(a-b)n+(a+b)$$
$$\therefore\ a=\frac{1}{2},\ b=\frac{1}{2}$$

Therefore, let S_n be the partial sum of the first n terms.

$$S_n=\frac{1}{2}\left(\frac{1}{1}-\frac{1}{3}\right)+\frac{1}{2}\left(\frac{1}{3}-\frac{1}{5}\right)+\frac{1}{2}\left(\frac{1}{5}-\frac{1}{7}\right)+\cdots+\frac{1}{2}\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)$$
$$=\frac{1}{2}\left(1-\frac{1}{2n+1}\right)$$
$$\therefore\ \lim_{n\to\infty}S_n=\lim_{n\to\infty}\frac{1}{2}\left(1-\frac{1}{2n+1}\right)=\frac{1}{2}$$

Thus, the series **converges** and the sum is $\frac{1}{2}$.

(3) $\dfrac{1}{1}+\dfrac{1}{1+2}+\dfrac{1}{1+2+3}+\cdots+\dfrac{1}{1+2+\cdots+n}+\cdots$

[Sol] $\dfrac{1}{1+2+\cdots+n}=\dfrac{1}{\frac{1}{2}n(n+1)}=\dfrac{2}{n(n+1)}$ $\leftarrow$ $1+2+\cdots+n=\sum k$

Let $\dfrac{2}{n(n+1)}=\dfrac{a}{n}-\dfrac{b}{n+1}$.

$$2=a(n+1)-bn$$
$$=(a-b)n+a$$
$$\therefore\ a=2,\ b=2$$

Therefore, let S_n be the partial sum of the first n terms.

$$S_n=2\left(\frac{1}{1}-\frac{1}{2}\right)+2\left(\frac{1}{2}-\frac{1}{3}\right)+2\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+2\left(\frac{1}{n}-\frac{1}{n+1}\right)$$
$$=2\left(1-\frac{1}{n+1}\right)$$
$$\therefore\ \lim_{n\to\infty}S_n=\lim_{n\to\infty}2\left(1-\frac{1}{n+1}\right)=2$$

Thus, the series **converges** and the sum is 2.

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KUMON

Infinite Series

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Determine whether each given infinite series converges or diverges.
If it converges, find the sum.

(1) $\frac{1}{1\cdot2\cdot3}+\frac{1}{2\cdot3\cdot4}+\frac{1}{3\cdot4\cdot5}+\cdots+\frac{1}{n(n+1)(n+2)}+\cdots$

[Sol] Let $\frac{1}{n(n+1)(n+2)}=\frac{a}{n(n+1)}-\frac{b}{(n+1)(n+2)}$.

$$1=a(n+2)-bn$$
$$=(a-b)n+2a$$
$$\therefore\ a=\frac{1}{2},\ b=\frac{1}{2}$$

Therefore, let S_n be the partial sum of the first n terms.

$$S_n=\frac{1}{2}\left(\frac{1}{2}-\frac{1}{6}\right)+\frac{1}{2}\left(\frac{1}{6}-\frac{1}{12}\right)+\frac{1}{2}\left(\frac{1}{12}-\frac{1}{20}\right)+\cdots+\frac{1}{2}\left[\frac{1}{n(n+1)}-\frac{1}{(n+1)(n+2)}\right]$$
$$=\frac{1}{2}\left[\frac{1}{2}-\frac{1}{(n+1)(n+2)}\right]$$
$$\therefore\ \lim_{n\to\infty}S_n=\lim_{n\to\infty}\frac{1}{2}\left[\frac{1}{2}-\frac{1}{(n+1)(n+2)}\right]=\frac{1}{4}$$

Thus, the series **converges** and the sum is $\frac{1}{4}$.

N93b

(2) $\dfrac{1}{1\cdot2}+\dfrac{1}{1\cdot2+2\cdot3}+\dfrac{1}{1\cdot2+2\cdot3+3\cdot4}+\cdots+\dfrac{1}{1\cdot2+2\cdot3+3\cdot4+\cdots+n(n+1)}+\cdots$

[Sol] $\dfrac{1}{1\cdot2+2\cdot3+3\cdot4+\cdots+n(n+1)}=\dfrac{1}{\sum_{k=1}^{n}k(k+1)}$

$$=\frac{1}{\frac{1}{6}n(n+1)(2n+1)+\frac{1}{2}n(n+1)}$$

$$=\frac{1}{\frac{1}{6}n(n+1)\{(2n+1)+3\}}$$

$$=\frac{3}{n(n+1)(n+2)}$$

Let $\dfrac{3}{n(n+1)(n+2)}=\dfrac{a}{n(n+1)}-\dfrac{b}{(n+1)(n+2)}$.

$3=a(n+2)-bn$

$\quad=(a-b)n+2a$

$\therefore\ a=\dfrac{3}{2},\ b=\dfrac{3}{2}$

Therefore, let S_n be the partial sum of the first n terms.

$$S_n=\frac{3}{2}\left(\frac{1}{2}-\frac{1}{6}\right)+\frac{3}{2}\left(\frac{1}{6}-\frac{1}{12}\right)+\frac{3}{2}\left(\frac{1}{12}-\frac{1}{20}\right)+\cdots+\frac{3}{2}\left[\frac{1}{n(n+1)}-\frac{1}{(n+1)(n+2)}\right]$$

$$=\frac{3}{2}\left[\frac{1}{2}-\frac{1}{(n+1)(n+2)}\right]$$

$$\therefore\ \lim_{n\to\infty}S_n=\lim_{n\to\infty}\frac{3}{2}\left[\frac{1}{2}-\frac{1}{(n+1)(n+2)}\right]=\frac{3}{4}$$

Thus, the series **converges** and the sum is $\dfrac{3}{4}$.

$=\dfrac{1}{6}n(n+1)(2n+1)$

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KUMON

Infinite Series

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Find the sum of the following infinite series.

Ex. $\sum_{k=1}^{\infty}\left(\cos\frac{\pi}{k}-\cos\frac{\pi}{k+1}\right)$

[Sol] Let S_n be the partial sum of the first n terms.

$$S_n=\left(\cos\pi-\cos\frac{\pi}{2}\right)+\left(\cos\frac{\pi}{2}-\cos\frac{\pi}{3}\right)+\cdots+\left(\cos\frac{\pi}{n}-\cos\frac{\pi}{n+1}\right)$$

$$=\cos\pi-\cos\frac{\pi}{n+1}$$

$$=-1-\cos\frac{\pi}{n+1}$$

$$\therefore \lim_{n\to\infty}S_n=\lim_{n\to\infty}\left(-1-\cos\frac{\pi}{n+1}\right)=-2$$

$\lim_{n\to\infty}\cos\frac{\pi}{n+1}=\cos 0=$

$$\therefore \sum_{k=1}^{\infty}\left(\cos\frac{\pi}{k}-\cos\frac{\pi}{k+1}\right)=-2$$

(1) $\sum_{k=1}^{\infty}\left(\sin\frac{\pi}{2k}-\sin\frac{\pi}{2k+2}\right)$

[Sol] Let S_n be the partial sum of the first n terms.

$$S_n=\left(\sin\frac{\pi}{2}-\sin\frac{\pi}{4}\right)+\left(\sin\frac{\pi}{4}-\sin\frac{\pi}{6}\right)+\cdots+\left(\sin\frac{\pi}{2n}-\sin\frac{\pi}{2n+}\right.$$

$$=\sin\frac{\pi}{2}-\sin\frac{\pi}{2n+2}$$

$$=1-\sin\frac{\pi}{2n+2}$$

$$\therefore \lim_{n\to\infty}S_n=\lim_{n\to\infty}\left(1-\sin\frac{\pi}{2n+2}\right)=1$$

$$\therefore \sum_{k=1}^{\infty}\left(\sin\frac{\pi}{2k}-\sin\frac{\pi}{2k+2}\right)=1$$

N94b

(2) $\sum_{k=1}^{\infty} 2\cos\frac{3\pi}{2^{k+1}}\sin\frac{\pi}{2^{k+1}}$

[Sol] $2\cos\frac{3\pi}{2^{k+1}}\sin\frac{\pi}{2^{k+1}}$

$$=2\cdot\frac{1}{2}\left[\sin\left(\frac{\boxed{3\pi}}{2^{k+1}}+\frac{\boxed{\pi}}{2^{k+1}}\right)-\sin\left(\frac{\boxed{3\pi}}{2^{k+1}}-\frac{\boxed{\pi}}{2^{k+1}}\right)\right]$$

$$=\sin\frac{\boxed{\pi}}{2^{k-1}}-\sin\frac{\boxed{\pi}}{2^{k}}$$

Therefore, let S_n be the partial sum of the first n terms.

$$S_n=\left(\sin\pi-\sin\frac{\pi}{2}\right)+\left(\sin\frac{\pi}{2}-\sin\frac{\pi}{4}\right)+\cdots+\left(\sin\frac{\pi}{2^{n-1}}-\sin\frac{\pi}{2^n}\right)$$

$$=\sin\pi-\sin\frac{\pi}{2^n}$$

$$=-\sin\frac{\pi}{2^n}$$

$\therefore\ \lim_{n\to\infty}S_n=\lim_{n\to\infty}\left(-\sin\frac{\pi}{2^n}\right)=0$

$\therefore\ \sum_{k=1}^{\infty} 2\cos\frac{3\pi}{2^{k+1}}\sin\frac{\pi}{2^{k+1}}=0$

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Infinite Series

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Properties of Infinite Series

When the infinite series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge, where $\sum_{n=1}^{\infty} a_n = S$ and $\sum_{n=1}^{\infty} b_n = T$, the following properties are true:

$$\sum_{n=1}^{\infty} ka_n = kS \qquad (k \text{ is a constant})$$

$$\sum_{n=1}^{\infty} (a_n + b_n) = S + T$$

$$\sum_{n=1}^{\infty} (a_n - b_n) = S - T$$

Find the sum of the following infinite series.

Ex. $\sum_{n=1}^{\infty} \left(\frac{1}{2^{n-1}} - \frac{5}{4^n} \right)$

[Sol] $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = \frac{1}{1 - \frac{1}{2}} = 2$ ← Since the series is the infinite geometric series with 1st term 1 and common ratio $\frac{1}{2}$, it converges.

$\sum_{n=1}^{\infty} \frac{5}{4^n} = \frac{\frac{5}{4}}{1 - \frac{1}{4}} = \frac{5}{3}$ ← Since the series is the infinite geometric series with 1st term $\frac{5}{4}$ and common ratio $\frac{1}{4}$, it converges.

$$\therefore \sum_{n=1}^{\infty} \left(\frac{1}{2^{n-1}} - \frac{5}{4^n} \right) = \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} - \sum_{n=1}^{\infty} \frac{5}{4^n} = 2 - \frac{5}{3} = \frac{1}{3}$$

(1) $\sum_{n=1}^{\infty} \left(\frac{2}{3^n} - \frac{1}{6^{n-1}} \right)$

[Sol] $\sum_{n=1}^{\infty} \frac{2}{3^n} = \frac{\frac{2}{3}}{1 - \frac{1}{3}} = 1$

$\sum_{n=1}^{\infty} \frac{1}{6^{n-1}} = \frac{1}{1 - \frac{1}{6}} = \frac{6}{5}$

$$\therefore \sum_{n=1}^{\infty} \left(\frac{2}{3^n} - \frac{1}{6^{n-1}} \right) = \sum_{n=1}^{\infty} \frac{2}{3^n} - \sum_{n=1}^{\infty} \frac{1}{6^{n-1}} = 1 - \frac{6}{5} = -\frac{1}{5}$$

N95b

(2) $\displaystyle\sum_{n=1}^{\infty}\frac{2^n-3^n}{4^n}$

[Sol] $\displaystyle\sum_{n=1}^{\infty}\frac{2^n}{4^n}=\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n=\frac{\frac{1}{2}}{1-\frac{1}{2}}=1$

$\displaystyle\sum_{n=1}^{\infty}\frac{3^n}{4^n}=\sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^n=\frac{\frac{3}{4}}{1-\frac{3}{4}}=3$

$\displaystyle\therefore\ \sum_{n=1}^{\infty}\frac{2^n-3^n}{4^n}=\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n-\sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^n=1-3=-2$

(3) $\displaystyle\sum_{n=1}^{\infty}\frac{3^{n+1}+4^n}{5^n}$

[Sol] $\displaystyle\sum_{n=1}^{\infty}\frac{3^{n+1}}{5^n}=\sum_{n=1}^{\infty}3\left(\frac{3}{5}\right)^n=\frac{\frac{9}{5}}{1-\frac{3}{5}}=\frac{9}{2}$

$\displaystyle\sum_{n=1}^{\infty}\frac{4^n}{5^n}=\sum_{n=1}^{\infty}\left(\frac{4}{5}\right)^n=\frac{\frac{4}{5}}{1-\frac{4}{5}}=4$

$\displaystyle\therefore\ \sum_{n=1}^{\infty}\frac{3^{n+1}+4^n}{5^n}=\sum_{n=1}^{\infty}3\left(\frac{3}{5}\right)^n+\sum_{n=1}^{\infty}\left(\frac{4}{5}\right)^n=\frac{9}{2}+4=\frac{17}{2}$

(4) $\displaystyle\sum_{n=1}^{\infty}\frac{3^{-n}-(-1)^n}{2^{3n}}$

[Sol] $\displaystyle\sum_{n=1}^{\infty}\frac{3^{-n}}{2^{3n}}=\sum_{n=1}^{\infty}\frac{(3^{-1})^n}{8^n}=\sum_{n=1}^{\infty}\left(\frac{1}{24}\right)^n=\frac{\frac{1}{24}}{1-\frac{1}{24}}=\frac{1}{23}$

$\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{2^{3n}}=\sum_{n=1}^{\infty}\frac{(-1)^n}{8^n}=\sum_{n=1}^{\infty}\left(-\frac{1}{8}\right)^n=\frac{-\frac{1}{8}}{1-\left(-\frac{1}{8}\right)}=-\frac{1}{9}$

$\displaystyle\therefore\ \sum_{n=1}^{\infty}\frac{3^{-n}-(-1)^n}{2^{3n}}=\sum_{n=1}^{\infty}\left(\frac{1}{24}\right)^n-\sum_{n=1}^{\infty}\left(-\frac{1}{8}\right)^n=\frac{1}{23}+\frac{1}{9}=\frac{32}{207}$

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Infinite Series

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Let S_n be the partial sum of the first n terms of infinite series $\sum_{n=1}^{\infty} a_n$.

When $n \geqq 2$, $a_n = S_n - S_{n-1}$

When the infinite series $\sum_{n=1}^{\infty} a_n$ converges, let S be the sum. Then,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = S - S = 0$$

Therefore, when $\sum_{n=1}^{\infty} a_n$ converges, $\lim_{n\to\infty} a_n = 0$

Also, when the sequence $\{a_n\}$ does not converge to 0, $\sum_{n=1}^{\infty} a_n$ diverges.

From the above, the following statement is true.

Convergence and Divergence of Infinite Series

If the infinite series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

If the sequence $\{a_n\}$ does not converge to 0, then the infinite series $\sum_{n=1}^{\infty} a_n$ diverges.

Determine whether each given infinite series converges or diverges.

Ex. $\frac{1}{1} + \frac{2}{3} + \frac{3}{5} + \frac{4}{7} + \cdots$

[Sol] The n^{th} term a_n is $a_n = \frac{n}{2n-1}$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{2 - \frac{1}{n}} = \frac{1}{2}$$

The sequence $\{a_n\}$ does not converge to 0.
Therefore, the series diverges.

(1) $\frac{1}{3} + \frac{2}{4} + \frac{3}{5} + \frac{4}{6} + \cdots$

[Sol] The n^{th} term a_n is $a_n = \frac{n}{n+2}$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{1 + \frac{2}{n}} = 1$$

The sequence $\{a_n\}$ does not converge to 0.
Therefore, the series diverges.

N96b

(2) $\frac{1}{3}+\frac{3}{6}+\frac{5}{9}+\frac{7}{12}+\cdots$

[Sol] The n^{th} term a_n is $a_n=\frac{2n-1}{3n}$.

$$\lim_{n\to\infty} a_n=\lim_{n\to\infty}\frac{2-\frac{1}{n}}{3}=\frac{2}{3}$$

The sequence $\{a_n\}$ does not converge to 0.
Therefore, the series **diverges**.

(3) $\frac{1}{2}+\frac{4}{5}+\frac{9}{10}+\frac{16}{17}+\cdots$

[Sol] The n^{th} term a_n is $a_n=\frac{n^2}{n^2+1}$.

$$\lim_{n\to\infty} a_n=\lim_{n\to\infty}\frac{1}{1+\frac{1}{n^2}}=1$$

The sequence $\{a_n\}$ does not converge to 0.
Therefore, the series **diverges**.

(4) $1-2+3-4+\cdots$

[Sol] The n^{th} term a_n is $a_n=(\boxed{-1})^{n-1}n$.

The sequence $\{a_n\}$ oscillates and does not converge to 0.
Therefore, the series $\boxed{\textbf{diverges}}$.

(5) $-1+3-5+7-\cdots$

[Sol] The n^{th} term a_n is $a_n=(-1)^n(2n-1)$.
The sequence $\{a_n\}$ oscillates and does not converge to 0.
Therefore, the series diverges.

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Determine whether each given infinite series converges or diverges.
If it converges, find the sum.

Ex. $\frac{1}{2}-\frac{2}{3}+\frac{2}{3}-\frac{3}{4}+\frac{3}{4}-\cdots-\frac{n}{n+1}+\frac{n}{n+1}-\frac{n+1}{n+2}+\cdots$

[Sol] Let S_n be the partial sum of the first n terms and m be a natural number.

(i) When $n=2m-1$,

$$S_n=S_{2m-1}$$

$$=\frac{1}{2}+\left(-\frac{2}{3}+\frac{2}{3}\right)+\left(-\frac{3}{4}+\frac{3}{4}\right)+\cdots+\left(-\frac{m}{m+1}+\frac{m}{m+1}\right)$$

$$=\boxed{\frac{1}{2}}$$

$$S_1=\frac{1}{2}$$
$$S_3=\frac{1}{2}+\left(-\frac{2}{3}+\frac{2}{3}\right)$$
$$S_5=\frac{1}{2}+\left(-\frac{2}{3}+\frac{2}{3}\right)+\left(-\frac{3}{4}+\frac{3}{4}\right)$$
⋮

$$\therefore \lim_{m\to\infty} S_{2m-1}=\boxed{\frac{1}{2}}$$

(ii) When $n=2m$,

$$S_n=S_{2m}$$

$S_{2m}=S_{2m-1}+$(the $2m^{\text{th}}$ term)

$$=S_{2m-1}+\left(-\frac{m+1}{m+2}\right)$$

From (i), $\lim_{m\to\infty} S_{2m-1}=\frac{1}{2}$

$$\therefore \lim_{m\to\infty} S_{2m}=\lim_{m\to\infty}\left(S_{2m-1}-\frac{1+\frac{1}{m}}{1+\frac{2}{m}}\right)=\boxed{-\frac{1}{2}}$$

From (i) and (ii), $\lim_{m\to\infty} S_{2m-1}\neq\lim_{m\to\infty} S_{2m}$

Therefore, the series diverges.

$\{S_n\}$ oscillates and diverges.

N97b

(1) $3-2+2-\dfrac{5}{3}+\dfrac{5}{3}-\cdots-\dfrac{n+2}{n}+\dfrac{n+2}{n}-\dfrac{n+3}{n+1}+\cdots$

[Sol] Let S_n be the partial sum of the first n terms and m be a natural number.

(i) When $n=2m-1$,

$$S_n=S_{2m-1}$$

$$=3+(-2+2)+\left(-\frac{5}{3}+\frac{5}{3}\right)+\cdots+\left(-\frac{m+2}{m}+\frac{m+2}{m}\right)$$

$$=3$$

$$\therefore\ \lim_{m\to\infty} S_{2m-1}=3$$

(ii) When $n=2m$,

$$S_n=S_{2m}$$

$$=S_{2m-1}+\left(-\frac{m+3}{m+1}\right)$$

$$\therefore\ \lim_{m\to\infty} S_{2m}=\lim_{m\to\infty}\left(S_{2m-1}-\frac{1+\frac{3}{m}}{1+\frac{1}{m}}\right)=2$$

From (i) and (ii), $\lim_{m\to\infty} S_{2m-1}\neq\lim_{m\to\infty} S_{2m}$

Therefore, the series diverges.

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Infinite Series

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Determine whether each given infinite series converges or diverges.
If it converges, find the sum.

Ex. $\frac{1}{2}+\frac{1}{3}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots$

[Sol] Let S_n be the partial sum of the first n terms and m be a natural number.

(i) When $n=2m$,

$$S_n=S_{2m}$$
$$=\left[\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\cdots+\left(\frac{1}{2}\right)^m\right]+\left[\frac{1}{3}+\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^3+\cdots+\left(\frac{1}{3}\right)^m\right]$$
$$=\frac{\frac{1}{2}\left[1-\left(\frac{1}{2}\right)^m\right]}{1-\frac{1}{2}}+\frac{\frac{1}{3}\left[1-\left(\frac{1}{3}\right)^m\right]}{1-\frac{1}{3}}$$
$$=\left[1-\left(\frac{1}{2}\right)^m\right]+\frac{1}{2}\left[1-\left(\frac{1}{3}\right)^m\right]$$
$$\therefore \lim_{m\to\infty} S_{2m}=\boxed{\frac{3}{2}}$$

(ii) When $n=2m-1$,

$$S_n=S_{2m-1}$$
$$=S_{2m}-\left(\boxed{\frac{1}{3}}\right)^m$$

$S_{2m-1}=S_{2m}-$(the $2m^{\text{th}}$ term)

$$\therefore \lim_{m\to\infty} S_{2m-1}=\lim_{m\to\infty}\left[S_{2m}-\left(\boxed{\frac{1}{3}}\right)^m\right]=\boxed{\frac{3}{2}}$$

From (i), $\lim_{m\to\infty} S_{2m}=$

From (i) and (ii), $\lim_{m\to\infty} S_{2m}=\lim_{m\to\infty} S_{2m-1}$

Therefore, the series converges and the sum is $\boxed{\frac{3}{2}}$.

N98b

(1) $1+2+\frac{1}{2}-\frac{2}{3}+\frac{1}{4}+\frac{2}{9}+\frac{1}{8}-\frac{2}{27}+\cdots$

[Sol] Let S_n be the partial sum of the first n terms and m be a natural number.

(i) When $n=2m$,

$$S_n=S_{2m}$$
$$=\left[1+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\cdots+\left(\frac{1}{2}\right)^{m-1}\right]+\left[2+2\cdot\left(-\frac{1}{3}\right)+2\left(-\frac{1}{3}\right)^2+\cdots+2\left(-\frac{1}{3}\right)^{m}\right]$$
$$=\frac{1-\left(\frac{1}{2}\right)^m}{1-\frac{1}{2}}+\frac{2\left[1-\left(-\frac{1}{3}\right)^m\right]}{1-\left(-\frac{1}{3}\right)}$$
$$=2\left[1-\left(\frac{1}{2}\right)^m\right]+\frac{3}{2}\left[1-\left(-\frac{1}{3}\right)^m\right]$$
$$\therefore \lim_{m\to\infty}S_{2m}=\frac{7}{2}$$

(ii) When $n=2m-1$,

$$S_n=S_{2m-1}$$
$$=S_{2m}-2\left(-\frac{1}{3}\right)^{m-1}$$
$$\therefore \lim_{m\to\infty}S_{2m-1}=\lim_{m\to\infty}\left[S_{2m}-2\left(-\frac{1}{3}\right)^{m-1}\right]=\frac{7}{2}$$

From (i) and (ii), $\lim_{m\to\infty}S_{2m}=\lim_{m\to\infty}S_{2m-1}$

Therefore, the series converges and the sum is $\frac{7}{2}$.

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Infinite Series

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1. Given the sequence $\{a_n\}$: $\frac{3}{1\cdot2\cdot2^1}$, $\frac{4}{2\cdot3\cdot2^2}$, $\frac{5}{3\cdot4\cdot2^3}$, $\frac{6}{4\cdot5\cdot2^4}$, ...

solve the following questions.

(1) Find the general term of $\{a_n\}$.

[Sol] $a_n=\dfrac{n+2}{n(n+1)\cdot2^n}$

(2) Find the constants b and c which satisfy $2^na_n=\dfrac{b}{n}+\dfrac{c}{n+1}$.

[Sol] From (1), $2^na_n=\dfrac{n+2}{n(n+1)}$ ...①

Since $2^na_n=\dfrac{b}{n}+\dfrac{c}{n+1}$,

$2^na_n=\dfrac{(b+c)n+b}{n(n+1)}$ ...②

From ① and ②, $n+2=(b+c)n+b$

$\therefore$ $b=2$, $c=-1$

Since $n+2=(b+c)n+b$, $\begin{cases} b+c=1 \\ b=2 \end{cases}$

(3) Find the sum S_n of the first n terms of $\{a_n\}$ and its limit value $\lim\limits_{n\to\infty}S_n$.

[Sol] From (2), $a_n=\left(\dfrac{2}{n}-\dfrac{1}{n+1}\right)\dfrac{1}{2^n}=\dfrac{1}{n\cdot2^{n-1}}-\dfrac{1}{(n+1)\cdot2^n}$

$$\therefore\ S_n=\left(\frac{1}{1}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{12}\right)+\left(\frac{1}{12}-\frac{1}{32}\right)+\cdots+\left[\frac{1}{n\cdot2^{n-1}}-\frac{1}{(n+1)\cdot2^n}\right]$$

$$=1-\frac{1}{(n+1)\cdot2^n}$$

$$\therefore\ \lim_{n\to\infty}S_n=\lim_{n\to\infty}\left[1-\frac{1}{(n+1)\cdot2^n}\right]=1$$

N99b

2. Find the sum of the following infinite series.

$$1+\frac{2}{\pi}+\frac{3}{\pi^2}+\frac{4}{\pi^3}+\cdots+\frac{n}{\pi^{n-1}}+\cdots$$

(Use the condition $\lim_{n\to\infty} nx^n=0$ when $|x|<1$.)

[Sol] Let S_n be the partial sum of the first n terms.

$$S_n=1+\frac{2}{\pi}+\frac{3}{\pi^2}+\frac{4}{\pi^3}+\cdots+\frac{n}{\pi^{n-1}} \quad \cdots①$$

Multiplying both sides of ① by $\frac{1}{\pi}$,

$$\frac{1}{\pi}S_n=\frac{1}{\pi}+\frac{2}{\pi^2}+\frac{3}{\pi^3}+\cdots+\frac{n-1}{\pi^{n-1}}+\frac{n}{\pi^n} \quad \cdots②$$

N35

From ① − ②,

$$\left(1-\frac{1}{\pi}\right)S_n=\left(1+\frac{1}{\pi}+\frac{1}{\pi^2}+\frac{1}{\pi^3}+\cdots+\frac{1}{\pi^{n-1}}\right)-\frac{n}{\pi^n}$$

$$=\frac{1-\left(\frac{1}{\pi}\right)^n}{1-\frac{1}{\pi}}-\frac{n}{\pi^n}$$

Sum of the geometric series with 1st term 1, common ratio $\frac{1}{\pi}$ and number of terms n

$$\therefore\ S_n=\frac{1-\left(\frac{1}{\pi}\right)^n}{\left(1-\frac{1}{\pi}\right)^2}-\frac{1}{1-\frac{1}{\pi}}\cdot\frac{n}{\pi^n}$$

$$=\left(\frac{\pi}{\pi-1}\right)^2\left[1-\left(\frac{1}{\pi}\right)^n\right]-\frac{\pi}{\pi-1}\cdot\frac{n}{\pi^n}$$

Therefore, the sum of the infinite series is

$$\lim_{n\to\infty}S_n=\lim_{n\to\infty}\left\{\left(\frac{\pi}{\pi-1}\right)^2\left[1-\left(\frac{1}{\pi}\right)^n\right]-\frac{\pi}{\pi-1}\cdot\frac{n}{\pi^n}\right\}$$

$$=\left(\frac{\pi}{\pi-1}\right)^2$$

From the condition $\lim_{n\to\infty} nx^n=0$

Also, $\left|\frac{1}{\pi}\right|<1$ from $\pi=3.14$,

$\therefore\ \lim_{n\to\infty}\frac{n}{\pi^n}=\lim_{n\to\infty}n\left(\frac{1}{\pi}\right)^n=0$

N100a

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Infinite Series

Name

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1. Determine whether each given infinite series converges or diverges.
If it converges, find the sum.

➡ N9

(1) $\displaystyle\sum_{n=1}^{\infty}\frac{2}{\sqrt{n+1}+\sqrt{n+2}}$

[Sol] $$\frac{2}{\sqrt{n+1}+\sqrt{n+2}}=\frac{2(\sqrt{n+1}-\sqrt{n+2})}{(\sqrt{n+1}+\sqrt{n+2})(\sqrt{n+1}-\sqrt{n+2})}$$
$$=-2(\sqrt{n+1}-\sqrt{n+2})$$

Therefore, let S_n be the partial sum of the first n terms.

$$S_n=-2(\sqrt{2}-\sqrt{3})-2(\sqrt{3}-\sqrt{4})-2(\sqrt{4}-\sqrt{5})-\cdots-2(\sqrt{n+1}-\sqrt{n+2})$$
$$=-2(\sqrt{2}-\sqrt{n+2})$$
$$\therefore\ \lim_{n\to\infty}S_n=\lim_{n\to\infty}[-2(\sqrt{2}-\sqrt{n+2})]=\infty$$

Thus, the series **diverges.**

➡ N

(2) $\displaystyle\frac{4}{1\cdot5}+\frac{4}{5\cdot9}+\frac{4}{9\cdot13}+\cdots+\frac{4}{(4n-3)(4n+1)}+\cdots$

[Sol] Let $\displaystyle\frac{4}{(4n-3)(4n+1)}=\frac{a}{4n-3}-\frac{b}{4n+1}$.

$$4=a(4n+1)-b(4n-3)$$
$$=4(a-b)n+(a+3b)$$
$$\therefore\ a=1,\ b=1$$

Therefore, let S_n be the partial sum of the first n terms.

$$S_n=\left(\frac{1}{1}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{9}\right)+\left(\frac{1}{9}-\frac{1}{13}\right)+\cdots+\left(\frac{1}{4n-3}-\frac{1}{4n+1}\right)$$
$$=1-\frac{1}{4n+1}$$
$$\therefore\ \lim S_n=\lim\left(1-\frac{1}{4n+1}\right)=1$$

Thus, the series **converges** and the sum is 1.

2. Find the sum of the following infinite series.

$$\sum_{n=1}^{\infty}\frac{5^n-2^n}{10^n}$$

➡ N95

[Sol] $\displaystyle\sum_{n=1}^{\infty}\frac{5^n}{10^n}=\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n=\frac{\frac{1}{2}}{1-\frac{1}{2}}=1$

$$\sum_{n=1}^{\infty}\frac{2^n}{10^n}=\sum_{n=1}^{\infty}\left(\frac{1}{5}\right)^n=\frac{\frac{1}{5}}{1-\frac{1}{5}}=\frac{1}{4}$$

$$\therefore \sum_{n=1}^{\infty}\frac{5^n-2^n}{10^n}=\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n-\sum_{n=1}^{\infty}\left(\frac{1}{5}\right)^n=1-\frac{1}{4}=\frac{3}{4}$$

3. Determine whether the following infinite series converges or diverges. If it converges, find the sum.

➡ N97

$$\frac{2}{3}-\frac{4}{5}+\frac{4}{5}-\frac{6}{7}+\frac{6}{7}-\cdots-\frac{2n}{2n+1}+\frac{2n}{2n+1}-\frac{2n+2}{2n+3}+\cdots$$

[Sol] Let S_n be the partial sum of the first n terms and m be a natural number.

(ⅰ) When $n=2m-1$,

$$S_n=S_{2m-1}$$

$$=\frac{2}{3}+\left(-\frac{4}{5}+\frac{4}{5}\right)+\left(-\frac{6}{7}+\frac{6}{7}\right)+\cdots+\left(-\frac{2m}{2m+1}+\frac{2m}{2m+1}\right)$$

⬅ N92

$$=\frac{2}{3}$$

$$\therefore \lim_{m\to\infty}S_{2m-1}=\frac{2}{3}$$

(ⅱ) When $n=2m$,

$$S_n=S_{2m}$$

$$=S_{2m-1}+\left(-\frac{2m+2}{2m+3}\right)$$

$$\therefore \lim_{m\to\infty}S_{2m}=\lim_{m\to\infty}\left(S_{2m-1}-\frac{2+\frac{2}{m}}{2+\frac{3}{m}}\right)=-\frac{1}{3}$$

From (ⅰ) and (ⅱ), $\lim_{m\to\infty}S_{2m-1}\neq\lim_{m\to\infty}S_{2m}$

Therefore, the series diverges.

N101a

KUMON

Limits of Functions 1

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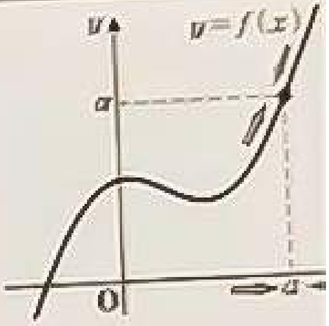
Given the function $f(x)$, if $f(x)$ approaches a constant value α as x approaches a, then $f(x)$ is said to ***converge*** to α, where it is expressed as:



$\lim_{x \to a} f(x)=\alpha$ or $f(x) \to \alpha$ as $x \to a$.

The value α is called the ***limit*** or the ***limit value*** of the function $f(x)$ as $x \to a$. As with limits of sequences, the following are true for limits of functions.

Properties of Limits of Functions

If $\lim_{x \to a} f(x)=\alpha$ and $\lim_{x \to a} g(x)=\beta$, then

$\lim_{x \to a} kf(x)=k\alpha$ (k is a constant)

$\lim_{x \to a} [f(x)+g(x)]=\alpha+\beta$, $\lim_{x \to a} [f(x)-g(x)]=\alpha-\beta$

$\lim_{x \to a} f(x)g(x)=\alpha\beta$

$\lim_{x \to a} \dfrac{f(x)}{g(x)}=\dfrac{\alpha}{\beta}$ ($\beta \neq 0$)

Find the following limits.

Ex. $\lim_{x \to 3} \sqrt{2x-1}=\sqrt{2 \cdot 3-1}=\sqrt{5}$

(1) $\lim_{x \to 2} \sqrt{3x+5}=\sqrt{3 \cdot 2+5}=\sqrt{11}$

(2) $\lim_{x \to 1} 3^x=3^1=3$

(3) $\lim_{x \to -1} \dfrac{x+2}{(x+4)(x-3)}=\dfrac{-1+2}{(-1+4)(-1-3)}=-\dfrac{1}{12}$

01b

$$\lim_{x\to 3}\frac{x^2-9}{x-3}=\lim_{x\to 3}\frac{(x+3)(x-3)}{x-3}=\lim_{x\to 3}(x+3)=6$$

$$\lim_{x\to -5}\frac{x^2+2x-15}{x+5}=\lim_{x\to -5}\frac{(x+5)(x-3)}{x+5}=\lim_{x\to -5}(x-3)=-8$$

$$\lim_{x\to 0}\frac{x^3+x}{3x-x^2}=\lim_{x\to 0}\frac{x(x^2+1)}{x(3-x)}=\lim_{x\to 0}\frac{x^2+1}{3-x}=\frac{1}{3}$$

(6) $$\lim_{x\to -2}\frac{x^3+8}{x^2-x-6}=\lim_{x\to -2}\frac{(x+2)(x^2-2x+4)}{(x+2)(x-3)}=\lim_{x\to -2}\frac{x^2-2x+4}{x-3}=-\frac{12}{5}$$

$$a^3+b^3=(a+b)(a^2-ab+b^2)$$

(7) $$\lim_{x\to 1}\frac{x^2-4x+3}{x^3-1}=\lim_{x\to 1}\frac{(x-1)(x-3)}{(x-1)(x^2+x+1)}=\lim_{x\to 1}\frac{x-3}{x^2+x+1}=-\frac{2}{3}$$

$$a^3-b^3=(a-b)(a^2+ab+b^2)$$

$$\lim_{x\to 0}\frac{1}{x}\left(1-\frac{1}{x+1}\right)=\lim_{x\to 0}\frac{1}{x}\cdot\frac{x}{x+1}=\lim_{x\to 0}\frac{1}{x+1}=1$$

$$1-\frac{1}{x+1}=\frac{(x+1)-1}{x+1}=\frac{x}{x+1}$$

N102a

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Limits of Functions 1

Name ____________________

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Find the following limits.

Ex.

$$\lim_{x\to 3}\frac{\sqrt{x+1}-2}{x-3}=\lim_{x\to 3}\frac{(\sqrt{x+1}-2)(\sqrt{x+1}+2)}{(x-3)(\sqrt{x+1}+2)}$$

← Multiplying the numerator and the denominator by $\sqrt{x+1}+2$

$$=\lim_{x\to 3}\frac{x-3}{(x-3)(\sqrt{x+1}+2)}$$

$$=\lim_{x\to 3}\frac{1}{\sqrt{x+1}+2}$$

$$=\frac{1}{4}$$

(1) $$\lim_{x\to 4}\frac{\sqrt{x+5}-3}{x-4}=\lim_{x\to 4}\frac{(\sqrt{x+5}-3)(\sqrt{x+5}+3)}{(x-4)(\sqrt{x+5}+3)}$$

$$=\lim_{x\to 4}\frac{x-4}{(x-4)(\sqrt{x+5}+3)}$$

$$=\lim_{x\to 4}\frac{1}{\sqrt{x+5}+3}$$

$$=\frac{1}{6}$$

(2) $$\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x}=\lim_{x\to 0}\frac{(\sqrt{1+x}-\sqrt{1-x})(\sqrt{1+x}+\sqrt{1-x})}{x(\sqrt{1+x}+\sqrt{1-x})}$$

$$=\lim_{x\to 0}\frac{2x}{x(\sqrt{1+x}+\sqrt{1-x})}$$

$$=\lim_{x\to 0}\frac{2}{\sqrt{1+x}+\sqrt{1-x}}$$

$$=1$$

(3) $$\lim_{x\to 1}\frac{x-1}{\sqrt{x}-1}=\lim_{x\to 1}\frac{(x-1)(\sqrt{x}+1)}{(\sqrt{x}-1)(\sqrt{x}+1)}$$

← Multiplying the numer[...] and the denominator b[...] $\sqrt{x}+1$

$$=\lim_{x\to 1}\frac{(x-1)(\sqrt{x}+1)}{x-1}$$

$$=\lim_{x\to 1}(\sqrt{x}+1)$$

$$=2$$

(4) $\lim\limits_{x\to 0}\dfrac{x}{\sqrt{x+5}-\sqrt{5}}$

$=\lim\limits_{x\to 0}\dfrac{x(\sqrt{x+5}+\sqrt{5})}{(\sqrt{x+5}-\sqrt{5})(\sqrt{x+5}+\sqrt{5})}$

$=\lim\limits_{x\to 0}\dfrac{x(\sqrt{x+5}+\sqrt{5})}{x}$

$=\lim\limits_{x\to 0}(\sqrt{x+5}+\sqrt{5})$

$=2\sqrt{5}$

(5) $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x+x^2}-1}{\sqrt{1+x}-\sqrt{1-x}}$

$=\lim\limits_{x\to 0}\dfrac{(\sqrt{1+x+x^2}-1)(\sqrt{1+x}+\sqrt{1-x})}{(\sqrt{1+x}-\sqrt{1-x})(\sqrt{1+x}+\sqrt{1-x})}$

Multiplying the numerator and the denominator by $\sqrt{1+x}+\sqrt{1-x}$

$=\lim\limits_{x\to 0}\dfrac{(\sqrt{1+x+x^2}-1)(\sqrt{1+x}+\sqrt{1-x})}{2x}$

$=\lim\limits_{x\to 0}\dfrac{(\sqrt{1+x+x^2}-1)(\sqrt{1+x+x^2}+1)(\sqrt{1+x}+\sqrt{1-x})}{2x(\sqrt{1+x+x^2}+1)}$

Multiplying the numerator and the denominator by $\sqrt{1+x+x^2}+1$

$=\lim\limits_{x\to 0}\dfrac{x(1+x)(\sqrt{1+x}+\sqrt{1-x})}{2x(\sqrt{1+x+x^2}+1)}$

$=\lim\limits_{x\to 0}\dfrac{(1+x)(\sqrt{1+x}+\sqrt{1-x})}{2(\sqrt{1+x+x^2}+1)}$

$=\dfrac{1}{2}$

Alternative Solution

$\lim\limits_{x\to 0}\dfrac{\sqrt{1+x+x^2}-1}{\sqrt{1+x}-\sqrt{1-x}}=\lim\limits_{x\to 0}\dfrac{(\sqrt{1+x+x^2}-1)(\sqrt{1+x+x^2}+1)(\sqrt{1+x}+\sqrt{1-x})}{(\sqrt{1+x}-\sqrt{1-x})(\sqrt{1+x}+\sqrt{1-x})(\sqrt{1+x+x^2}+1)}$

$=\lim\limits_{x\to 0}\dfrac{x(1+x)(\sqrt{1+x}+\sqrt{1-x})}{2x(\sqrt{1+x+x^2}+1)}=\lim\limits_{x\to 0}\dfrac{(1+x)(\sqrt{1+x}+\sqrt{1-x})}{2(\sqrt{1+x+x^2}+1)}=\dfrac{1}{2}$

...te Summary

...limit becomes an indeterminate form $\dfrac{0}{0}$, the expression has to be

N103a

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Limits of Functions 1

Name ______________

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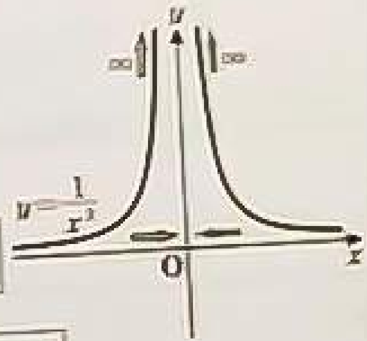
Given $f(x)=\frac{1}{x^2}$, solve the following questions. Circle the correct term in the brackets ( ) below.



[1] $f\left(\frac{1}{2}\right)=4$

[2] $f\left(\frac{1}{10}\right)=\boxed{100}$

[3] $f\left(\frac{1}{100}\right)=\boxed{10000}$

[4] $f\left(-\frac{1}{2}\right)=\boxed{4}$

[5] $f\left(-\frac{1}{10}\right)=\boxed{100}$

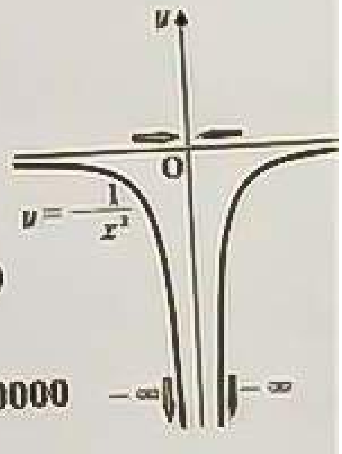
[6] $f\left(-\frac{1}{100}\right)=\boxed{10000}$

[7] From the answers above, when the value of x approaches 0, the value $f(x)=\frac{1}{x^2}$ becomes (larger , smaller).

$$\therefore \lim_{x\to 0}\frac{1}{x^2}=\boxed{\infty}$$

swers: 100, 10000, 4, 100, 10000, larger, ∞

1. Given $f(x)=-\frac{1}{x^2}$, solve the following questions.



(1) $f\left(\frac{1}{2}\right)=-4$

(2) $f\left(\frac{1}{10}\right)=-100$

(3) $f\left(\frac{1}{100}\right)=-10000$

(4) $f\left(-\frac{1}{2}\right)=-4$

(5) $f\left(-\frac{1}{10}\right)=-100$

(6) $f\left(-\frac{1}{100}\right)=-10000$

2. Using the answers from question 1, find the following limit.

$$\lim_{x\to 0}\left(-\frac{1}{x^2}\right)=-\infty$$

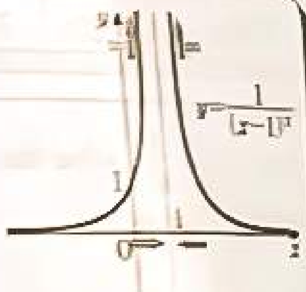
03b

nd the following limits.

$$\lim_{x\to 1}\frac{1}{(x-1)^2}$$

[Sol] As $x\to 1$, $(x-1)^2\to 0$

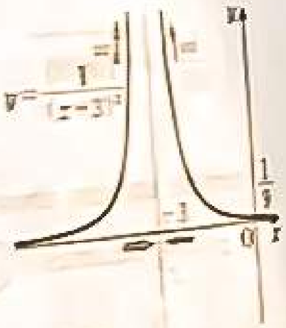
$$\therefore \lim_{x\to 1}\frac{1}{(x-1)^2}=\infty$$



$$\lim_{x\to -3}\frac{1}{(x+3)^2}$$

ol] As $x\to -3$, $(x+3)^2\to 0$

$$\therefore \lim_{x\to -3}\frac{1}{(x+3)^2}=\infty$$



(2) $\lim_{x\to 2}\left[-\frac{1}{(x-2)^2}\right]$

[Sol] As $x\to 2$, $(x-2)^2\to 0$

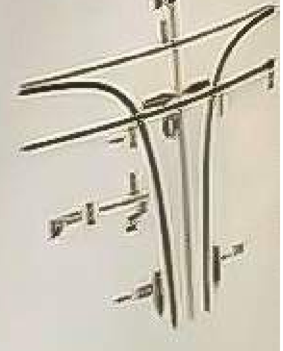
$$\therefore \lim_{x\to 2}\left[-\frac{1}{(x-2)^2}\right]=-\infty$$

(3) $\lim_{x\to 0}\left(1-\frac{1}{x^2}\right)$

[Sol] As $x\to 0$, $x^2\to \boxed{0}$

$$\therefore \lim_{x\to 0}\frac{1}{x^2}=\boxed{\infty}$$

$$\therefore \lim_{x\to 0}\left(1-\frac{1}{x^2}\right)=\boxed{-\infty}$$



N104a

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Limits of Functions 1

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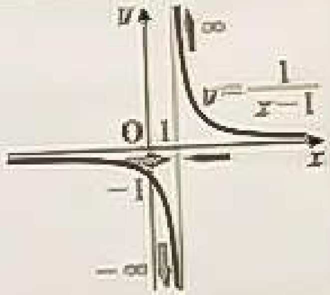
1. Determine the following limits for each different case of x.

Ex.

$\lim_{x \to 1} \frac{1}{x-1}$

[Sol] (i) When $x > 1$, as $x \to 1$, $\frac{1}{x-1} \to \infty$

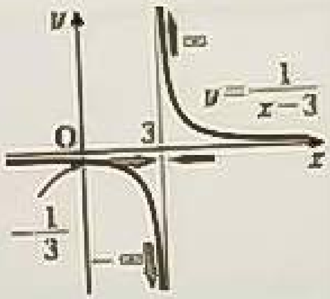
(ii) When $x < 1$, as $x \to 1$, $\frac{1}{x-1} \to -\infty$



(1) $\lim_{x \to 3} \frac{1}{x-3}$

[Sol] (i) When $x > 3$, as $x \to 3$, $\frac{1}{x-3} \to \infty$

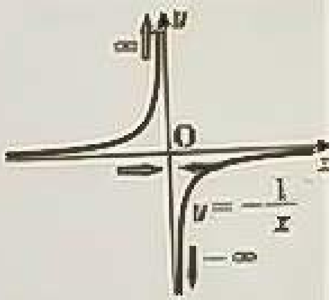
(ii) When $x < 3$, as $x \to 3$, $\frac{1}{x-3} \to -\infty$



(2) $\lim_{x \to 0}\left(-\frac{1}{x}\right)$

[Sol] (i) When $x > 0$, as $x \to 0$, $-\frac{1}{x} \to -\infty$

(ii) When $x < 0$, as $x \to 0$, $-\frac{1}{x} \to \infty$

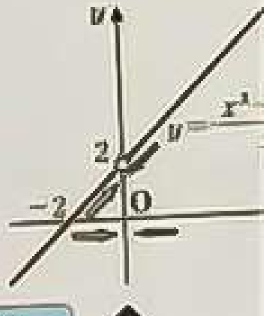


(3) $\lim_{x \to 0} \frac{x^2+2x}{x}$

[Sol] When $x \neq 0$, $\frac{x^2+2x}{x} = \boxed{x+2}$

(i) When $x > 0$, as $x \to 0$, $\frac{x^2+2x}{x} = x+2 \to 2$

(ii) When $x < 0$, as $x \to 0$, $\frac{x^2+2x}{x} = x+2 \to 2$



When $x=0$, $\frac{x^2+2x}{x}$ is not defined.

104b

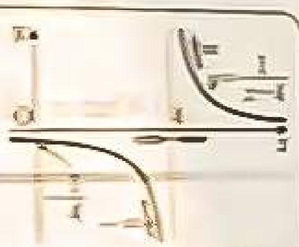
r the function $f(x)$, the limit when x approaches a m the right is called the ***right-handed limit*** and is pressed as $\lim_{x \to a+0} f(x)$.

e limit when x approaches a from the left is called ***left-handed limit*** and is expressed as $\lim_{x \to a-0} f(x)$.

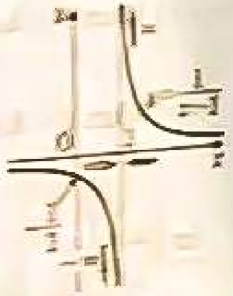
Find the following limits.

$\lim_{x \to 4+0} \frac{1}{x-4} = \infty$

$\lim_{x \to 4-0} \frac{1}{x-4} = -\infty$



) $\lim_{x \to 2+0} \frac{1}{x-2} = \infty$



) $\lim_{x \to 2-0} \frac{1}{x-2} = -\infty$

) $\lim_{x \to -1+0} \frac{1}{(x+1)^2} = \infty$

) $\lim_{x \to -1-0} \frac{1}{(x+1)^2} = \infty$

N105a

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Limits of Functions 1

Name

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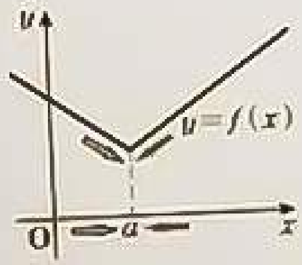
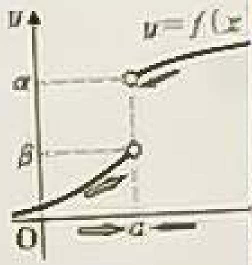
If $\lim\limits_{x \to a+0} f(x)$ is equal to $\lim\limits_{x \to a-0} f(x)$, then $\lim\limits_{x \to a} f(x)$ exists.
Also, if $\lim\limits_{x \to a+0} f(x)$ is not equal to $\lim\limits_{x \to a-0} f(x)$, then $\lim\limits_{x \to a} f(x)$ does not exist.

$\lim\limits_{x \to a+0} f(x) = \lim\limits_{x \to a-0} f(x)$



$\lim\limits_{x \to a+0} f(x) \neq \lim\limits_{x \to a-0} f(x)$



From the above, the following statements are true.

Existence of a Limit

If $\lim\limits_{x \to a+0} f(x) = \lim\limits_{x \to a-0} f(x) = \alpha$, then $\lim\limits_{x \to a} f(x) = \alpha$ exists.
If $\lim\limits_{x \to a+0} f(x) \neq \lim\limits_{x \to a-0} f(x)$, then $\lim\limits_{x \to a} f(x)$ does not exist.

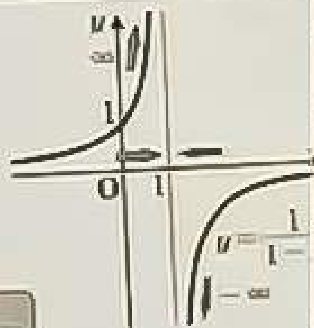
Determine if the following limits exist by finding the right-handed and left-hand limits. If they exist, find the limits.

Ex. $\lim\limits_{x \to 1} \dfrac{1}{1-x}$

[Sol] $\lim\limits_{x \to 1+0} \dfrac{1}{1-x} = -\infty$, $\lim\limits_{x \to 1-0} \dfrac{1}{1-x} = \infty$

Therefore, $\lim\limits_{x \to 1} \dfrac{1}{1-x}$ does not exist.

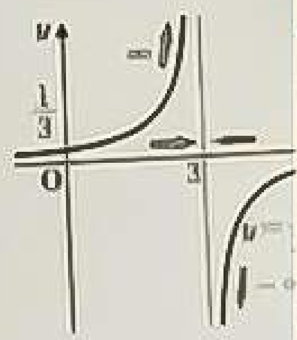
$\lim\limits_{x \to 1+0} \dfrac{1}{1-x} \neq \lim\limits_{x \to 1-0} \dfrac{1}{1-x}$



(1) $\lim\limits_{x \to 3} \dfrac{1}{3-x}$

[Sol] $\lim\limits_{x \to 3+0} \dfrac{1}{3-x} = -\infty$, $\lim\limits_{x \to 3-0} \dfrac{1}{3-x} = \infty$

Therefore, $\lim\limits_{x \to 3} \dfrac{1}{3-x}$ does not exist.

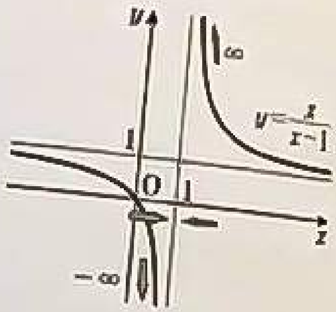


N105b

(2) $\lim_{x \to 1} \frac{x}{x-1}$

[Sol] $\lim_{x \to 1+} \frac{x}{x-1} = \infty$, $\lim_{x \to 1-} \frac{x}{x-1} = -\infty$

Therefore, $\lim_{x \to 1} \frac{x}{x-1}$ does not exist.

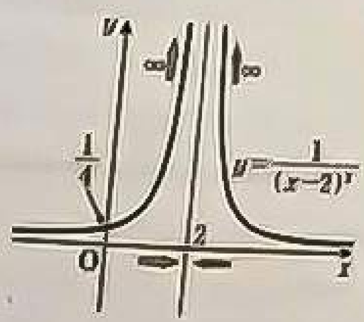


3) $\lim_{x \to 2} \frac{1}{(x-2)^2}$

Sol] $\lim_{x \to 2+} \frac{1}{(x-2)^2} = \infty$, $\lim_{x \to 2-} \frac{1}{(x-2)^2} = \infty$

Therefore, $\lim_{x \to 2} \frac{1}{(x-2)^2} = \infty$

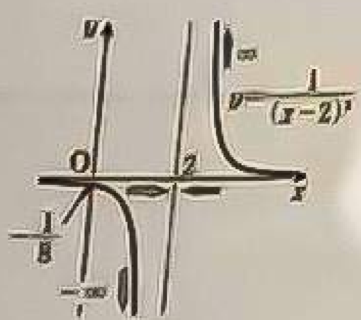
$\boxed{\lim_{x \to 2+} \frac{1}{(x-2)^2} = \lim_{x \to 2-} \frac{1}{(x-2)^2}}$



$\lim_{x \to 2} \frac{1}{(x-2)^3}$

$\lim_{x \to 2+} \frac{1}{(x-2)^3} = \infty$, $\lim_{x \to 2-} \frac{1}{(x-2)^3} = -\infty$

Therefore, $\lim_{x \to 2} \frac{1}{(x-2)^3}$ does not exist.



N 106a

KUMON

Limits of Functions 1

Name

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Time : to :

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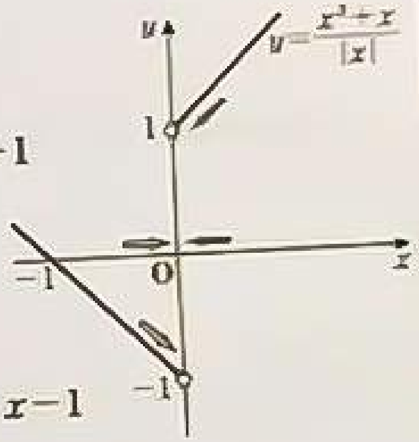
Determine if the following limits exist. If they exist, find the limits.

Ex. $\lim\limits_{x\to 0}\dfrac{x^2+x}{|x|}$



[Sol] (i) When $x>0$, $\dfrac{x^2+x}{|x|}=\dfrac{x(x+1)}{x}=x+1$

$\therefore\ \lim\limits_{x\to 0^+}\dfrac{x^2+x}{|x|}=\lim\limits_{x\to 0^+}(x+1)=1$

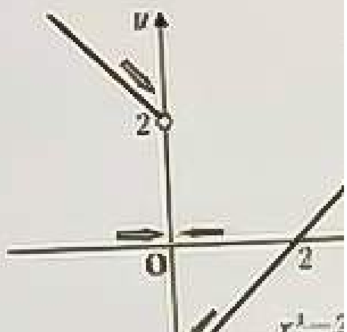
(ii) When $x<0$, $\dfrac{x^2+x}{|x|}=\dfrac{x(x+1)}{-x}=-x-1$

$\therefore\ \lim\limits_{x\to 0^-}\dfrac{x^2+x}{|x|}=\lim\limits_{x\to 0^-}(-x-1)=-1$

$\lim\limits_{x\to 0^+}\dfrac{x^2+x}{|x|}\neq\lim\limits_{x\to 0^-}\dfrac{x^2+x}{|x|}$

From (i) and (ii), $\lim\limits_{x\to 0}\dfrac{x^2+x}{|x|}$ does not exist.

(1) $\lim\limits_{x\to 0}\dfrac{x^2-2x}{|x|}$



[Sol] (i) When $x>0$, $\dfrac{x^2-2x}{|x|}=\dfrac{x(x-2)}{x}=x-2$

$\therefore\ \lim\limits_{x\to 0^+}\dfrac{x^2-2x}{|x|}=\lim\limits_{x\to 0^+}(x-2)=-2$

(ii) When $x<0$, $\dfrac{x^2-2x}{|x|}=\dfrac{x(x-2)}{-x}=-x+2$

$\therefore\ \lim\limits_{x\to 0^-}\dfrac{x^2-2x}{|x|}=\lim\limits_{x\to 0^-}(-x+2)=2$

From (i) and (ii), $\lim\limits_{x\to 0}\dfrac{x^2-2x}{|x|}$ does not exist.

$\lim\limits_{x\to 1} \dfrac{|x-1|}{x-1}$

Sol (i) When $x-1>0$, i.e. $x>1$, $\dfrac{|x-1|}{x-1}=\dfrac{x-1}{x-1}=1$

$\therefore \lim\limits_{x\to 1^+} \dfrac{|x-1|}{x-1}=1$

(ii) When $x-1<0$, i.e. $x<1$, $\dfrac{|x-1|}{x-1}=\dfrac{-(x-1)}{x-1}=-1$

$\therefore \lim\limits_{x\to 1^-} \dfrac{|x-1|}{x-1}=-1$

From (i) and (ii), $\lim\limits_{x\to 1} \dfrac{|x-1|}{x-1}$ does not exist.

(b) $\lim\limits_{x\to 1} \dfrac{(x-1)^2}{|x^2-1|}$

Sol (i) When $x^2-1>0$, i.e. $x<-1$, $1<x$

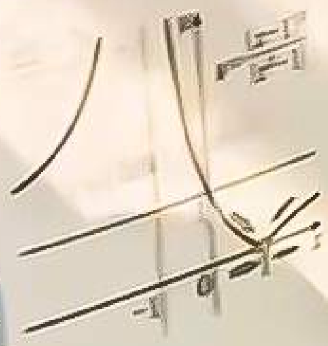
$\dfrac{(x-1)^2}{|x^2-1|}=\dfrac{(x-1)^2}{x^2-1}=\dfrac{(x-1)^2}{(x+1)(x-1)}=\dfrac{x-1}{x+1}$

$\therefore \lim\limits_{x\to 1^+} \dfrac{(x-1)^2}{|x^2-1|}=\lim\limits_{x\to 1^+}\dfrac{x-1}{x+1}=0$

(ii) When $x^2-1<0$, i.e. $-1<x<1$, $\dfrac{(x-1)^2}{-(x+1)(x-1)}=-\dfrac{x-1}{x+1}$

$\dfrac{(x-1)^2}{|x^2-1|}=\dfrac{(x-1)^2}{-(x^2-1)}$

$\therefore \lim\limits_{x\to 1^-} \dfrac{(x-1)^2}{|x^2-1|}=\lim\limits_{x\to 1^-}\left(-\dfrac{x-1}{x+1}\right)=0$



$\lim\limits_{x\to 1} \dfrac{(x-1)^2}{|x^2-1|}=0$

From (i) and (ii), $\lim\limits_{x\to 1^+} \dfrac{(x-1)^2}{|x^2-1|}$

$\boxed{\lim\limits_{x\to 1^-} \dfrac{(x-1)^2}{|x^2-1|}=\lim\limits_{x\to 1} \dfrac{(x-1)^2}{|x^2-1|}}$

KUMON

N107a

Limits of Functions 1

Name ______________________

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Time   :   to   :

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$[x]$ denotes the greatest integer less than or equal to a real number x.
This can be expressed as follows:

If n is an integer and $n \leqslant x < n+1$, then $[x]=n$

For example, $\left[\frac{7}{2}\right]=3$, $[2]=2$, $[0.99]=0$, $\left[-\frac{1}{10}\right]=-1$.

Determine if the following limits exist. If they exist, find the limits.

Ex. $\lim\limits_{x\to 2}[x]$

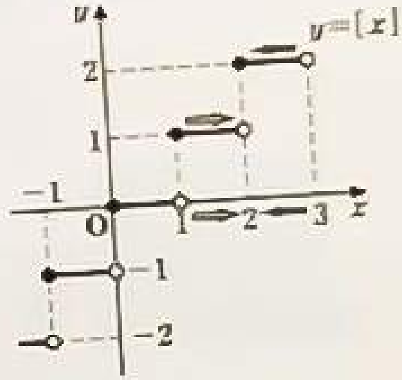
[Sol] (i) When $1 \leqslant x < 2$, $[x]=1$

$\therefore \lim\limits_{x\to 2^-}[x]=1$ Considering $x \to 2^-$

(ii) When $2 \leqslant x < 3$, $[x]=2$

$\therefore \lim\limits_{x\to 2^+}[x]=2$ Considering $x \to 2^+$

From (i) and (ii), $\lim\limits_{x\to 2}[x]$ does not exist. ← $\lim\limits_{x\to 2^-}[x] \neq \lim\limits_{x\to 2^+}[x]$



(1) $\lim\limits_{x\to -1}[x]$

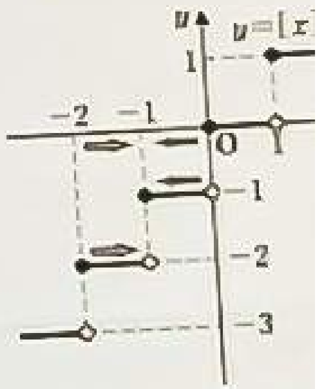
[Sol] (i) When $-2 \leqslant x < -1$, $[x]=-2$

$\therefore \lim\limits_{x\to -1^-}[x]=-2$

(ii) When $-1 \leqslant x < 0$, $[x]=-1$

$\therefore \lim\limits_{x\to -1^+}[x]=-1$

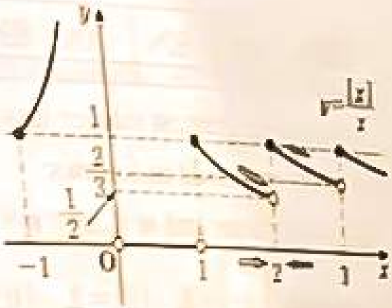
From (i) and (ii), $\lim\limits_{x\to -1}[x]$ does not exist.



The symbol [   ] is called the *Gauss symbol*. $[x]$ is read as "Gauss x."

N107b

(2) $\lim_{x \to 2} \frac{[x]}{x}$



[Sol] (i) When $1 \leqq x < 2$, $[x]=1$

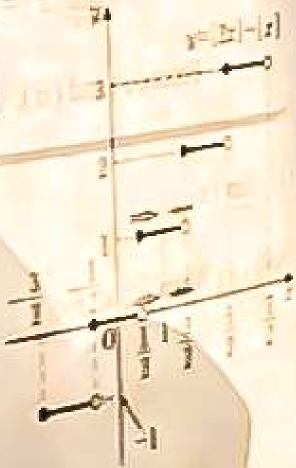
$$\therefore \lim_{x \to 2^-} \frac{[x]}{x} = \frac{1}{2}$$

(ii) When $2 \leqq x < 3$, $[x]=2$

$$\therefore \lim_{x \to 2^+} \frac{[x]}{x} = \frac{2}{2} = 1$$

From (i) and (ii), $\lim_{x \to 2} \frac{[x]}{x}$ does not exist.

(3) $\lim_{x \to 1} ([2x]-[x])$



[Sol] (i) When $\frac{1}{2} \leqq x < 1$, $[x]=0$ ← ※

Since $1 \leqq 2x < 2$, $[2x]=1$

$$\therefore \lim_{x \to 1^-} ([2x]-[x]) = 1-0 = 1$$

(ii) When $1 \leqq x < \frac{3}{2}$, $[x]=1$

Since $2 \leqq 2x < 3$, $[2x]=2$

$$\therefore \lim_{x \to 1^+} ([2x]-[x]) = 2-1 = 1$$ ←

From (i) and (ii), $\lim_{x \to 1} ([2x]-[x]) = 1$

※ As for the interval $0 \leqq x < 1$,
when $0 \leqq x < \frac{1}{2}$, $[2x]=0$
when $\frac{1}{2} \leqq x < 1$, $[2x]=1$
Therefore, $[2x]$ is not defined.
Thus, it is necessary to consider an interval where $[2x]$ is defined.

N108a

KUMON

Limits of Functions 1

Name ____________________

Date ____ / ____ / ____

Time ____ : ____ to ____ : ____

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Find the constants a and b which satisfy each given equation.

Ex.

$$\lim_{x\to 4}\frac{ax+b}{\sqrt{x}-2}=8 \quad \cdots①$$

[Sol] Since $\lim_{x\to 4}(\sqrt{x}-2)=0$,

$\lim_{x\to 4}(ax+b)=0$, i.e. $4a+b=0$

As $x\to 4$, the denominator $\to 0$; therefore, unless the numerator $\to 0$, the limit is either ∞ or $-\infty$, not 8.

$\therefore\ b=-4a \quad \cdots②$

Then,

$$\lim_{x\to 4}\frac{ax+b}{\sqrt{x}-2}=\lim_{x\to 4}\frac{a(x-4)}{\sqrt{x}-2}$$

$$=\lim_{x\to 4}\frac{a(x-4)(\sqrt{x}+2)}{(\sqrt{x}-2)(\sqrt{x}+2)}$$

Multiplying the numerator and the denominator by $\sqrt{x}+2$

$$=\lim_{x\to 4}\frac{a(x-4)(\sqrt{x}+2)}{x-4}$$

$$=\lim_{x\to 4}a(\sqrt{x}+2)$$

$$=4a \quad \cdots③$$

From ① and ③, $4a=\boxed{8}$; therefore, $a=\boxed{2} \quad \cdots④$

From ② and ④, $b=\boxed{-8}$

$\therefore\ a=\boxed{2}$, $b=\boxed{-8}$

8, 2, −8, 2, −8

$$\frac{\sqrt{x-1}+b}{x-2}=3 \quad \cdots ①$$

$\lim_{x\to 2}(x-2)=0,$

$a\sqrt{x-1}+b)=0$, i.e. $a+b=0$

$b=-a \quad \cdots ②$

n,

$$\lim_{x\to 2}\frac{a\sqrt{x-1}+b}{x-2}=\lim_{x\to 2}\frac{a(\sqrt{x-1}-1)}{x-2}$$

$$=\lim_{x\to 2}\frac{a(\sqrt{x-1}-1)(\sqrt{x-1}+1)}{(x-2)(\sqrt{x-1}+1)}$$

$$=\lim_{x\to 2}\frac{a(x-2)}{(x-2)(\sqrt{x-1}+1)}$$

$$=\lim_{x\to 2}\frac{a}{\sqrt{x-1}+1}$$

$$=\frac{a}{2} \quad \cdots ③$$

rom ① and ③, $\frac{a}{2}=3$; therefore, $a=6 \quad \cdots ④$

rom ② and ④, $b=-6$

$\therefore a=6,\ b=-6$

e Summary

en the functions $f(x)$, $g(x)$ and constant α,

n $\lim_{x\to a}\frac{f(x)}{g(x)}=\alpha$ and also $\lim_{x\to a}g(x)=0$,

$\lim_{x\to a}f(x)=\lim_{x\to a}\left[\frac{f(x)}{g(x)}\cdot g(x)\right]=\alpha\cdot 0=0$

erefore, if $\lim_{x\to a}\frac{f(x)}{g(x)}=\alpha$ and $\lim_{x\to a}g(x)=0$, then $\lim_{x\to a}f(x)=0$

N109a

KUMON

Limits of Functions 1

Name ______________________

Date ____ / ____ / ____

Time ____ : ____ to ____ : ____

100%	~90%	~80%	~70%	69%~
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1. Given $\displaystyle\lim_{x\to 0}\frac{\sqrt{1+x+x^2}-(1+ax)}{x^2}=b$ $(a\neq 0)$, find the values of a and b.

[Sol] $\displaystyle\lim_{x\to 0}\frac{\sqrt{1+x+x^2}-(1+ax)}{x^2}$

$$=\lim_{x\to 0}\frac{\left[\sqrt{1+x+x^2}-(1+ax)\right]\left[\sqrt{1+x+x^2}+(1+ax)\right]}{x^2\left[\sqrt{1+x+x^2}+(1+ax)\right]}$$

Multiplying the numerator and the denominator by $\sqrt{1+x+x^2}+(1+ax)$

$$=\lim_{x\to 0}\frac{(1-a^2)x^2+(1-2a)x}{x^2(\sqrt{1+x+x^2}+1+ax)}$$

$$=\lim_{x\to 0}\frac{(1-a^2)x+1-2a}{x(\sqrt{1+x+x^2}+1+ax)}$$

Since $\displaystyle\lim_{x\to 0}x(\sqrt{1+x+x^2}+1+ax)=0$,

$\displaystyle\lim_{x\to 0}[(1-a^2)x+1-2a]=0$, i.e. $1-2a=0$

$\therefore\ a=\dfrac{1}{2}$

Then,

$$\lim_{x\to 0}\frac{(1-a^2)x+1-2a}{x(\sqrt{1+x+x^2}+1+ax)}=\lim_{x\to 0}\frac{\left[1-\left(\frac{1}{2}\right)^2\right]x+1-2\cdot\frac{1}{2}}{x\left(\sqrt{1+x+x^2}+1+\frac{1}{2}x\right)}$$

$$=\lim_{x\to 0}\frac{3}{4\left(\sqrt{1+x+x^2}+1+\frac{1}{2}x\right)}$$

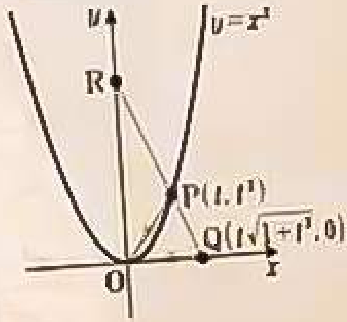
$$=\frac{3}{8}$$

$\therefore\ b=\dfrac{3}{8}$

$\therefore\ a=\dfrac{1}{2},\ b=\dfrac{3}{8}$

1109b

When point P on curve $y=x^2$ and point Q on the positive x-axis move while satisfying OP=OQ, let R be the point where line PQ intersects with the y-axis. Given that point P is in the 1st Quadrant and approaches origin O, find the coordinates of the point which point R approaches.



[Sol] Let (t, t^2) be the coordinates of point P. $(t>0)$

$$OP=\sqrt{t^2+(t^2)^2}=\sqrt{t^2+t^4}=t\sqrt{1+t^2}$$

Since OP=OQ, the coordinates of point Q are $(t\sqrt{1+t^2}, 0)$.

Therefore, the equation of line PQ is

$$y-t^2=\frac{0-t^2}{t\sqrt{1+t^2}-t}(x-t) \quad \leftarrow$$ Equation of a Line II (M I I)

$\leftarrow$ Since point R lies on the y-axis, the x-coordinate is 0.

When $x=0$,

$$y=t^2+\frac{t^2}{\sqrt{1+t^2}-1}=\frac{t^2\sqrt{1+t^2}}{\sqrt{1+t^2}-1}$$

Thus, the coordinates of point R are $\left(0, \dfrac{t^2\sqrt{1+t^2}}{\sqrt{1+t^2}-1}\right)$.

Then,

$$\lim_{t\to 0^+}\frac{t^2\sqrt{1+t^2}}{\sqrt{1+t^2}-1}=\lim_{t\to 0^+}\frac{t^2\sqrt{1+t^2}(\sqrt{1+t^2}+1)}{(\sqrt{1+t^2}-1)(\sqrt{1+t^2}+1)}$$ $\leftarrow$ When point P in the 1st Quadrant approaches the origin, $t\to 0^+$

$$=\lim_{t\to 0^+}\frac{t^2\sqrt{1+t^2}(\sqrt{1+t^2}+1)}{t^2}$$

$$=\lim_{t\to 0^+}\sqrt{1+t^2}(\sqrt{1+t^2}+1)$$

$$=2$$

the coordinates of the point which point R approaches are (0, 2).

N110a

KUMON

Limits of Functions 1

Name ______________________

Date   /   /

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0	—	—	1	2

1. Find the following limits.

➡ N10

(1) $\displaystyle \lim_{x\to1}\frac{2x^2+x-3}{x^2+2x-3}=\lim_{x\to1}\frac{(2x+3)(x-1)}{(x+3)(x-1)}$

$\displaystyle =\lim_{x\to1}\frac{2x+3}{x+3}$

$\displaystyle =\frac{5}{4}$

➡ N1(

(2) $\displaystyle \lim_{x\to-3}\frac{\sqrt{x+7}-2}{x+3}=\lim_{x\to-3}\frac{(\sqrt{x+7}-2)(\sqrt{x+7}+2)}{(x+3)(\sqrt{x+7}+2)}$

$\displaystyle =\lim_{x\to-3}\frac{x+3}{(x+3)(\sqrt{x+7}+2)}$

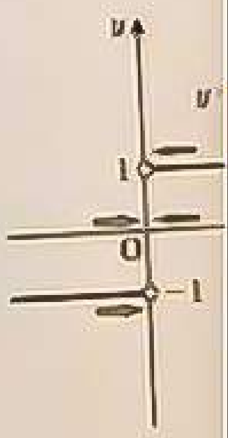
$\displaystyle =\lim_{x\to-3}\frac{1}{\sqrt{x+7}+2}$

$\displaystyle =\frac{1}{4}$

2. Determine if the following limit exists. If it exists, find the limit.

➡ N

$\displaystyle \lim_{x\to0}\frac{|x|}{x}$



[Sol] (i) When $x>0$, $\dfrac{|x|}{x}=\dfrac{x}{x}=1$

$\therefore\ \displaystyle \lim_{x\to0+}\frac{|x|}{x}=1$

(ii) When $x<0$, $\dfrac{|x|}{x}=\dfrac{-x}{x}=-1$

$\therefore\ \displaystyle \lim_{x\to0-}\frac{|x|}{x}=-1$

From (i) and (ii), $\displaystyle \lim_{x\to0}\frac{|x|}{x}$ does not exist.

3. Find the constants a and b which satisfy the following equation. ➡ N108

$$\lim_{x\to 1}\frac{a\sqrt{x+1}-b}{x-1}=\sqrt{2}\quad\cdots①$$

[Sol] Since $\lim_{x\to 1}(x-1)=0$,

$$\lim_{x\to 1}(a\sqrt{x+1}-b)=0\text{, i.e. }\sqrt{2}a-b=0$$

$$\therefore\ b=\sqrt{2}a\quad\cdots②$$

Then,

$$\begin{aligned}\lim_{x\to 1}\frac{a\sqrt{x+1}-b}{x-1}&=\lim_{x\to 1}\frac{a(\sqrt{x+1}-\sqrt{2})}{x-1}\\&=\lim_{x\to 1}\frac{a(\sqrt{x+1}-\sqrt{2})(\sqrt{x+1}+\sqrt{2})}{(x-1)(\sqrt{x+1}+\sqrt{2})}\\&=\lim_{x\to 1}\frac{a(x-1)}{(x-1)(\sqrt{x+1}+\sqrt{2})}\\&=\lim_{x\to 1}\frac{a}{\sqrt{x+1}+\sqrt{2}}\\&=\frac{a}{2\sqrt{2}}\\&=\frac{\sqrt{2}}{4}a\quad\cdots③\end{aligned}$$

From ① and ③, $\frac{\sqrt{2}}{4}a=\sqrt{2}$; therefore, $a=4\quad\cdots④$

From ② and ④, $b=4\sqrt{2}$

$$\therefore\ a=4,\ b=4\sqrt{2}$$

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KUMON

Limits of Functions 2

Name ______________________

Date ______ / ______ / ______

Time ______ : ______ to ______ : ______

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Find the following limits.

Ex.

$$\lim_{x\to\infty}\frac{x+1}{x^2-3}=\lim_{x\to\infty}\frac{\frac{1}{x}+\frac{1}{x^2}}{1-\frac{3}{x^2}}=0$$

← Dividing the numerator and the denominator by the term with the highest power in the denominator (x^2 in this case)

(1) $\lim_{x\to\infty}\frac{2x+1}{x^3+x-1}=\lim_{x\to\infty}\frac{\frac{2}{x^2}+\frac{1}{x^3}}{1+\frac{1}{x^2}-\frac{1}{x^3}}=0$

(2) $\lim_{x\to\infty}\frac{2x^2-x-6}{3x^2-2x-8}=\lim_{x\to\infty}\frac{2-\frac{1}{x}-\frac{6}{x^2}}{3-\frac{2}{x}-\frac{8}{x^2}}=\frac{2}{3}$

(3) $\lim_{x\to\infty}\frac{5-7x+x^3}{4+3x^2-5x^3}=\lim_{x\to\infty}\frac{\frac{5}{x^3}-\frac{7}{x^2}+1}{\frac{4}{x^3}+\frac{3}{x}-5}=-\frac{1}{5}$

(4) $\lim_{x\to\infty}\frac{x^2+1}{x-1}=\lim_{x\to\infty}\frac{x+\frac{1}{x}}{1-\frac{1}{x}}=\infty$

(5) $\lim_{x\to\infty}\frac{x-4x^2}{3+x}=\lim_{x\to\infty}\frac{1-4x}{\frac{3}{x}+1}=-\infty$

← $\lim_{x\to\infty}(1-4x)=-\infty$

$\lim_{x\to\infty}\left(\frac{3}{x}+1\right)=1$

NIIIb

(6) $\lim_{x\to\infty}\dfrac{x}{x+\sqrt{x^2+1}}=\lim_{x\to\infty}\dfrac{1}{1+\sqrt{1+\dfrac{1}{x^2}}}=\dfrac{1}{2}$ ←

Dividing the denominator $x+\sqrt{x^2+1}$ by x.

$$\frac{x+\sqrt{x^2+1}}{x}=1+\sqrt{\frac{x^2+1}{x^2}}$$
$$=1+\sqrt{\frac{x^2}{x^2}+\frac{1}{x^2}}$$

(7) $\lim_{x\to\infty}\dfrac{x+\sqrt{5x^2+4}}{-2x+1}=\lim_{x\to\infty}\dfrac{1+\sqrt{5+\dfrac{4}{x^2}}}{-2+\dfrac{1}{x}}=-\dfrac{1+\sqrt{5}}{2}$

(8) $\lim_{x\to\infty}\dfrac{3x^2+\sqrt{x^3+2}}{x^2-x+1}=\lim_{x\to\infty}\dfrac{3+\sqrt{\dfrac{1}{x}+\dfrac{2}{x^4}}}{1-\dfrac{1}{x}+\dfrac{1}{x^2}}=3$

(9) $\lim_{x\to\infty}\dfrac{\sqrt{x+1}+\sqrt{3x-1}}{\sqrt{x-1}}=\lim_{x\to\infty}\dfrac{\sqrt{1+\dfrac{1}{x}}+\sqrt{3-\dfrac{1}{x}}}{\sqrt{1-\dfrac{1}{x}}}=1+\sqrt{3}$ ←

Dividing the numerator and the denominator by $\sqrt{x}$

(10) $\lim_{x\to\infty}(x^2-3x+2)=\lim_{x\to\infty}x^2\left(1-\dfrac{3}{x}+\dfrac{2}{x^2}\right)=\infty$ ←

Factoring by the term with the highest power

$\lim_{x\to\infty}(4+3x-2x^2-x^3)=\lim_{x\to\infty}x^3\left(\dfrac{4}{x^3}+\dfrac{3}{x^2}-\dfrac{2}{x}-1\right)=-\infty$ ←

$\lim_{x\to\infty}x^3=\infty$

$\lim_{x\to\infty}\left(\dfrac{4}{x^3}+\dfrac{3}{x^2}-\dfrac{2}{x}-1\right)=-1$

N 112a

KUMON

Limits of Functions 2

Name

Date　　/　　/

Time　　:　　to　　:

100%	~90%	~80%	~70%	69%~
0	—	—	1	2~

1. Find the following limits.

Ex.

$$\lim_{x\to\infty}(\sqrt{x^2+2x}-x)=\lim_{x\to\infty}\frac{(\sqrt{x^2+2x}-x)(\sqrt{x^2+2x}+x)}{\sqrt{x^2+2x}+x}$$

$$=\lim_{x\to\infty}\frac{2x}{\sqrt{x^2+2x}+x}$$

$$=\lim_{x\to\infty}\frac{2}{\sqrt{1+\frac{2}{x}}+1}$$

$$=1$$

Considering $\sqrt{x^2+2x}-x=\frac{\sqrt{x^2+2x}-x}{1}$, then multiplying the numerator and the denominator by $\sqrt{x^2+2x}+x$

(1) $$\lim_{x\to\infty}(\sqrt{x^2-x}-x)=\lim_{x\to\infty}\frac{(\sqrt{x^2-x}-x)(\sqrt{x^2-x}+x)}{\sqrt{x^2-x}+x}$$

$$=\lim_{x\to\infty}\frac{-x}{\sqrt{x^2-x}+x}$$

$$=\lim_{x\to\infty}\frac{-1}{\sqrt{1-\frac{1}{x}}+1}$$

$$=-\frac{1}{2}$$

(2) $$\lim_{x\to\infty}(\sqrt{x^2+2x}-\sqrt{x^2+1})=\lim_{x\to\infty}\frac{(\sqrt{x^2+2x}-\sqrt{x^2+1})(\sqrt{x^2+2x}+\sqrt{x^2+1})}{\sqrt{x^2+2x}+\sqrt{x^2+1}}$$

$$=\lim_{x\to\infty}\frac{2x-1}{\sqrt{x^2+2x}+\sqrt{x^2+1}}$$

$$=\lim_{x\to\infty}\frac{2-\frac{1}{x}}{\sqrt{1+\frac{2}{x}}+\sqrt{1+\frac{1}{x^2}}}$$

$$=1$$

N112b

(3) $\lim_{x\to\infty}\left(2x+1-\sqrt{4x^2+2x+1}\right)$

$=\lim_{x\to\infty}\dfrac{\left(2x+1-\sqrt{4x^2+2x+1}\right)\left(2x+1+\sqrt{4x^2+2x+1}\right)}{2x+1+\sqrt{4x^2+2x+1}}$

$=\lim_{x\to\infty}\dfrac{2x}{2x+1+\sqrt{4x^2+2x+1}}$

$=\lim_{x\to\infty}\dfrac{2}{2+\frac{1}{x}+\sqrt{4+\frac{2}{x}+\frac{1}{x^2}}}$

$=\dfrac{1}{2}$

Find the constant a which satisfies $\lim_{x\to\infty}\left(\sqrt{x^2+ax+5}-\sqrt{x^2-3}\right)=4$

$\lim_{x\to\infty}\left(\sqrt{x^2+ax+5}-\sqrt{x^2-3}\right)$

$=\lim_{x\to\infty}\dfrac{\left(\sqrt{x^2+ax+5}-\sqrt{x^2-3}\right)\left(\sqrt{x^2+ax+5}+\sqrt{x^2-3}\right)}{\sqrt{x^2+ax+5}+\sqrt{x^2-3}}$

$=\lim_{x\to\infty}\dfrac{ax+8}{\sqrt{x^2+ax+5}+\sqrt{x^2-3}}$

$=\lim_{x\to\infty}\dfrac{a+\frac{8}{x}}{\sqrt{1+\frac{a}{x}+\frac{5}{x^2}}+\sqrt{1-\frac{3}{x^2}}}$

$=\dfrac{a}{2}$

$\therefore\ \dfrac{a}{2}=4$

$\therefore\ a=8$

[illegible]

Limits of Functions 2

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Find the following limits.

Ex. $\lim_{x\to-\infty}(4x^5+x^2+4)$

[Sol] Let $x=-t$. As $x\to-\infty$, $t\to\infty$; therefore,

$$\lim_{x\to-\infty}(4x^5+x^2+4)=\lim_{t\to\infty}(-4t^5+t^2+4)$$

$x^5=(-t)^5=$
$x^2=(-t)^2=t$

$$=\lim_{t\to\infty}t^5\left(-4+\frac{1}{t^3}+\frac{4}{t^5}\right)$$

$$=-\infty$$

(1) $\lim_{x\to-\infty}(2x^4-x^3+2x-1)$

[Sol] Let $x=-t$. As $x\to-\infty$, $t\to\infty$; therefore,

$$\lim_{x\to-\infty}(2x^4-x^3+2x-1)=\lim_{t\to\infty}(2t^4+t^3-2t-1)$$

$$=\lim_{t\to\infty}t^4\left(2+\frac{1}{t}-\frac{2}{t^3}-\frac{1}{t^4}\right)$$

$$=\infty$$

(2) $\lim_{x\to-\infty}\frac{4x^2-x+2}{2x-3}$

[Sol] Let $x=-t$. As $x\to-\infty$, $t\to\infty$; therefore,

$$\lim_{x\to-\infty}\frac{4x^2-x+2}{2x-3}=\lim_{t\to\infty}\frac{4t^2+t+2}{-2t-3}$$

$$=\lim_{t\to\infty}\frac{4t+1+\frac{2}{t}}{-2-\frac{3}{t}}$$

$$=-\infty$$

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$\frac{\sqrt{1+x^2}-1}{x}$

t $x=-t$. As $x\to-\infty$, $t\to\infty$; therefore,

$$\lim_{x\to-\infty}\frac{\sqrt{1+x^2}-1}{x}=\lim_{t\to\infty}\frac{\sqrt{1+t^2}-1}{-t}$$
$$=\lim_{t\to\infty}\left(-\sqrt{\frac{1}{t^2}+1}+\frac{1}{t}\right)$$
$$=-1$$

$\lim_{x\to-\infty}\left(\sqrt{x^2+3x-1}-\sqrt{x^2-x+1}\right)$

Let $x=-t$. As $x\to-\infty$, $t\to\infty$; therefore,

$$\lim_{x\to-\infty}\left(\sqrt{x^2+3x-1}-\sqrt{x^2-x+1}\right)$$
$$=\lim_{t\to\infty}\left(\sqrt{t^2-3t-1}-\sqrt{t^2+t+1}\right)$$
$$=\lim_{t\to\infty}\frac{\left(\sqrt{t^2-3t-1}-\sqrt{t^2+t+1}\right)\left(\sqrt{t^2-3t-1}+\sqrt{t^2+t+1}\right)}{\sqrt{t^2-3t-1}+\sqrt{t^2+t+1}}$$
$$=\lim_{t\to\infty}\frac{-4t-2}{\sqrt{t^2-3t-1}+\sqrt{t^2+t+1}}$$
$$=\lim_{t\to\infty}\frac{-4-\frac{2}{t}}{\sqrt{1-\frac{3}{t}-\frac{1}{t^2}}+\sqrt{1+\frac{1}{t}+\frac{1}{t^2}}}$$
$$=-2$$

Considering $\sqrt{t^2-3t-1}-\sqrt{t^2+t+1}=\frac{\sqrt{t^2-3t-1}-\sqrt{t^2+t+1}}{1}$, then multiplying the numerator and the denominator by $\sqrt{t^2-3t-1}+\sqrt{t^2+t+1}$

Note Summary

When $x\to-\infty$, it is ...ier to find the answer by letting $x=-t$ and considering the case when ... $\sqrt{x^2}=-x$ when $x\leq 0$, and finding the correct answer bec...

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Limits of Functions 2

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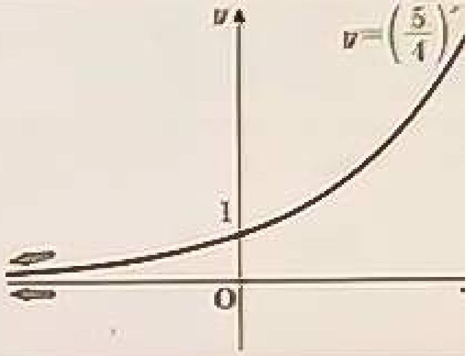
0	—	1	2	3~

Find the following limits.

Ex.

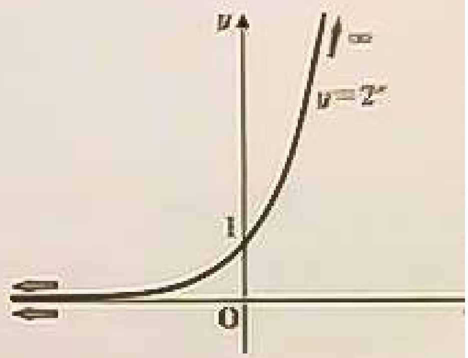
$$\lim_{x\to\infty}\left(\frac{5}{4}\right)^x=\infty$$

$$\lim_{x\to-\infty}\left(\frac{5}{4}\right)^x=0$$



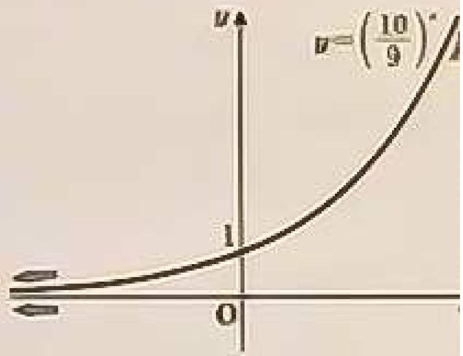
(1) $\lim_{x\to\infty}2^x=\infty$

(2) $\lim_{x\to-\infty}2^x=0$



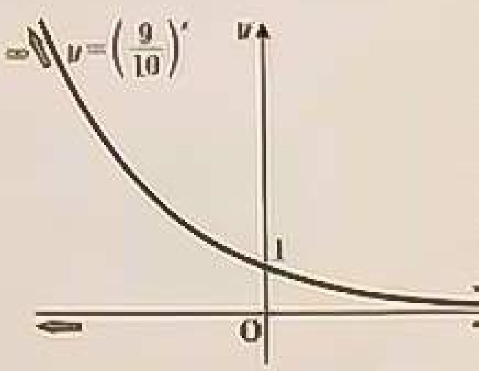
(3) $\lim_{x\to\infty}\left(\frac{10}{9}\right)^x=\infty$

(4) $\lim_{x\to-\infty}\left(\frac{10}{9}\right)^x=0$



(5) $\lim_{x\to\infty}\left(\frac{9}{10}\right)^x=0$

(6) $\lim_{x\to-\infty}\left(\frac{9}{10}\right)^x=\infty$



N114b

(7) $\lim_{x \to \infty} a^x \quad (a > 0)$

[Sol] (i) When $a > 1$,

$$\lim_{x \to \infty} a^x = \boxed{\infty}$$

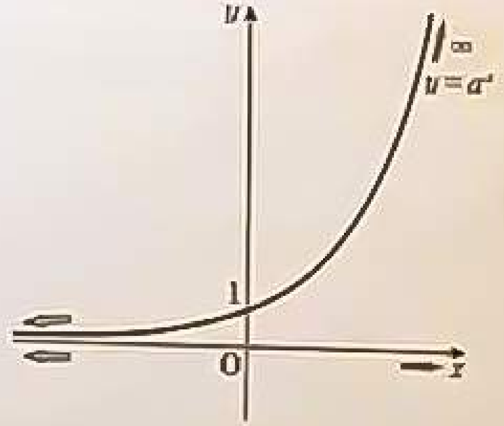
(ii) When $a = 1$,

$$\lim_{x \to \infty} a^x = \boxed{1}$$

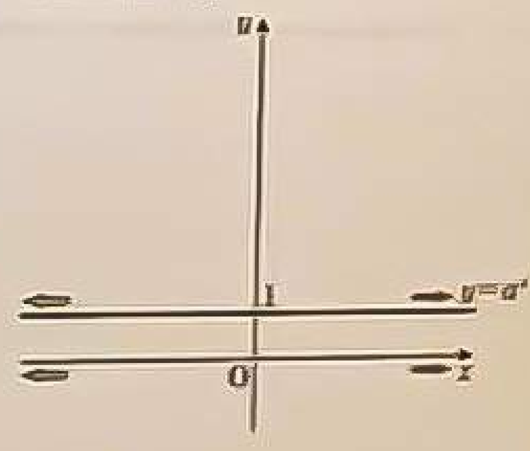
(iii) When $0 < a < 1$,

$$\lim_{x \to \infty} a^x = \boxed{0}$$

[When $a > 1$]



[When $a = 1$]



(8) $\lim_{x \to -\infty} a^x \quad (a > 0)$

[Sol] (i) When $a > 1$,

$$\lim_{x \to -\infty} a^x = 0$$

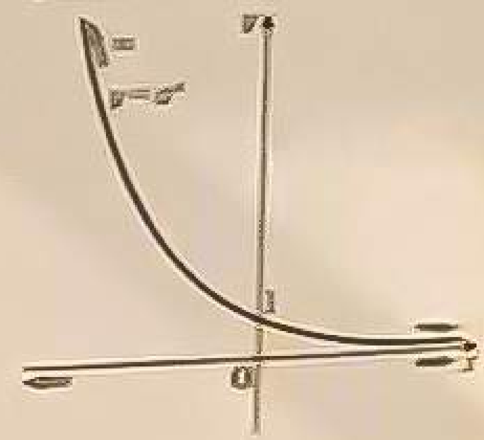
(ii) When $a = 1$,

$$\lim_{x \to -\infty} a^x = 1$$

(iii) When $0 < a < 1$,

$$\lim_{x \to -\infty} a^x = \infty$$

[When $0 < a < 1$]



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Limits of Functions 2

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Find the following limits.

Ex.
$$\lim_{x\to\infty}(5^x-3^x)=\lim_{x\to\infty}5^x\left[1-\left(\frac{3}{5}\right)^x\right]=\infty \leftarrow$$

Factoring by the term with the largest absolute value of the b

(1) $\lim_{x\to\infty}(3^x-2^x)=\lim_{x\to\infty}3^x\left[1-\left(\frac{2}{3}\right)^x\right]=\infty$

(2) $\lim_{x\to\infty}(10^x-3^x)=\lim_{x\to\infty}10^x\left[1-\left(\frac{3}{10}\right)^x\right]=\infty$

(3) $\lim_{x\to\infty}(2^x-5^x)=\lim_{x\to\infty}5^x\left[\left(\frac{2}{5}\right)^x-1\right]=-\infty \leftarrow$

$\lim_{x\to\infty}5^x=\infty$

$\lim_{x\to\infty}\left[\left(\frac{2}{5}\right)^x-1\right]=-1$

(4) $\lim_{x\to\infty}(3^x-2^{2x})=\lim_{x\to\infty}(3^x-4^x)=\lim_{x\to\infty}4^x\left[\left(\frac{3}{4}\right)^x-1\right]=-\infty$

(5) $\lim_{x\to\infty}(2^{2x}-3^x)=\lim_{x\to\infty}(4^x-3^x)=\lim_{x\to\infty}4^x\left[1-\left(\frac{3}{4}\right)^x\right]=\infty$

(6) $\lim_{x\to\infty}(5^x-2^{-x})=\lim_{x\to\infty}\left[5^x-\left(\frac{1}{2}\right)^x\right]=\infty \leftarrow$

$\lim_{x\to\infty}5^x=\infty$

$\lim_{x\to\infty}\left(\frac{1}{2}\right)^x=0$

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x.

$$\lim_{x\to\infty}\frac{2^{x-1}}{1+2^x}=\lim_{x\to\infty}\frac{\frac{1}{2}}{\left(\frac{1}{2}\right)^x+1}=\frac{1}{2}$$

← Dividing the numerator and the denominator by the term with the largest absolute value of the base in the denominator (2^x in this case)

$$\lim_{x\to\infty}\frac{4^x-1}{4^x+1}=\lim_{x\to\infty}\frac{1-\left(\frac{1}{4}\right)^x}{1+\left(\frac{1}{4}\right)^x}=1$$

$$\lim_{x\to\infty}\frac{2^x-3^{x+2}}{2^x+3^x}=\lim_{x\to\infty}\frac{\left(\frac{2}{3}\right)^x-9}{\left(\frac{2}{3}\right)^x+1}=-9$$

$$\lim_{x\to\infty}\frac{\left(\frac{2}{3}\right)^x-\left(\frac{2}{3}\right)^{-x}}{\left(\frac{2}{3}\right)^x+\left(\frac{2}{3}\right)^{-x}}=\lim_{x\to\infty}\frac{\left(\frac{2}{3}\right)^x-\left(\frac{3}{2}\right)^x}{\left(\frac{2}{3}\right)^x+\left(\frac{3}{2}\right)^x}=\lim_{x\to\infty}\frac{\left(\frac{2}{3}\right)^{2x}-1}{\left(\frac{2}{3}\right)^{2x}+1}=-1$$

Since $\left(\frac{2}{3}\right)^{-x}=\left(\frac{3}{2}\right)^x$, dividing the numerator and the denominator by $\left(\frac{3}{2}\right)^x$

$$\lim_{x\to\infty}\frac{a^x+1}{a^x+a}\quad (a>0)$$

(i) When $a>1$,

$$\lim_{x\to\infty}\frac{a^x+1}{a^x+a}=\lim_{x\to\infty}\frac{1+\left(\frac{1}{a}\right)^x}{1+\left(\frac{1}{a}\right)^{x-1}}=1$$

← $\lim_{x\to\infty}\left(\frac{1}{a}\right)^x=0$

(ii) When $a=1$,

$$\lim_{x\to\infty}\frac{a^x+1}{a^x+a}=1$$

← $\lim_{x\to\infty}a^x=1$

(iii) When $0<a<1$,

$$\lim_{x\to\infty}\frac{a^x+1}{a^x+a}=\frac{1}{a}$$

← $\lim_{x\to\infty}a^x=0$

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Limits of Functions 2

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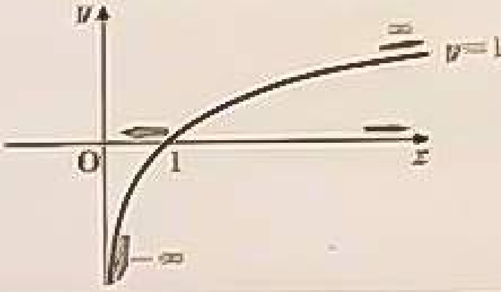
0	—	1	2	3~

Find the following limits.

Ex.

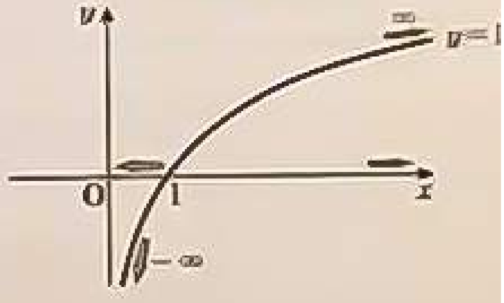
$\lim_{x \to \infty} \log_3 x = \infty$

$\lim_{x \to 0^+} \log_3 x = -\infty$



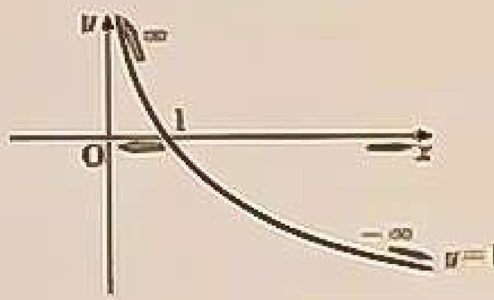
(1) $\lim_{x \to \infty} \log_2 x = \infty$

(2) $\lim_{x \to 0^+} \log_2 x = -\infty$



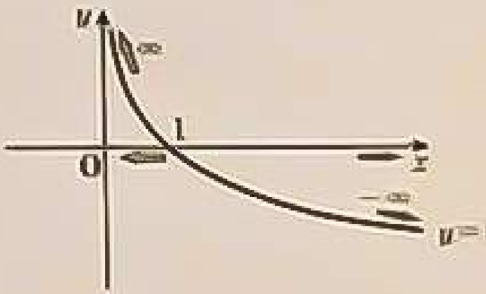
(3) $\lim_{x \to \infty} \log_{\frac{1}{2}} x = -\infty$

(4) $\lim_{x \to 0^+} \log_{\frac{1}{2}} x = \infty$



(5) $\lim_{x \to \infty} \log_{\frac{1}{3}} x = -\infty$

(6) $\lim_{x \to 0^+} \log_{\frac{1}{3}} x = \infty$



【Reference】L 15-16 When $a^y = x$, $y = \log_a x$

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) $\lim_{x\to\infty} \log_a x \quad (a>0,\ a\neq 1)$

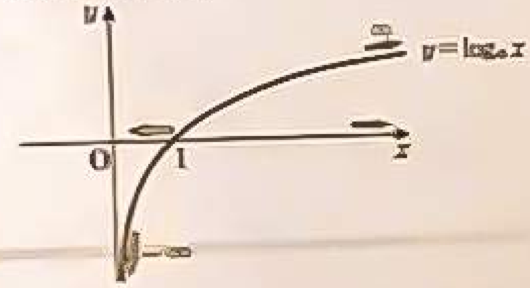
[Sol] (i) When $a>1$,

$$\lim_{x\to\infty} \log_a x = \boxed{\infty}$$

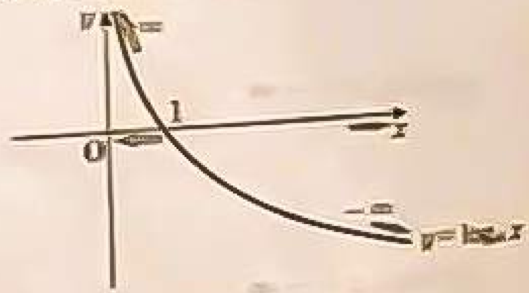
(ii) When $0<a<1$,

$$\lim_{x\to\infty} \log_a x = \boxed{-\infty}$$

[When $a>1$]



[When $0<a<1$]



(8) $\lim_{x\to 0+} \log_a x \quad (a>0,\ a\neq 1)$

[Sol] (i) When $a>1$,

$$\lim_{x\to 0+} \log_a x = -\infty$$

(ii) When $0<a<1$,

$$\lim_{x\to 0+} \log_a x = \infty$$

Limits of Functions 2

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Time : to :

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Find the following limits.

Ex.

$$\lim_{x\to\infty}[\log_2(4x+1)-\log_2(x-1)]=\lim_{x\to\infty}\log_2\frac{4x+1}{x-1}$$

$\log_a M-\log_a N=\log_a\frac{M}{N}$

$$=\lim_{x\to\infty}\log_2\frac{4+\frac{1}{x}}{1-\frac{1}{x}}$$

$$=\log_2 4$$

$$=2$$

$\log_a a^m=m$

(1) $$\lim_{x\to\infty}[\log_4(8x-2)-\log_4(2x+3)]=\lim_{x\to\infty}\log_4\frac{8x-2}{2x+3}$$

$$=\lim_{x\to\infty}\log_4\frac{8-\frac{2}{x}}{2+\frac{3}{x}}$$

$$=\log_4 4$$

$$=1$$

$\log_a a=1$

(2) $$\lim_{x\to\infty}\left[\log_{\frac{1}{3}}(x+1)-\log_{\frac{1}{3}}x\right]=\lim_{x\to\infty}\log_{\frac{1}{3}}\frac{x+1}{x}$$

$$=\lim_{x\to\infty}\log_{\frac{1}{3}}\left(1+\frac{1}{x}\right)$$

$$=\log_{\frac{1}{3}}1$$

$$=0$$

$\log_a 1=0$

$\log_a a=1,\quad \log_a 1=0,\quad \log_a M-\log_a N=\log_a\frac{M}{N}$

$$\lim_{x\to\infty}(\log_2\sqrt{2x^2+1}-\log_2 x)$$
$$=\lim_{x\to\infty}\log_2\frac{\sqrt{2x^2+1}}{x}$$
$$=\lim_{x\to\infty}\log_2\sqrt{2+\frac{1}{x^2}}$$
$$=\log_2\sqrt{2}$$
$$=\frac{1}{2}$$

$\log_a\sqrt[n]{M}=\frac{1}{n}\log_a M$

$$\lim_{x\to 2}(\log_3|x^2+5x-14|-\log_3|x-2|)$$
$$=\lim_{x\to 2}\log_3\left|\frac{x^2+5x-14}{x-2}\right|$$
$$=\lim_{x\to 2}\log_3\left|\frac{(x+7)(x-2)}{x-2}\right|$$
$$=\lim_{x\to 2}\log_3|x+7|$$
$$=\log_3 9$$
$$=2$$

$$\lim_{x\to\infty}\log_2(x+2-\sqrt{x^2+3x+4})$$
$$=\lim_{x\to\infty}\log_2\frac{(x+2-\sqrt{x^2+3x+4})(x+2+\sqrt{x^2+3x+4})}{x+2+\sqrt{x^2+3x+4}}$$
$$=\lim_{x\to\infty}\log_2\frac{x}{x+2+\sqrt{x^2+3x+4}}$$
$$=\lim_{x\to\infty}\log_2\frac{1}{1+\frac{2}{x}+\sqrt{1+\frac{3}{x}+\frac{4}{x^2}}}$$
$$=\log_2\frac{1}{2}$$
$$=-1$$

Considering $x+2-\sqrt{x^2+3x+4}=\frac{x+2-\sqrt{x^2+3x+4}}{1}$, then multiplying the numerator and the denominator of the antilogarithm by $x+2+\sqrt{x^2+3x+4}$

$\log_a\frac{1}{N}=-\log_a N$

$\log_a\sqrt[n]{M}=\frac{1}{n}\log_a M$, $\log_a\frac{1}{N}=-\log_a N$

Limits of Functions 2

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0	–	–	–	1~

Ex.

Find the cubic function $f(x)$ which satisfies $\lim\limits_{x\to\infty}\dfrac{f(x)-2x^3+3}{x^2}=4$

and $\lim\limits_{x\to 0}\dfrac{f(x)-5}{x}=3$ …②.

[Sol] Let $f(x)=ax^3+bx^2+cx+d$ $(a\neq 0)$.

From ①,

$$\lim_{x\to\infty}\frac{f(x)-2x^3+3}{x^2}=\lim_{x\to\infty}\frac{(a-2)x^3+bx^2+cx+d+3}{x^2}$$

$$=\lim_{x\to\infty}\left[(a-2)x+b+\frac{c}{x}+\frac{d+3}{x^2}\right]$$

$$=4$$

Therefore, the values must be $a=2$ and $b=4$.

If $a\neq 2$, $\lim\limits_{x\to\infty}\dfrac{f(x)-2x}{x^2}$ does not conve

$\therefore\ f(x)=2x^3+4x^2+cx+d$

From ②, $\lim\limits_{x\to 0}\dfrac{2x^3+4x^2+cx+d-5}{x}=3$ …③

Since $\lim\limits_{x\to 0}x=0$,

$\lim\limits_{x\to 0}(2x^3+4x^2+cx+d-5)=0$, i.e. $d-5=0$ ← NI08

$\therefore\ d=\boxed{5}$

Then,

$$\lim_{x\to 0}\frac{2x^3+4x^2+cx+d-5}{x}=\lim_{x\to 0}\frac{\boxed{2x^3+4x^2+cx}}{x}$$

$$=\lim_{x\to 0}(\boxed{2x^2+4x+c})$$

$$=c \quad \cdots ④$$

From ③ and ④, $c=\boxed{3}$

$\therefore\ f(x)=\boxed{2x^3+4x^2+3x+5}$

Ex. 5. $2x^3+4x^2+cx$. $2x^2+4x+c$. 3. $2x^3+4x^2+3x+5$

N118b

1. Find the cubic function $f(x)$ which satisfies $\lim_{x\to\infty}\frac{f(x)-x^3}{x^2}=1$ …① and $\lim_{x\to1}\frac{f(x)}{x-1}=-3$ …②.

[Sol] Let $f(x)=ax^3+bx^2+cx+d$ $(a\neq0)$.

From ①,

$$\lim_{x\to\infty}\frac{f(x)-x^3}{x^2}=\lim_{x\to\infty}\frac{(a-1)x^3+bx^2+cx+d}{x^2}$$

$$=\lim_{x\to\infty}\left[(a-1)x+b+\frac{c}{x}+\frac{d}{x^2}\right]$$

$$=1$$

Therefore, the values must be $a=1$ and $b=1$.

$\therefore f(x)=x^3+x^2+cx+d$

From ②, $\lim_{x\to1}\frac{x^3+x^2+cx+d}{x-1}=-3$ …③

Since $\lim_{x\to1}(x-1)=0$,

$\lim_{x\to1}(x^3+x^2+cx+d)=0$, i.e. $2+c+d=0$

$\therefore d=-c-2$ …④

Then,

$$\lim_{x\to1}\frac{x^3+x^2+cx+d}{x-1}=\lim_{x\to1}\frac{x^3+x^2+cx-c-2}{x-1}$$

$$=\lim_{x\to1}\frac{(x-1)(x^2+2x+c+2)}{x-1}$$

$$=\lim_{x\to1}(x^2+2x+c+2)$$

$$=c+5$$ …⑤

The Factor Theorem (J172)

From ③ and ⑤, $c+5=-3$, therefore, $c=-8$ …⑥

From ④ and ⑥, $d=6$

$\therefore f(x)=x^3+x^2-8x+6$

N119a

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Limits of Functions 2

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1. Find the constants a and b which satisfy the following equation.

$$\lim_{x\to\infty}\left[\sqrt{4x^2-12x+1}-(ax+b)\right]=0 \quad \cdots①$$

[Sol] When $a \leqq 0$, $\lim_{x\to\infty}\left[\sqrt{4x^2-12x+1}-(ax+b)\right]=\infty$; therefore,

it does not satisfy ①.

$\therefore\ a>0 \quad \cdots②$

Then,

$$\lim_{x\to\infty}\left[\sqrt{4x^2-12x+1}-(ax+b)\right]$$

$$=\lim_{x\to\infty}\frac{\left[\sqrt{4x^2-12x+1}-(ax+b)\right]\left[\sqrt{4x^2-12x+1}+(ax+b)\right]}{\sqrt{4x^2-12x+1}+(ax+b)}$$

$$=\lim_{x\to\infty}\frac{(4-a^2)x^2-(12+2ab)x+1-b^2}{\sqrt{4x^2-12x+1}+(ax+b)}$$

$$=\lim_{x\to\infty}\frac{(4-a^2)x-(12+2ab)+\frac{1-b^2}{x}}{\sqrt{4-\frac{12}{x}+\frac{1}{x^2}}+a+\frac{b}{x}}$$

Considering $\sqrt{4x^2-12x+1}-(ax+b$
$=\frac{\sqrt{4x^2-12x+1}-(ax+}{1}$
then multiplying the num
and the denominator by
$\sqrt{4x^2-12x+1}+(ax+b$

From ①,

$$\begin{cases} 4-a^2=0 & \cdots③ \\ 12+2ab=0 & \cdots④ \end{cases}$$

If $4-a^2\neq0$,
$\lim_{x\to\infty}\left[\sqrt{4x^2-12x+1}-(ax+b)\right]$
does not converge.

From ② and ③, $a=2 \quad \cdots⑤$

From ④ and ⑤, $b=-3$

$\therefore\ a=2,\ b=-3$

NI19b

2. Let l be line $y=ax\ (a>0)$ and P be the point on curve $y=x^2(0\leqslant x\leqslant a)$ such that the distance to l is maximized.

(1) Find the coordinates of point $P(p,\ p^2)$.

[Sol] Let d be the distance from point $(t,\ t^2)\ (0\leqslant t\leqslant a)$ on curve $y=x^2$ to line $y=ax$.

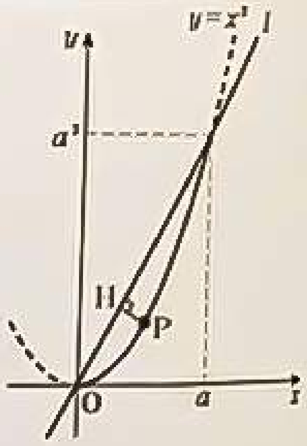
$$d=\frac{|at-t^2|}{\sqrt{a^2+(-1)^2}}$$

Distance from a Point to a Line (M25)

$$=\frac{at-t^2}{\sqrt{a^2+1}}$$

Since $0\leqslant t\leqslant a$, $|at-t^2|=at-t^2$

$$=-\frac{1}{\sqrt{a^2+1}}\left[\left(t-\frac{a}{2}\right)^2-\frac{a^2}{4}\right]$$



Therefore, when $t=\frac{a}{2}$, d is maximized.

$$\therefore\ p=\frac{a}{2}$$

$$\therefore\ P\left(\frac{a}{2},\ \frac{a^2}{4}\right)$$

(2) Let $H(h,\ ah)$ be the point of intersection of the perpendicular from P to l and l. Find $\lim\limits_{a\to\infty}\frac{h}{p}$.

[Sol] From (1), the equation of line PH is

$$y-\frac{a^2}{4}=-\frac{1}{a}\left(x-\frac{a}{2}\right)$$

Since lines PH and l are perpendicular, the slope of line PH is $-\frac{1}{a}$.

Since this line passes through point $H(h,\ ah)$,

$$ah-\frac{a^2}{4}=-\frac{1}{a}\left(h-\frac{a}{2}\right)$$

$$\therefore\ h=\frac{a(a^2+2)}{4(a^2+1)}$$

Therefore,

$$\lim_{a\to\infty}\frac{h}{p}=\lim_{a\to\infty}\frac{\dfrac{a(a^2+2)}{4(a^2+1)}}{\dfrac{a}{2}}=\lim_{a\to\infty}\frac{a^2+2}{2(a^2+1)}=\lim_{a\to\infty}\frac{1+\dfrac{2}{a^2}}{2\left(1+\dfrac{1}{a^2}\right)}=\frac{1}{2}$$

N I 20a

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Limits of Functions 2

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Name

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Find the following limits.

(1) $\lim_{x\to\infty}\frac{2x+3}{\sqrt{x^2-1}+3}=\lim_{x\to\infty}\frac{2+\frac{3}{x}}{\sqrt{1-\frac{1}{x^2}}+\frac{3}{x}}=2$

(2) $\lim_{x\to\infty}(\sqrt{4x^2+x}-2x)=\lim_{x\to\infty}\frac{(\sqrt{4x^2+x}-2x)(\sqrt{4x^2+x}+2x)}{\sqrt{4x^2+x}+2x}$

$$=\lim_{x\to\infty}\frac{x}{\sqrt{4x^2+x}+2x}$$

$$=\lim_{x\to\infty}\frac{1}{\sqrt{4+\frac{1}{x}}+2}$$

$$=\frac{1}{4}$$

(3) $\lim_{x\to-\infty}\frac{x}{\sqrt{x^2+x}+2x}$

[Sol] Let $x=-t$. As $x\to-\infty$, $t\to\infty$; therefore,

$$\lim_{x\to-\infty}\frac{x}{\sqrt{x^2+x}+2x}=\lim_{t\to\infty}\frac{-t}{\sqrt{t^2-t}-2t}$$

$$=\lim_{t\to\infty}\frac{-1}{\sqrt{1-\frac{1}{t}}-2}$$

$$=1$$

N120b

(4) $\displaystyle\lim_{x\to\infty}\frac{a^{x-1}}{1+a^x}$ $(a>0)$

➡ N1

[Sol] (i) **When $a>1$,**

$$\lim_{x\to\infty}\frac{a^{x-1}}{1+a^x}=\lim_{x\to\infty}\frac{\frac{1}{a}}{\left(\frac{1}{a}\right)^x+1}=\frac{1}{a}$$

(ii) **When $a=1$,**

$$\lim_{x\to\infty}\frac{a^{x-1}}{1+a^x}=\frac{1}{2}$$

(iii) **When $0<a<1$,**

$$\lim_{x\to\infty}\frac{a^{x-1}}{1+a^x}=0$$

(5) $\displaystyle\lim_{x\to\infty}[\log_3(9x-1)-\log_3(x+1)]$

➡ N117

$$=\lim_{x\to\infty}\log_3\frac{9x-1}{x+1}$$

$$=\lim_{x\to\infty}\log_3\frac{9-\frac{1}{x}}{1+\frac{1}{x}}$$

$$=\log_3 9$$

$$=2$$

Limits of Trigonometric Functions

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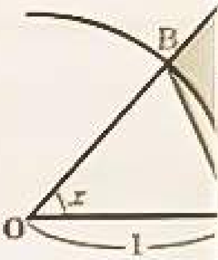
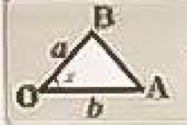
Prove the equation $\lim_{x \to 0} \frac{\sin x}{x} = 1$.



[Sol] (i) When $0 < x < \frac{\pi}{2}$, ← Considering $\lim_{x \to 0^+} \frac{\sin x}{x}$

consider sector OAB with radius 1 and central angle x as shown in the diagram. AT is a tangent to the circle at point A.

As for the areas,



The area of $\triangle$OAB is $\frac{1}{2}ab\sin$

$\triangle OAB = \frac{1}{2}\sin x$ ←

The area of a circle with radius r is πr^2

The area of sector OAB is $\pi r^2 \times \frac{x}{2\pi} =$

Sector OAB $= \frac{1}{2}x$ ←

$\triangle OAT = \frac{1}{2}\tan x$ ← AT = OA tan x = tan x

Also, since $\triangle$OAB < sector OAB < $\triangle$OAT,

$\boxed{\frac{1}{2}\sin x} < \frac{1}{2}x < \boxed{\frac{1}{2}\tan x}$, i.e. $\boxed{\sin x} < x < \boxed{\tan x}$

Since $\sin x > 0$, dividing all terms by $\sin x$,

$\boxed{1} < \frac{x}{\sin x} < \frac{\boxed{1}}{\cos x}$ ← $\frac{\tan x}{\sin x} = \frac{\frac{\sin x}{\cos x}}{\sin x} = \frac{1}{\cos x}$

Since all the terms are positive,

$\boxed{1} > \frac{\sin x}{x} > \cos x$ ← When $0 < a < b < c$, $\frac{1}{a} > \frac{1}{b}$

Since $\lim_{x \to 0^+} \cos x = \boxed{1}$, $\lim_{x \to 0^+} \frac{\sin x}{x} = 1$ ← Limits of Functio… Their Relationshi… (※ Refer to side…

(ii) When $-\frac{\pi}{2} < x < 0$, ← Considering $\lim_{x \to 0^-} \frac{\sin x}{x}$

let $x = -\theta$. As $x \to 0^-$, $\theta \to 0^+$; therefore, from (i),

$\lim_{x \to 0^-} \frac{\sin x}{x} = \lim_{\theta \to 0^+} \frac{\sin(-\theta)}{\boxed{-\theta}} = \lim_{\theta \to 0^+} \frac{\sin\theta}{\boxed{\theta}} = 1$ ← $\sin(-\theta) =$

From (i) and (ii), $\lim_{x \to 0} \frac{\sin x}{x} = 1$

$\frac{1}{2}\sin x$, $\frac{1}{2}\tan x$, $\sin x$, $\tan x$, 1, 1, 1, 1, $-\theta$, θ

N121b

From the result on side a, the following equation is true.

Limit of $\frac{\sin x}{x}$

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

Find the following limits.

Ex.

$$\lim_{x \to 0} \frac{\sin 5x}{x} = \lim_{x \to 0} \frac{\sin 5x}{5x} \cdot 5 = 1 \cdot 5 = 5 \leftarrow$$

Rearranging the denominator into $5x$, then rewriting as $\frac{\sin \bullet}{\bullet}$

(1) $\lim_{x \to 0} \frac{\sin 2x}{x} = \lim_{x \to 0} \frac{\sin 2x}{2x} \cdot 2 = 1 \cdot 2 = 2$

(2) $\lim_{x \to 0} \frac{\sin(-3x)}{x} = \lim_{x \to 0} \frac{\sin(-3x)}{-3x} \cdot (-3) = 1 \cdot (-3) = -3$

(3) $\lim_{x \to 0} \frac{\sin 2x}{3x} = \lim_{x \to 0} \frac{\sin 2x}{2x} \cdot \frac{2}{3} = 1 \cdot \frac{2}{3} = \frac{2}{3}$

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s for ※ on side a, the following statements are true.

Limits of Functions and Their Relationships

For all x close to a, when $f(x) \leqq g(x)$,

if $\lim_{x \to a} f(x) = \alpha$ and $\lim_{x \to a} g(x) = \beta$, then $\alpha \leqq \beta$

if $\lim_{x \to a} f(x) = \infty$, then $\lim_{x \to a} g(x) = \infty$

For all x close to a, when $f(x) \leqq h(x) \leqq g(x)$,

$(x) = \lim_{x \to a} g(x) = \alpha$, then $\lim_{x \to a} h(x) = \alpha$

e when $f(x) < g(x)$, and statement 2 is also true

$g(x)$, $f(x) < h(x) \leqq g(x)$, $f(x) < h(x) < g(x)$.

Limits of Trigonometric Functions

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Find the following limits.

(1) $$\lim_{x\to 0}\frac{x}{\sin 4x}=\lim_{x\to 0}\frac{4x}{\sin 4x}\cdot\frac{1}{\boxed{4}}$$

$$=\lim_{x\to 0}\frac{1}{\frac{\sin 4x}{4x}}\cdot\frac{1}{\boxed{4}}$$

$$=\frac{1}{1}\cdot\frac{1}{4}$$

$$=\frac{1}{4}$$

(2) $$\lim_{x\to 0}\frac{2x}{\sin 3x}=\lim_{x\to 0}\frac{3x}{\sin 3x}\cdot\frac{2}{3}$$

$$=\lim_{x\to 0}\frac{1}{\frac{\sin 3x}{3x}}\cdot\frac{2}{3}$$

$$=\frac{1}{1}\cdot\frac{2}{3}$$

$$=\frac{2}{3}$$

(3) $$\lim_{x\to 0}\frac{\sin 3x}{\sin 2x}=\lim_{x\to 0}\frac{\sin 3x}{3x}\cdot\frac{2x}{\sin 2x}\cdot\frac{3}{2}$$

$$=\lim_{x\to 0}\frac{\sin 3x}{3x}\cdot\frac{1}{\frac{\sin 2x}{2x}}\cdot\frac{3}{2}$$

$$=1\cdot\frac{1}{1}\cdot\frac{3}{2}$$

$$=\frac{3}{2}$$

Since $\lim_{x\to 0}\frac{x}{\sin x}=\lim_{x\to 0}\frac{1}{\frac{\sin x}{x}}=\frac{1}{1}=1$, it is possible to solve by using $\lim_{x\to 0}\frac{x}{\sin x}=1$.

Ex.

$$\lim_{x\to 0}\frac{\tan x}{x}=\lim_{x\to 0}\frac{1}{x}\cdot\frac{\sin x}{\cos x} \quad \longleftarrow \quad \boxed{\tan x=\frac{\sin x}{\cos x}}$$

$$=\lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{1}{\cos x} \qquad \boxed{\lim_{x\to 0}\cos x=1}$$

$$=1\cdot\frac{1}{1}$$

$$=1$$

(4) $$\lim_{x\to 0}\frac{\tan 4x}{x}=\lim_{x\to 0}\frac{1}{x}\cdot\frac{\sin 4x}{\cos 4x}$$

$$=\lim_{x\to 0}\frac{\sin 4x}{4x}\cdot\frac{4}{\cos 4x}$$

$$=1\cdot\frac{4}{1}$$

$$=4$$

(5) $$\lim_{x\to 0}\frac{\sin 3x}{\tan 2x}=\lim_{x\to 0}\frac{\sin 3x}{\frac{\sin 2x}{\cos 2x}}$$

$$=\lim_{x\to 0}\frac{\frac{\sin 3x}{3x}\cdot 3}{\frac{\sin 2x}{2x}\cdot\frac{2}{\cos 2x}}$$

$$=\frac{1\cdot 3}{1\cdot\frac{2}{1}}$$

$$=\frac{3}{2}$$

(6) $$\lim_{x\to 0}\frac{\tan^2 x}{1-\cos x}=\lim_{x\to 0}\frac{1}{1-\cos x}\cdot\frac{\sin^2 x}{\cos^2 x}$$

$$=\lim_{x\to 0}\frac{1-\cos^2 x}{(1-\cos x)\cos^2 x} \qquad \boxed{\sin^2 x=1-\cos^2 x}$$

$$=\lim_{x\to 0}\frac{(1+\cos x)(1-\cos x)}{(1-\cos x)\cos^2 x}$$

$$=\lim_{x\to 0}\frac{1+\cos x}{\cos^2 x}$$

$$=\frac{1+1}{1^2}$$

$$=2$$

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Limits of Trigonometric Functions

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Find the following limits.

Ex.

$$\lim_{x\to0}\frac{\sin 3x}{x+\sin x}=\lim_{x\to0}\frac{\frac{\sin 3x}{x}}{1+\frac{\sin x}{x}}$$

$$=\lim_{x\to0}\frac{\frac{\sin 3x}{3x}\cdot 3}{1+\frac{\sin x}{x}}$$

$$=\frac{1\cdot 3}{1+1}$$

$$=\frac{3}{2}$$

(1) $\displaystyle\lim_{x\to0}\frac{\sin x}{x+\sin x}=\lim_{x\to0}\frac{\frac{\sin x}{x}}{1+\frac{\sin x}{x}}$

$$=\frac{1}{1+1}$$

$$=\frac{1}{2}$$

(2) $\displaystyle\lim_{x\to0}\frac{\sin 2x}{x^2+2x}=\lim_{x\to0}\frac{\frac{\sin 2x}{x}}{x+2}$

$$=\lim_{x\to0}\frac{\frac{\sin 2x}{2x}\cdot 2}{x+2}$$

$$=\frac{1\cdot 2}{0+2}$$

$$=1$$

N123b

(3) $$\lim_{x\to 0}\frac{x-\sin 2x}{\sin 3x}=\lim_{x\to 0}\frac{1-\dfrac{\sin 2x}{x}}{\dfrac{\sin 3x}{x}}$$

$$=\lim_{x\to 0}\frac{1-\dfrac{\sin 2x}{2x}\cdot 2}{\dfrac{\sin 3x}{3x}\cdot 3}$$

$$=\frac{1-1\cdot 2}{1\cdot 3}$$

$$=-\frac{1}{3}$$

Alternative Solution

$$\lim_{x\to 0}\frac{x-\sin 2x}{\sin 3x}$$

$$=\lim_{x\to 0}\left(\frac{x}{\sin 3x}-\frac{\sin 2x}{\sin 3x}\right)$$

$$=\lim_{x\to 0}\left(\frac{3x}{\sin 3x}\cdot\frac{1}{3}-\frac{\sin 2x}{2x}\cdot\frac{3x}{\sin 3x}\cdot\frac{2}{3}\right)$$

$$=\lim_{x\to 0}\left(\frac{1}{\dfrac{\sin 3x}{3x}}\cdot\frac{1}{3}-\frac{\sin 2x}{2x}\cdot\frac{1}{\dfrac{\sin 3x}{3x}}\cdot\frac{2}{3}\right)$$

$$=\frac{1}{1}\cdot\frac{1}{3}-1\cdot\frac{1}{1}\cdot\frac{2}{3}$$

$$=-\frac{1}{3}$$

(4) $$\lim_{x\to 0}\frac{\sin x}{x+\tan x}=\lim_{x\to 0}\frac{\dfrac{\sin x}{x}}{1+\dfrac{1}{x}\cdot\dfrac{\sin x}{\cos x}}$$

$$=\lim_{x\to 0}\frac{\dfrac{\sin x}{x}}{1+\dfrac{\sin x}{x}\cdot\dfrac{1}{\cos x}}$$

$$=\frac{1}{1+1\cdot\dfrac{1}{1}}$$

$$=\frac{1}{2}$$

(5) $$\lim_{x\to 0}\frac{\sin x-\tan 2x}{x}=\lim_{x\to 0}\left(\frac{\sin x}{x}-\frac{1}{x}\cdot\frac{\sin 2x}{\cos 2x}\right)$$

$$=\lim_{x\to 0}\left(\frac{\sin x}{x}-\frac{\sin 2x}{2x}\cdot\frac{2}{\cos 2x}\right)$$

$$=1-1\cdot\frac{2}{1}$$

$$=-1$$

Limits of Trigonometric Functions

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Find the following limits.

Ex.

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{(1-\cos x)(1+\cos x)}{x^2(1+\cos x)}$$

← Multiplying the numerator and the denominator by $1+\cos x$

$$=\lim_{x\to 0}\frac{1-\cos^2 x}{x^2(1+\cos x)}$$

$$=\lim_{x\to 0}\frac{\sin^2 x}{x^2(1+\cos x)}$$

$$=\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2\cdot\frac{1}{1+\cos x}$$

$$=\frac{1}{2}$$

(1) $$\lim_{x\to 0}\frac{x\sin x}{1-\cos x}=\lim_{x\to 0}\frac{x\sin x(1+\cos x)}{(1-\cos x)(1+\cos x)}$$

$$=\lim_{x\to 0}\frac{x\sin x(1+\cos x)}{1-\cos^2 x}$$

$$=\lim_{x\to 0}\frac{x\sin x(1+\cos x)}{\sin^2 x}$$

$$=\lim_{x\to 0}\frac{1+\cos x}{\frac{\sin x}{x}}$$

$$=2$$

(2) $$\lim_{x\to 0}\frac{\sin^3 x}{x(1-\cos x)}=\lim_{x\to 0}\frac{\sin^3 x(1+\cos x)}{x(1-\cos x)(1+\cos x)}$$

$$=\lim_{x\to 0}\frac{\sin^3 x(1+\cos x)}{x(1-\cos^2 x)}$$

$$=\lim_{x\to 0}\frac{\sin^3 x(1+\cos x)}{x\sin^2 x}$$

$$=\lim_{x\to 0}\frac{\sin x}{x}\cdot(1+\cos x)$$

$$=2$$

N124b

(3) $$\lim_{x\to 0}\frac{1-\cos 3x}{3x^2}=\lim_{x\to 0}\frac{(1-\cos 3x)(1+\cos 3x)}{3x^2(1+\cos 3x)}$$

$$=\lim_{x\to 0}\frac{1-\cos^2 3x}{3x^2(1+\cos 3x)}$$

$$=\lim_{x\to 0}\frac{\sin^2 3x}{3x^2(1+\cos 3x)}$$

$$=\lim_{x\to 0}\left(\frac{\sin 3x}{3x}\right)^2\cdot\frac{3}{1+\cos 3x}$$

$$=\frac{3}{2}$$

Alternative Solution

$$\lim_{x\to 0}\frac{1-\cos 3x}{3x^2}$$

$$=\lim_{x\to 0}\frac{2\sin^2\frac{3}{2}x}{3x^2}$$

$$=\lim_{x\to 0}\left(\frac{\sin\frac{3}{2}x}{\frac{3}{2}x}\right)^2\cdot\frac{3}{2}$$

$$=\frac{3}{2}$$

(4) $$\lim_{x\to 0}\frac{\tan x-\sin x}{x^3}=\lim_{x\to 0}\frac{\frac{\sin x}{\cos x}-\sin x}{x^3}$$

$$=\lim_{x\to 0}\frac{\sin x(1-\cos x)}{x^3\cos x}$$

$$=\lim_{x\to 0}\frac{\sin x(1-\cos x)(1+\cos x)}{x^3\cos x(1+\cos x)}$$

$$=\lim_{x\to 0}\frac{\sin x(1-\cos^2 x)}{x^3\cos x(1+\cos x)}$$

$$=\lim_{x\to 0}\frac{\sin^3 x}{x^3\cos x(1+\cos x)}$$

$$=\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^3\cdot\frac{1}{\cos x(1+\cos x)}$$

$$=\frac{1}{2}$$

Alternative Solution

$$\lim_{x\to 0}\frac{\tan x-\sin x}{x^3}=\lim_{x\to 0}\frac{\sin x(1-\cos x)}{x^3\cos x}=\lim_{x\to 0}\frac{\sin x\cdot 2\sin^2\frac{x}{2}}{x^3\cos x}=\lim_{x\to 0}\frac{\sin x}{x}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2\cdot\frac{1}{2\cos x}=\frac{1}{2}$$

... can also be solved by using the Half-Angle Formula.

... $\frac{x}{2}$ (84 | 64). In the case of $\boxed{\text{Ex}}$,

$$\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{x^2}=\lim_{x\to 0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2\cdot\frac{1}{2}=\frac{1}{2}$$

Limits of Trigonometric Functions

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Name

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Find the following limits.

(1) $\lim_{x\to0}\dfrac{2\tan x}{\sqrt{1+x}-1}=\lim_{x\to0}\dfrac{2\sin x(\sqrt{1+x}+1)}{\cos x(\sqrt{1+x}-1)(\sqrt{1+x}+1)}$

$$=\lim_{x\to0}\frac{2\sin x(\sqrt{1+x}+1)}{x\cos x}$$

$$=\lim_{x\to0}\frac{\sin x}{x}\cdot\frac{2(\sqrt{1+x}+1)}{\cos x}$$

$$=4$$

(2) $\lim_{x\to0}\dfrac{\sqrt{1+x\sin x}-1}{x^2}=\lim_{x\to0}\dfrac{(\sqrt{1+x\sin x}-1)(\sqrt{1+x\sin x}+1)}{x^2(\sqrt{1+x\sin x}+1)}$

$$=\lim_{x\to0}\frac{x\sin x}{x^2(\sqrt{1+x\sin x}+1)}$$

$$=\lim_{x\to0}\frac{\sin x}{x}\cdot\frac{1}{\sqrt{1+x\sin x}+1}$$

$$=\frac{1}{2}$$

N125b

(3)
$$\lim_{x\to 0}\frac{\sqrt{2-\cos x}-1}{3x^2}=\lim_{x\to 0}\frac{(\sqrt{2-\cos x}-1)(\sqrt{2-\cos x}+1)}{3x^2(\sqrt{2-\cos x}+1)}$$
$$=\lim_{x\to 0}\frac{1-\cos x}{3x^2(\sqrt{2-\cos x}+1)}$$
$$=\lim_{x\to 0}\frac{(1-\cos x)(1+\cos x)}{3x^2(\sqrt{2-\cos x}+1)(1+\cos x)}$$
$$=\lim_{x\to 0}\frac{1-\cos^2 x}{3x^2(\sqrt{2-\cos x}+1)(1+\cos x)}$$
$$=\lim_{x\to 0}\frac{\sin^2 x}{3x^2(\sqrt{2-\cos x}+1)(1+\cos x)}$$
$$=\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2\cdot\frac{1}{3(\sqrt{2-\cos x}+1)(1+\cos x)}$$
$$=\frac{1}{12}$$

[Alternative Solution

$$\lim_{x\to 0}\frac{\sqrt{2-\cos x}-1}{3x^2}=\lim_{x\to 0}\frac{1-\cos x}{3x^2(\sqrt{2-\cos x}+1)}=\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{3x^2(\sqrt{2-\cos x}+1)}=\lim_{x\to 0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2\cdot\frac{1}{6(\sqrt{2-\cos x}+1)}=\frac{1}{12}$$]

(4)
$$\lim_{x\to 0}\frac{1-\cos x}{\sqrt{1+x^2}-\sqrt{1-x^2}}=\lim_{x\to 0}\frac{(1-\cos x)(\sqrt{1+x^2}+\sqrt{1-x^2})}{(\sqrt{1+x^2}-\sqrt{1-x^2})(\sqrt{1+x^2}+\sqrt{1-x^2})}$$
$$=\lim_{x\to 0}\frac{(1-\cos x)(\sqrt{1+x^2}+\sqrt{1-x^2})}{2x^2}$$
$$=\lim_{x\to 0}\frac{(1-\cos x)(1+\cos x)(\sqrt{1+x^2}+\sqrt{1-x^2})}{2x^2(1+\cos x)}$$
$$=\lim_{x\to 0}\frac{(1-\cos^2 x)(\sqrt{1+x^2}+\sqrt{1-x^2})}{2x^2(1+\cos x)}$$
$$=\lim_{x\to 0}\frac{\sin^2 x(\sqrt{1+x^2}+\sqrt{1-x^2})}{2x^2(1+\cos x)}$$
$$=\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2\cdot\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{2(1+\cos x)}=\frac{1}{2}$$

[illegible] $$=\lim\frac{2\sin^2\frac{x}{2}\cdot(\sqrt{1+x^2}+\sqrt{1-x^2})}{(\sqrt{1+x^2}-\sqrt{1-x^2})(\sqrt{1+x^2}+\sqrt{1-x^2})}=\lim\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2\cdot\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{4}=\frac{1}{2}$$]

N126a

KUMON

Limits of Trigonometric Functions

Name

Date / /

Time : to :

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0	—	—	1	2~

Find the following limits.

Ex.

$\lim_{x\to\infty} x\sin\frac{1}{x}$

[Sol] Let $\frac{1}{x}=\theta$. As $x\to\infty, \theta\to 0^+$

$$\therefore \lim_{x\to\infty} x\sin\frac{1}{x}=\lim_{\theta\to 0^+}\frac{1}{\theta}\sin\theta$$
$$=\lim_{\theta\to 0^+}\frac{\sin\theta}{\theta}$$
$$=1$$

(1) $\lim_{x\to\infty} x\tan\frac{1}{x}$

[Sol] Let $\frac{1}{x}=\theta$. As $x\to\infty, \theta\to 0^+$

$$\therefore \lim_{x\to\infty} x\tan\frac{1}{x}=\lim_{\theta\to 0^+}\frac{1}{\theta}\tan\theta$$
$$=\lim_{\theta\to 0^+}\frac{1}{\theta}\cdot\frac{\sin\theta}{\cos\theta}$$
$$=\lim_{\theta\to 0^+}\frac{\sin\theta}{\theta}\cdot\frac{1}{\cos\theta}$$
$$=1$$

(2) $\lim_{x\to 0}\frac{\sin(\sin x)}{\sin x}$

[Sol] Let $\sin x=\theta$. As $x\to 0, \theta\to 0$ ←

As $x\to 0^+, \theta\to 0^+$
As $x\to 0^-, \theta\to 0^-$
Therefore, use $\theta\to 0$ instead of $\theta\to 0^+$ or $\theta\to 0^-$.

$$\therefore \lim_{x\to 0}\frac{\sin(\sin x)}{\sin x}=\lim_{\theta\to 0}\frac{\sin\theta}{\theta}$$
$$=1$$

N126b

Ex. $\lim_{x\to\frac{\pi}{4}}\frac{2\pi-8x}{\sin(4x-\pi)}$

[Sol] Let $x-\frac{\pi}{4}=\theta$. $x=\theta+\frac{\pi}{4}$

As $x\to\frac{\pi}{4}$, $\theta\to 0$

$$\therefore \lim_{x\to\frac{\pi}{4}}\frac{2\pi-8x}{\sin(4x-\pi)}=\lim_{\theta\to 0}\frac{-8\theta}{\sin 4\theta}$$

$$=\lim_{\theta\to 0}\frac{-2}{\frac{\sin 4\theta}{4\theta}}$$

$$=-2$$

$2\pi-8x=2\pi-8\left(\theta+\frac{\pi}{4}\right)=-8\theta$

$4x-\pi=4\left(\theta+\frac{\pi}{4}\right)-\pi=4\theta$

(3) $\lim_{x\to\frac{\pi}{6}}\frac{6x-\pi}{\sin\left(x-\frac{\pi}{6}\right)}$

[Sol] Let $x-\frac{\pi}{6}=\theta$. $x=\theta+\frac{\pi}{6}$

As $x\to\frac{\pi}{6}$, $\theta\to 0$

$$\therefore \lim_{x\to\frac{\pi}{6}}\frac{6x-\pi}{\sin\left(x-\frac{\pi}{6}\right)}=\lim_{\theta\to 0}\frac{6\theta}{\sin\theta}$$

$$=\lim_{\theta\to 0}\frac{6}{\frac{\sin\theta}{\theta}}$$

$$=6$$

(4) $\lim_{x\to\frac{\pi}{2}}\frac{\cos x}{2x-\pi}$

[Sol] Let $x-\frac{\pi}{2}=\theta$. $x=\theta+\frac{\pi}{2}$

As $x\to\frac{\pi}{2}$, $\theta\to 0$

$$\lim_{x\to\frac{\pi}{2}}\frac{\cos x}{2x-\pi}=\lim_{\theta\to 0}\frac{\cos\left(\theta+\frac{\pi}{2}\right)}{2\theta}$$

$$=\lim_{\theta\to 0}\frac{-\sin\theta}{2\theta}$$

$$=\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\cdot\left(-\frac{1}{2}\right)$$

$$=-\frac{1}{2}$$

$\cos\left(\theta+\frac{\pi}{2}\right)=-\sin\theta$

Limits of Trigonometric Functions

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Find the following limits.

Ex. $\lim_{x\to 0} x\sin\frac{1}{x}$

$|ab|=|a||b|$

[Sol] Since $0 \leqslant \left|\sin\frac{1}{x}\right| \leqslant 1$, $0 \leqslant \left|x\sin\frac{1}{x}\right| = |x|\left|\sin\frac{1}{x}\right| \leqslant |x|$

Then, since $\lim_{x\to 0}|x|=0$,

$\lim_{x\to 0}\left|x\sin\frac{1}{x}\right|=0$

Limits of Functions and Their Relationships (N121)

$\therefore \lim_{x\to 0} x\sin\frac{1}{x}=0$

If $\lim_{x\to a}|f(x)|=0$, then $\lim_{x\to a} f(x)=0$

(1) $\lim_{x\to 0} x\cos\frac{1}{x}$

[Sol] Since $0 \leqslant \left|\cos\frac{1}{x}\right| \leqslant 1$, $0 \leqslant \left|x\cos\frac{1}{x}\right| = |x|\left|\cos\frac{1}{x}\right| \leqslant |x|$

Then, since $\lim_{x\to 0}|x|=0$,

$\lim_{x\to 0}\left|x\cos\frac{1}{x}\right|=0$

$\therefore \lim_{x\to 0} x\cos\frac{1}{x}=0$

(2) $\lim_{x\to 0} x^2\sin\frac{1}{x}$

[Sol] Since $0 \leqslant \left|\sin\frac{1}{x}\right| \leqslant 1$, $0 \leqslant \left|x^2\sin\frac{1}{x}\right| = x^2\left|\sin\frac{1}{x}\right| \leqslant x^2$

Then, since $\lim_{x\to 0}x^2=0$,

$\lim_{x\to 0}\left|x^2\sin\frac{1}{x}\right|=0$

$\therefore \lim_{x\to 0} x^2\sin\frac{1}{x}=0$

N127b

(3) $\lim_{x\to\infty}\frac{\cos x}{x}$

[Sol] Since $0 \leqslant |\cos x| \leqslant 1$, $0 \leqslant \left|\frac{\cos x}{x}\right| = \left|\frac{1}{x}\right| |\cos x| \leqslant \left|\frac{1}{x}\right|$

Then, since $\lim_{x\to\infty}\left|\frac{1}{x}\right| = 0$,

$$\lim_{x\to\infty}\left|\frac{\cos x}{x}\right| = 0$$

$$\therefore \lim_{x\to\infty}\frac{\cos x}{x} = 0$$

(4) $\lim_{x\to\infty}\frac{x}{x+\sin x}$

[Sol] $\lim_{x\to\infty}\frac{x}{x+\sin x} = \lim_{x\to\infty}\frac{1}{1+\frac{\sin x}{x}}$

Since $0 \leqslant |\sin x| \leqslant 1$, $0 \leqslant \left|\frac{\sin x}{x}\right| = \left|\frac{1}{x}\right| |\sin x| \leqslant \left|\frac{1}{x}\right|$

Then, since $\lim_{x\to\infty}\left|\frac{1}{x}\right| = 0$,

$$\lim_{x\to\infty}\left|\frac{\sin x}{x}\right| = 0$$

$$\therefore \lim_{x\to\infty}\frac{\sin x}{x} = 0$$

$$\therefore \lim_{x\to\infty}\frac{x}{x+\sin x} = 1$$

If the absolute values are not used as in N67, the different conditions need to be considered

In the case of Ex, since $-1 \leqslant \sin\frac{1}{x} \leqslant 1$, the conditions are:

When $x > 0$, $-x \leqslant x\sin\frac{1}{x} \leqslant x$

When $x < 0$, $x \leqslant x\sin\frac{1}{x} \leqslant -x$

Limits of Trigonometric Functions

Name

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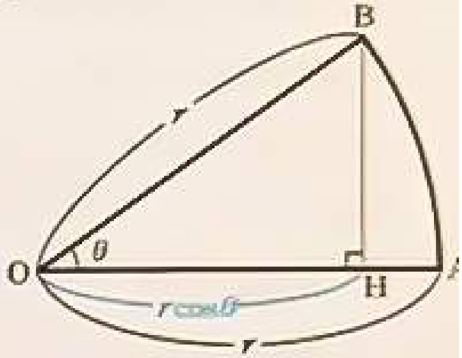
1. Given sector OAB below, drop a perpendicular BH from point B to OA.

Let $\angle AOB = \theta \left(0 < \theta < \frac{\pi}{2}\right)$ and $OA = OB = r$.



(1) Express AH in terms of θ.

[Sol] $AH = OA - OH$

$$= r - \boxed{r\cos\theta}$$

$$= r\left(\boxed{1 - \cos\theta}\right)$$

(2) Find $\lim\limits_{\theta \to 0} \frac{AH}{\theta^2}$.

[Sol] $\lim\limits_{\theta \to 0} \frac{AH}{\theta^2} = \lim\limits_{\theta \to 0} \frac{r(1-\cos\theta)}{\theta^2}$

$$= \lim_{\theta \to 0} \frac{r(1-\cos\theta)(1+\cos\theta)}{\theta^2(1+\cos\theta)}$$

$$= \lim_{\theta \to 0} \frac{r(1-\cos^2\theta)}{\theta^2(1+\cos\theta)}$$

$$= \lim_{\theta \to 0} \frac{r\sin^2\theta}{\theta^2(1+\cos\theta)}$$

$$= \lim_{\theta \to 0} \left(\frac{\sin\theta}{\theta}\right)^2 \cdot \frac{r}{1+\cos\theta}$$

$$= \frac{r}{2}$$

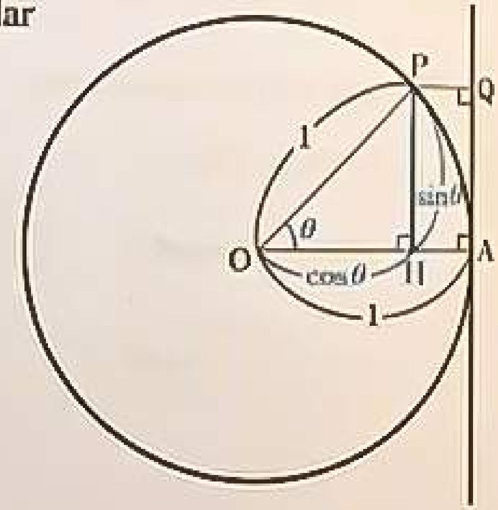
Alternative Solution

$$\lim_{\theta \to 0} \frac{AH}{\theta^2} = \lim_{\theta \to 0} \frac{r(1-\cos\theta)}{\theta^2} = \lim_{\theta \to 0} \frac{2r\sin^2\frac{\theta}{2}}{\theta^2} = \lim_{\theta \to 0} \left(\frac{\sin\frac{\theta}{2}}{\frac{\theta}{2}}\right)^2 \cdot \frac{r}{2} = \frac{r}{2}$$

N128b

2. Given fixed point A on the circumference of a circle with center at point O and radius 1, drop a perpendicular PQ from another distinct point P on the circumference of the circle to the tangent at point A $\left(0<\angle AOP<\frac{\pi}{2}\right)$. Find the limit of $\frac{AQ^2}{PQ}$ when P approaches A.

[Sol] Let $\angle AOP=\theta$, and PH be a perpendicular dropped from P to OA.



$AQ=HP=\sin\theta$

$PQ=HA$

$=OA-OH$

$=1-\cos\theta$

When P approaches A, $\theta\to0^+$

Therefore,

$$\lim_{\theta\to0^+}\frac{AQ^2}{PQ}=\lim_{\theta\to0^+}\frac{\sin^2\theta}{1-\cos\theta}$$

$$=\lim_{\theta\to0^+}\frac{1-\cos^2\theta}{1-\cos\theta}$$

$$=\lim_{\theta\to0^+}\frac{(1+\cos\theta)(1-\cos\theta)}{1-\cos\theta}$$

$$=\lim_{\theta\to0^+}(1+\cos\theta)$$

$$=2$$

Alternative Solution 1

$$\lim_{\theta\to0^+}\frac{AQ^2}{PQ}=\lim_{\theta\to0^+}\frac{\sin^2\theta}{1-\cos\theta}=\lim_{\theta\to0^+}\frac{\sin^2\theta}{2\sin^2\frac{\theta}{2}}=\lim_{\theta\to0^+}\frac{\left(\frac{\sin\theta}{\theta}\right)^2\cdot2}{\left(\frac{\sin\frac{\theta}{2}}{\frac{\theta}{2}}\right)^2}=2$$

Alternative Solution 2

$$\lim_{\theta\to0^+}\frac{AQ^2}{PQ}=\lim_{\theta\to0^+}\frac{\sin^2\theta}{1-\cos\theta}=\lim_{\theta\to0^+}\frac{\sin^2\theta(1+\cos\theta)}{(1-\cos\theta)(1+\cos\theta)}=\lim_{\theta\to0^+}\frac{\sin^2\theta(1+\cos\theta)}{1-\cos^2\theta}=\lim_{\theta\to0^+}\frac{\sin^2\theta(1+\cos\theta)}{\sin^2\theta}$$

$$=\lim_{\theta\to0^+}(1+\cos\theta)=2$$

N129a KUMON

Limits of Trigonometric Functions

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1. Find the constants a and b which satisfy the following equation.

$$\lim_{x\to\frac{\pi}{2}}\frac{ax+b}{\cos x}=\frac{1}{2} \quad \cdots①$$

[Sol] Since $\lim_{x\to\frac{\pi}{2}}\cos x=0$,

$$\lim_{x\to\frac{\pi}{2}}(ax+b)=0, \text{ i.e. } \frac{1}{2}\pi a+b=0 \quad \leftarrow \boxed{\text{N108}}$$

$$\therefore\ b=-\frac{1}{2}\pi a \quad \cdots②$$

Then,

$$\lim_{x\to\frac{\pi}{2}}\frac{ax+b}{\cos x}=\lim_{x\to\frac{\pi}{2}}\frac{a\left(x-\frac{\pi}{2}\right)}{\cos x}$$

Let $x-\frac{\pi}{2}=\theta$. $x=\theta+\frac{\pi}{2}$

As $x\to\frac{\pi}{2}$, $\theta\to 0$

Therefore,

$$\lim_{x\to\frac{\pi}{2}}\frac{a\left(x-\frac{\pi}{2}\right)}{\cos x}=\lim_{\theta\to 0}\frac{a\theta}{\cos\left(\theta+\frac{\pi}{2}\right)}$$

$\boxed{\cos\left(\theta+\frac{\pi}{2}\right)=-}$

$$=\lim_{\theta\to 0}\frac{a\theta}{-\sin\theta}$$

$$=\lim_{\theta\to 0}\frac{-a}{\frac{\sin\theta}{\theta}}$$

$$=-a \quad \cdots③$$

From ① and ③, $-a=\frac{1}{2}$; therefore, $a=-\frac{1}{2}$ $\cdots$④

From ② and ④, $b=\frac{\pi}{4}$

$$\therefore\ a=-\frac{1}{2},\ b=\frac{\pi}{4}$$

N129b

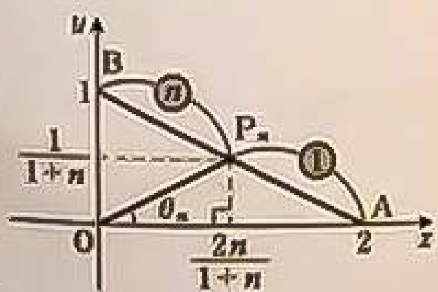
2. Given two points A$(2, 0)$ and B$(0, 1)$ on the coordinate plane with origin O, let $\angle AOP_n = \theta_n$ and P_n be the point internally dividing line segment AB in the ratio $1:n$ $\left(n \text{ is a natural number and } 0 < \theta_n < \frac{\pi}{2}\right)$. Let l_n be the length of line segment AP_n. Find the limit value of $\lim_{n\to\infty} \frac{l_n}{\theta_n}$.

[Sol] Let (x_n, y_n) be the coordinates of point P_n. Since point P_n internally divides line segment AB in the ratio $1:n$,

$$x_n = \frac{n\cdot 2 + 1\cdot 0}{1+n} = \frac{2n}{1+n},\quad y_n = \frac{n\cdot 0 + 1\cdot 1}{1+n} = \frac{1}{1+n}$$ ← Coordinates of Internal Dividing Points (M6)

Therefore, since $OP_n = \sqrt{\left(\frac{2n}{1+n}\right)^2 + \left(\frac{1}{1+n}\right)^2} = \frac{\sqrt{4n^2+1}}{1+n}$,

$$\sin\theta_n = \frac{\frac{1}{1+n}}{\frac{\sqrt{4n^2+1}}{1+n}} = \frac{1}{\sqrt{4n^2+1}}$$



Also, since $AB = \sqrt{5}$, $l_n = \frac{1}{n+1}AB = \frac{\sqrt{5}}{n+1}$

$$\therefore \lim_{n\to\infty} \frac{l_n}{\theta_n} = \lim_{n\to\infty} \frac{\sqrt{5}}{(n+1)\theta_n} = \lim_{n\to\infty} \frac{\sqrt{5}}{(n+1)\sin\theta_n}\cdot\frac{\sin\theta_n}{\theta_n}$$

Then, $$\lim_{n\to\infty} \frac{\sqrt{5}}{(n+1)\sin\theta_n} = \lim_{n\to\infty} \frac{\sqrt{5}\sqrt{4n^2+1}}{n+1} = \lim_{n\to\infty} \frac{\sqrt{5}\sqrt{4+\frac{1}{n^2}}}{1+\frac{1}{n}} = 2\sqrt{5}$$

Also, as $n \to \infty$, $\theta_n \to 0^+$; therefore,

$$\lim_{n\to\infty} \frac{\sin\theta_n}{\theta_n} = \lim_{\theta_n\to 0^+} \frac{\sin\theta_n}{\theta_n} = 1$$

$$\therefore \lim_{n\to\infty} \frac{l_n}{\theta_n} = 2\sqrt{5}$$

Alternative Solution

In $\triangle OAP_n$, $\frac{l_n}{\sin\theta_n} = \frac{2}{\sin\angle OP_nA}$ ...① ← Sine Rule ([illegible])

Since $AB = \sqrt{5}$, in $\triangle OBP_n$, $\frac{\sqrt{5} - l_n}{\sin\left(\frac{\pi}{2} - \theta_n\right)} = \frac{1}{\sin(\pi - \angle OP_nA)}$

$\therefore \frac{\sqrt{5} - l_n}{\cos\theta_n} = \frac{1}{\sin\angle OP_nA}$ ...② ← $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$, $\sin(\pi - \theta) = \sin\theta$

From ① and ②, $l_n = \frac{2\sqrt{5}\sin\theta_n}{2\sin\theta_n + \cos\theta_n}$

As $n \to \infty$, $\theta_n \to 0^+$

$\therefore \lim_{n\to\infty}\frac{l_n}{\theta_n} = \lim_{n\to\infty} \frac{2\sqrt{5}\sin\theta_n}{\theta_n(2\sin\theta_n + \cos\theta_n)} = \lim_{n\to\infty} \frac{\sin\theta_n}{\theta_n}\cdot\frac{2\sqrt{5}}{2\sin\theta_n + \cdots}$

KUMON

Limits of Trigonometric Functions

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Find the following limits.

(1) $\lim_{x\to 0}\frac{\tan 2x}{x}=\lim_{x\to 0}\frac{1}{x}\cdot\frac{\sin 2x}{\cos 2x}$

$=\lim_{x\to 0}\frac{\sin 2x}{2x}\cdot\frac{2}{\cos 2x}$

$=2$

➡ N

(2) $\lim_{x\to 0}\frac{\sin 4x}{x+\sin x}=\lim_{x\to 0}\frac{\frac{\sin 4x}{x}}{1+\frac{\sin x}{x}}$

$=\lim_{x\to 0}\frac{\frac{\sin 4x}{4x}\cdot 4}{1+\frac{\sin x}{x}}$

$=2$

➡ N

(3) $\lim_{x\to 0}\frac{1-\cos x}{x^2\cos x}=\lim_{x\to 0}\frac{(1-\cos x)(1+\cos x)}{x^2\cos x(1+\cos x)}$

$=\lim_{x\to 0}\frac{1-\cos^2 x}{x^2\cos x(1+\cos x)}$

$=\lim_{x\to 0}\frac{\sin^2 x}{x^2\cos x(1+\cos x)}$

$=\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2\cdot\frac{1}{\cos x(1+\cos x)}$

$=\frac{1}{2}$

➡ N124

[**Alternative Solution**

$\lim_{x\to 0}\frac{1-\cos x}{x^2\cos x}=\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{x^2\cos x}=\lim_{x\to 0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2\cdot\frac{1}{2\cos x}=\frac{1}{2}$]

N130b

(4) $\lim_{x \to \frac{\pi}{6}} \frac{3\pi - 18x}{\sin(6x - \pi)}$ ➡ N126

[Sol] Let $x - \frac{\pi}{6} = \theta$. $x = \theta + \frac{\pi}{6}$

As $x \to \frac{\pi}{6}$, $\theta \to 0$

$$\therefore \lim_{x \to \frac{\pi}{6}} \frac{3\pi - 18x}{\sin(6x - \pi)} = \lim_{\theta \to 0} \frac{-18\theta}{\sin 6\theta}$$

$$= \lim_{\theta \to 0} \frac{-3}{\frac{\sin 6\theta}{6\theta}}$$

$$= -3$$

(5) $\lim_{x \to \infty} \frac{\sin x}{x}$ ➡ N127

[Sol] Since $0 \leqslant |\sin x| \leqslant 1$, $\quad 0 \leqslant \left| \frac{\sin x}{x} \right| = \left| \frac{1}{x} \right| |\sin x| \leqslant \left| \frac{1}{x} \right|$

Then, since $\lim_{x \to \infty} \left| \frac{1}{x} \right| = 0$,

$$\lim_{x \to \infty} \left| \frac{\sin x}{x} \right| = 0$$

$$\therefore \lim_{x \to \infty} \frac{\sin x}{x} = 0$$

Continuous and Discontinuous Functions

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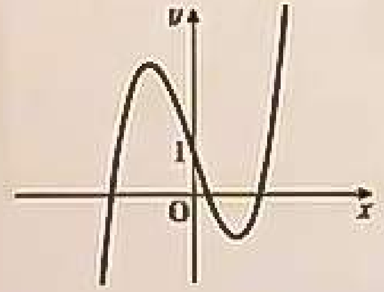
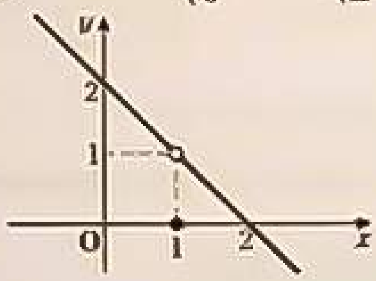
[I] Choose two graphs that have a break(s) within the domain from (A) ~ (F) and circle the letters.

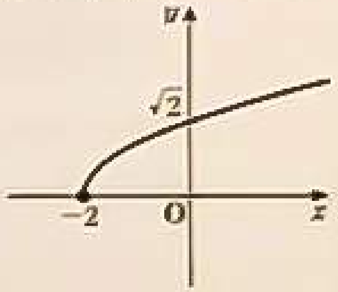
(A) $f(x)=x^3-3x+1$



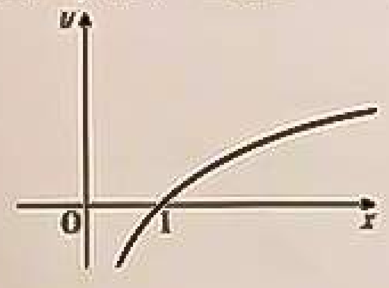
(B) $f(x)=\begin{cases} -x+2 & (x \neq 1) \\ 0 & (x=1) \end{cases}$



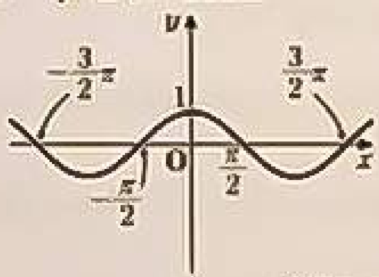
(C) $f(x)=\sqrt{x+2}$



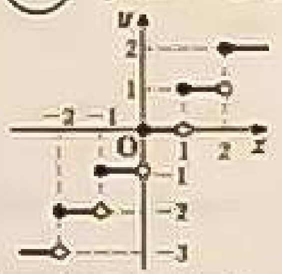
(D) $f(x)=\log_2 x$



(E) $f(x)=\cos x$



(F) $f(x)=[x]$



($[x]$ represents the greatest integer less than or equal to x.)

Answers: (B), (F)

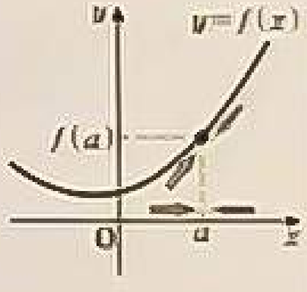
(The domain of (C) is $x \geqq -2$, and the domain of (D) is $x > 0$. Therefore, they do not break within the domain.)

Continuous and Discontinuous Functions

Generally, the function $f(x)$ is said to be ***continuous*** at $x=a$ if $f(x)$ satisfies the following two conditions with respect to a, which is the value of x within the domain.

(i) $\lim_{x \to a} f(x)$ exists.

(ii) $\lim_{x \to a} f(x) = f(a)$ is true.



With these conditions, the graph of $y=f(x)$ does not break at $x=a$. If the function $f(x)$ is not continuous at $x=a$, $f(x)$ is said to be ***discontinuous*** at $x=a$.

As shown in [I], most of the functions introduced thus far are continuous for all values of x within the domain.

In (B), since $\lim_{x \to 1} f(x)=1$ and $f(1)=0$, $\lim_{x \to 1} f(x) \neq f(1)$.

In (F), since $\lim_{x \to n-0} f(x)=n-1$ and $\lim_{x \to n+0} f(x)=n$ when n is an integer, $\lim_{x \to n} f(x)$ does not exist.

N I 3 I b

Examine if the following functions are continuous at $x=0$.

Ex.

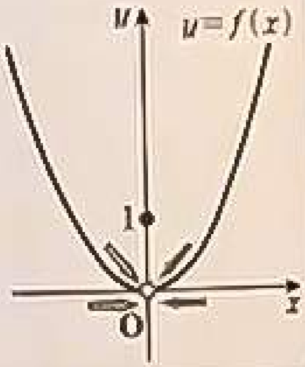
$$f(x)=\begin{cases} x^2 & (x \neq 0) \\ 1 & (x = 0) \end{cases}$$

[Sol] $\lim_{x\to0} f(x)=\lim_{x\to0} x^2=0$ ← Examining if $\lim_{x\to a} f(x)$ exists

Also, $f(0)=1$ ← Examining if $\lim_{x\to a} f(x)=f(a)$ is true

$\therefore \lim_{x\to0} f(x) \neq f(0)$

Therefore, $f(x)$ is discontinuous at $x=0$.



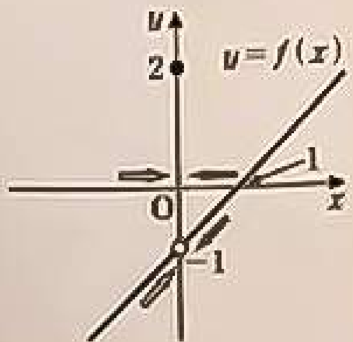
(1) $f(x)=\begin{cases} x-1 & (x \neq 0) \\ 2 & (x = 0) \end{cases}$

[Sol] $\lim_{x\to0} f(x)=\lim_{x\to0}(x-1)=-1$

Also, $f(0)=2$

$\therefore \lim_{x\to0} f(x) \neq f(0)$

Therefore, $f(x)$ is discontinuous at $x=0$.



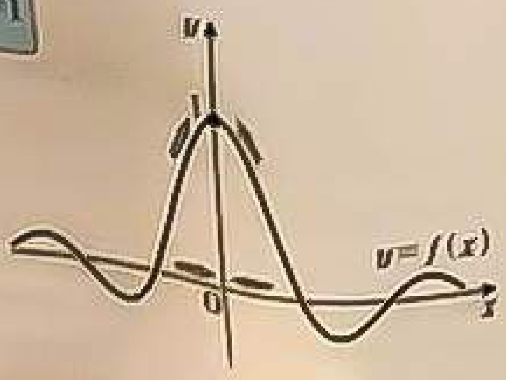
(2) $f(x)=\begin{cases} \dfrac{\sin x}{x} & (x \neq 0) \\ 1 & (x = 0) \end{cases}$

[Sol] $\lim_{x\to0} f(x)=\lim_{x\to0}\dfrac{\sin x}{x}=1$ ← $\lim_{x\to0}\dfrac{\sin x}{x}=1$

Also, $f(0)=1$

$\therefore \lim_{x\to0} f(x)=f(0)$

Therefore, $f(x)$ is continuous at $x=0$.



Continuous and Discontinuous Functions

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Name

Date / /

Time : to :

Examine if the following functions are continuous at $x=0$.
($[x]$ represents the greatest integer less than or equal to x.)

Ex. $f(x)=[x]$

[Sol] (i) When $-1 \leqq x < 0$, $[x]=-1$ ← Examining if $\lim_{x\to 0} f(x)$ exists

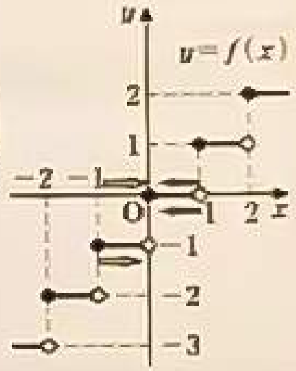
$\therefore \lim_{x\to 0^-} f(x)=-1$

(ii) When $0 \leqq x < 1$, $[x]=0$

$\therefore \lim_{x\to 0^+} f(x)=0$ $\boxed{\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)}$

From (i) and (ii), $\lim_{x\to 0} f(x)$ does not exist.

Therefore, $f(x)$ is discontinuous at $x=0$.



(1) $f(x)=x-[x]$

[Sol] (i) When $-1 \leqq x < 0$, $[x]=-1$

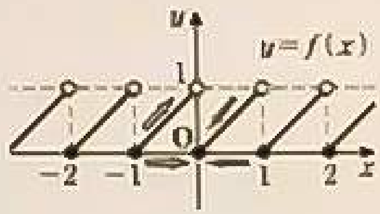
$\therefore \lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}[x-(-1)]=1$

(ii) When $0 \leqq x < 1$, $[x]=0$

$\therefore \lim_{x\to 0^+} f(x)=\lim_{x\to 0^+} x=0$

From (i) and (ii), $\lim_{x\to 0} f(x)$ does not exist.

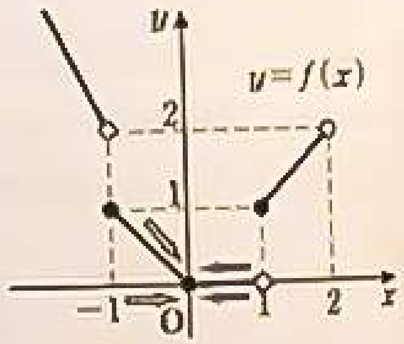
Therefore, $f(x)$ is **discontinuous** at $x=0$.



NI32b

(2) $f(x)=x[x]$

[Sol] (i) When $-1 \leqslant x < 0$, $[x]=-1$



$$\therefore \lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}(-x)=0$$

(ii) When $0 \leqslant x < 1$, $[x]=0$

$$\therefore \lim_{x\to 0^+} f(x)=0$$

From (i) and (ii), $\lim_{x\to 0} f(x)=0$ ← $\lim_{x\to 0^-} f(x)=\lim_{x\to 0^+} f(x)$

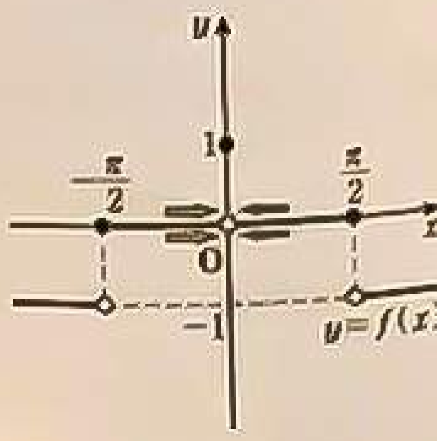
Also, $f(0)=0$ ← Examining if $\lim_{x\to a} f(x)=f(a)$ is true

$$\therefore \lim_{x\to 0} f(x)=f(0)$$

Therefore, $f(x)$ is **continuous** at $x=0$.

(3) $f(x)=[\cos x]$

[Sol] (i) When $-\frac{\pi}{2} \leqslant x < 0$, $[\cos x]=0$



$$\therefore \lim_{x\to 0^-} f(x)=0$$

(ii) When $0 < x \leqslant \frac{\pi}{2}$, $[\cos x]=0$

$$\therefore \lim_{x\to 0^+} f(x)=0$$

From (i) and (ii), $\lim_{x\to 0} f(x)=0$

Also, $f(0)=[\cos 0]=1$

$$\therefore \lim_{x\to 0} f(x) \neq f(0)$$

Therefore, $f(x)$ is **discontinuous** at $x=0$

Continuous and Discontinuous Functions

Name ____________

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Time : to :

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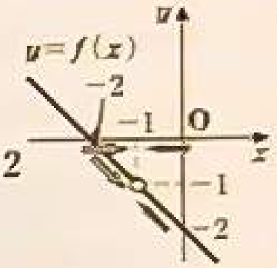
Examine the continuity of the following functions.

Ex. $f(x) = -\dfrac{x^2+3x+2}{x+1}$



[Sol] When $x \neq -1$, $f(x) = -\dfrac{(x+2)(x+1)}{x+1} = -x-2$

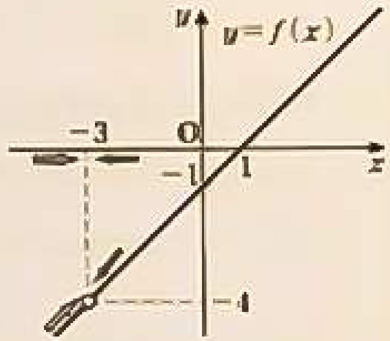
Therefore, $f(x)$ is continuous

when $x < -1$ and $-1 < x$.

Since $x = -1$ is not within the domain, ← Denominator $\neq 0$

$f(x)$ is continuous for all x, excluding the point at $x = -1$.

(1) $f(x) = \dfrac{x^2+2x-3}{x+3}$



[Sol] When $x \neq -3$, $f(x) = \dfrac{(x+3)(x-1)}{x+3} = x-1$

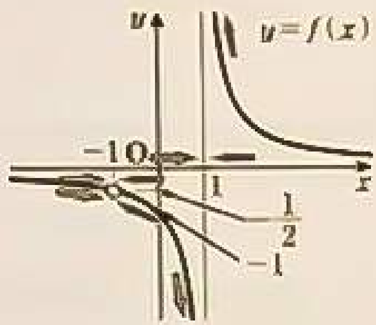
Therefore, $f(x)$ is continuous

when $x < -3$ and $-3 < x$.

Since $x = -3$ is not within the domain,

$f(x)$ is continuous for all x, excluding the point at $x = -3$.

(2) $f(x) = \dfrac{x+1}{x^2-1}$



[Sol] When $x \neq \pm 1$, $f(x) = \dfrac{x+1}{(x+1)(x-1)} = \dfrac{1}{x-1}$

Therefore, $f(x)$ is continuous

when $x < -1$, $-1 < x < 1$ and $1 < x$.

Since $x = \pm 1$ is not within the domain,

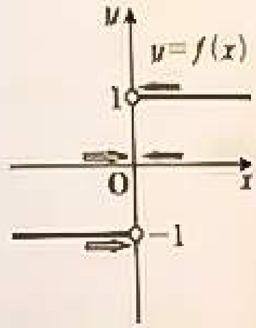
$f(x)$ is continuous for all x, excluding the point at $x = \pm 1$.

N133b

(3) $f(x)=\frac{|x|}{x}$

[Sol] (i) When $x>0$, $f(x)=\frac{x}{x}=\boxed{1}$

(ii) When $x<0$, $f(x)=\frac{-x}{x}=\boxed{-1}$



From (i) and (ii), $f(x)$ is continuous

when $x<0$ and $0<x$.

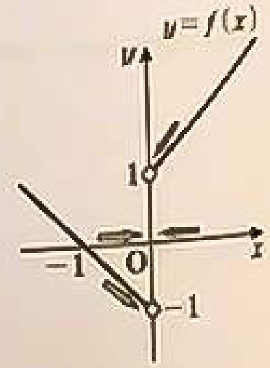
Since $x=0$ is not within the domain,

$f(x)$ is continuous for all x, excluding the point at $x=0$.

(4) $f(x)=\frac{x^2+x}{|x|}$

[Sol] (i) When $x>0$, $f(x)=\frac{x(x+1)}{x}=x+1$

(ii) When $x<0$, $f(x)=\frac{x(x+1)}{-x}=-x-1$



From (i) and (ii), $f(x)$ is continuous

when $x<0$ and $0<x$.

Since $x=0$ is not within the domain,

$f(x)$ is continuous for all x, excluding the point at $x=0$.

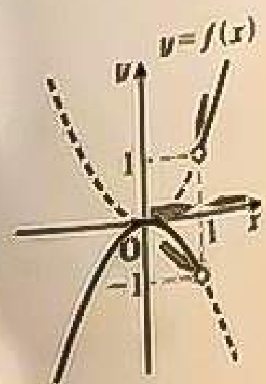
(5) $f(x)=\frac{(x-1)x^2}{|x-1|}$

[Sol] (i) When $x-1>0$, i.e. $x>1$,

$$f(x)=\frac{(x-1)x^2}{x-1}=x^2$$

(ii) When $x-1<0$, i.e. $x<1$,

$$f(x)=\frac{(x-1)x^2}{-(x-1)}=-x^2$$



From (i) and (ii), $f(x)$ is continuous

when $x<1$ and $1<x$.

Since $x=1$ is not within the domain,

$f(x)$ is continuous for all x, excluding the point at $x=1$.

Continuous and Discontinuous Functions

100%	~90%	~80%	~70%	69%~
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Name

Date / /

Time : to :

Graph each given function and find x at which $f(x)$ is discontinuous.

Ex.

$$f(x)=\lim_{n\to\infty}\frac{x^{2n}}{1+x^{2n}}$$

[Sol] (i) When $|x|<1$,

$$f(x)=\boxed{0} \quad \leftarrow \quad \boxed{\lim_{n\to\infty}x^{2n}=0}$$

(ii) When $x=\pm1$,

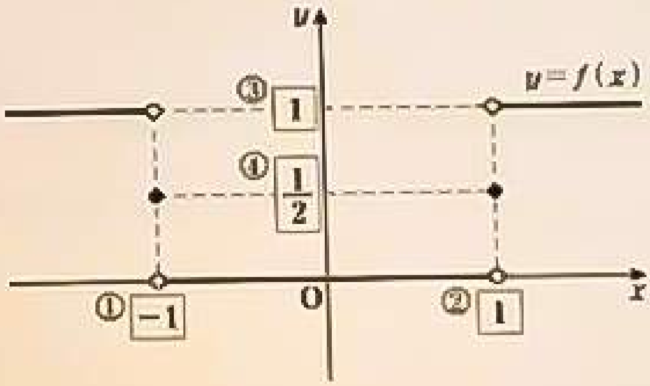
$$f(x)=\boxed{\frac{1}{2}} \quad \leftarrow \quad \boxed{\lim_{n\to\infty}x^{2n}=1}$$

(iii) When $|x|>1$,

$$f(x)=\lim_{n\to\infty}\frac{1}{\frac{1}{x^{2n}}+1}=\boxed{1} \quad \leftarrow \quad \boxed{\lim_{n\to\infty}\frac{1}{x^{2n}}=0}$$

From (i)~(iii), the graph of $y=f(x)$ is as shown in the diagram.

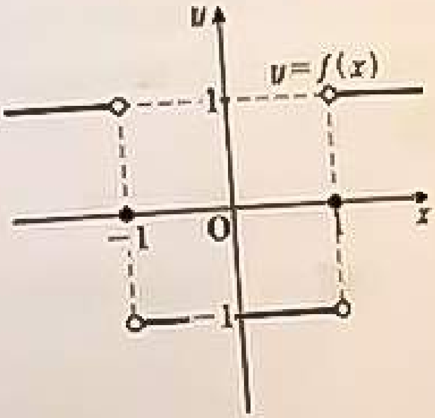
Also, $f(x)$ is discontinuous at $x=\boxed{\pm1}$.



Answers: 0, $\frac{1}{2}$, 1, ±1, (on the graph) ① −1, ② 1, ③ 1, ④ $\frac{1}{2}$

N134b

(1) $f(x)=\lim_{n\to\infty}\frac{|x|^n-1}{|x|^n+1}$



[Sol] (i) When $|x|<1$,

$$f(x)=-1$$

(ii) When $x=\pm 1$,

$$f(x)=0$$

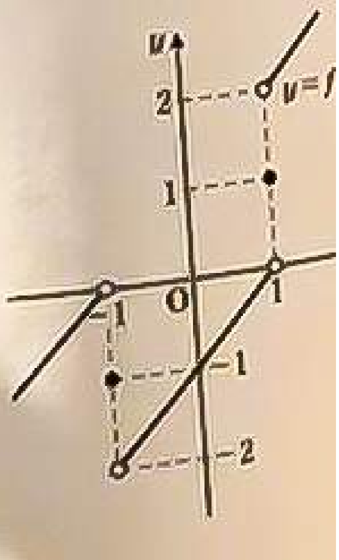
(iii) When $|x|>1$,

$$f(x)=\lim_{n\to\infty}\frac{1-\frac{1}{|x|^n}}{1+\frac{1}{|x|^n}}=1$$

From (i)~(iii), the graph of $y=f(x)$ is as shown in the diagram.

Also, $f(x)$ is discontinuous at $x=\pm 1$.

(2) $f(x)=\lim_{n\to\infty}\frac{|x|^n-1}{|x|^n+1}+x$



[Sol] (i) When $|x|<1$,

$$f(x)=x-1$$

(ii) When $x=1$,

$$f(1)=1$$

(iii) When $x=-1$,

$$f(-1)=-1$$

(iv) When $|x|>1$,

$$f(x)=\lim_{n\to\infty}\frac{1-\frac{1}{|x|^n}}{1+\frac{1}{|x|^n}}+x=x+1$$

From (i)~(iv), the graph of $y=f(x)$ is as shown in the diagram.

Also, $f(x)$ is discontinuous at $x=\pm 1$.

Continuous and Discontinuous Functions

100%	~90%	~80%	~70%	69%~

Name

Date / /

Time : to :

Graph each given function and find x at which $f(x)$ is discontinuous.

(1) $f(x)=\lim_{n\to\infty}\dfrac{x^{2n+1}+1}{x^{2n}+1}$

[Sol] (i) When $|x|<1$,

$$f(x)=1$$

(ii) When $x=1$,

$f(1)=1 \quad \leftarrow \quad \boxed{\lim_{n\to\infty}x^{2n}=\lim_{n\to\infty}x^{2n+1}=1}$

(iii) When $x=-1$,

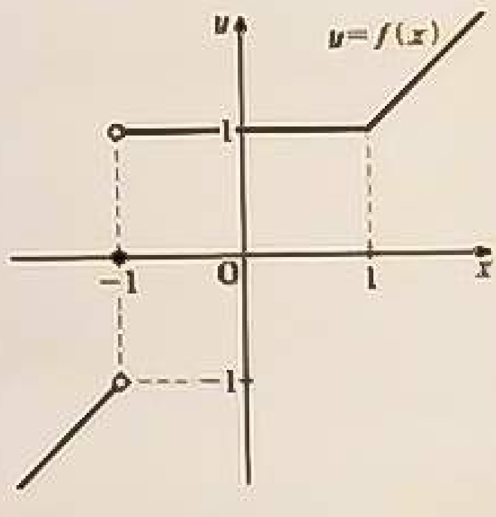
$f(-1)=0 \quad \leftarrow \quad \boxed{\lim_{n\to\infty}x^{2n}=1,\ \lim_{n\to\infty}x^{2n+1}=-1}$

(iv) When $|x|>1$,

$$f(x)=\lim_{n\to\infty}\frac{x+\dfrac{1}{x^{2n}}}{1+\dfrac{1}{x^{2n}}}=x$$

From (i) ~ (iv), the graph of $y=f(x)$ is as shown in the diagram.

Also, $f(x)$ is discontinuous at $x=-1$.



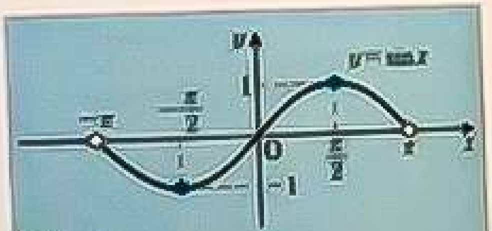
N135b

(2) $f(x)=\lim_{n\to\infty}(x+\sin^{2n}x) \quad (-\pi<x<\pi)$

[Sol] (i) When $|\sin x|<1$,

i.e. $-\pi<x<-\dfrac{\pi}{2},\ -\dfrac{\pi}{2}<x<\dfrac{\pi}{2},\ \dfrac{\pi}{2}<x<\pi,$

$f(x)=x \leftarrow \boxed{\lim_{n\to\infty}\sin^{2n}x=0}$



When

$-\pi<x<-\dfrac{\pi}{2},\ -\dfrac{\pi}{2}<x<\dfrac{\pi}{2},\ \dfrac{\pi}{2}<x<\pi,$

$|\sin x|<1$

When $x=\dfrac{\pi}{2}$, $\sin x=1$

When $x=-\dfrac{\pi}{2}$, $\sin x=-1$

(ii) When $\sin x=1$, i.e. $x=\dfrac{\pi}{2}$,

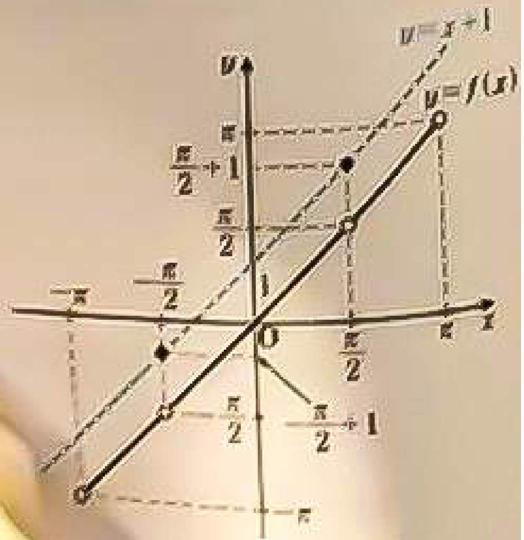
$f\left(\dfrac{\pi}{2}\right)=\dfrac{\pi}{2}+1 \leftarrow \boxed{\lim_{n\to\infty}\sin^{2n}x=1}$

(iii) When $\sin x=-1$, i.e. $x=-\dfrac{\pi}{2}$,

$f\left(-\dfrac{\pi}{2}\right)=-\dfrac{\pi}{2}+1 \leftarrow \boxed{\lim_{n\to\infty}\sin^{2n}x=1}$

From (i)~(iii), the graph of $y=f(x)$ is as shown in the diagram.

Also, $f(x)$ is discontinuous at $x=\pm\dfrac{\pi}{2}$.



Continuous and Discontinuous Functions

Name ______________

Date / /

Time : to :

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1. Find the constant a for which each given function $f(x)$ is continuous for all values of x.

Ex.

$$f(x)=\begin{cases} -ax^2-4a-1 & (x \leqq -2) \\ -ax-9 & (x > -2) \end{cases}$$

[Sol] If $f(x)$ is continuous for all x, then $f(x)$ must be continuous at $x=-2$. ← $f(-2)=\lim_{x\to-2^+} f(x)$ needs to be satisfied.

$f(-2)=-a\cdot(-2)^2-4a-1=-8a-1$

$\lim_{x\to-2^+} f(x)=\lim_{x\to-2^+}(-ax-9)=2a-9$ ← As $x\to-2^+$, $f(x)=-ax-9$

$\therefore\ -8a-1=2a-9$

$\therefore\ a=\dfrac{4}{5}$

(1) $f(x)=\begin{cases} x^2 & (x \leqq 2) \\ ax+1 & (x > 2) \end{cases}$

[Sol] If $f(x)$ is continuous for all x, then $f(x)$ must be continuous at $x=2$.

$f(2)=2^2=4$

$\lim_{x\to2^+} f(x)=\lim_{x\to2^+}(ax+1)=2a+1$

$\therefore\ 4=2a+1$

$\therefore\ a=\dfrac{3}{2}$

(2) $f(x)=\begin{cases} |2x-5|+9a & (x < 3) \\ a^2x^2+ax+2 & (x \geqq 3) \end{cases}$

[Sol] If $f(x)$ is continuous for all x, then $f(x)$ must be continuous at $x=3$.

$f(3)=a^2\cdot3^2+a\cdot3+2=9a^2+3a+2$

$\lim_{x\to3^-} f(x)=\lim_{x\to3^-}(|2x-5|+9a)=9a+1$

$\therefore\ 9a^2+3a+2=9a+1$

$9a^2-6a+1=0$

$(3a-1)^2=0$

$\therefore\ a=\dfrac{1}{3}$

N136b

2. Find the constants a and b for which each given function $f(x)$ is continuous for all values of x.

(1) $f(x)=\begin{cases} \dfrac{2}{x} & (x<-1,\ 2<x) \\ ax+b & (-1 \leqq x \leqq 2) \end{cases}$

[Sol] If $f(x)$ is continuous for all x, then $f(x)$ must be continuous at $x=-1,\ 2$

$$f(-1)=a\cdot(-1)+b=-a+b,\quad f(2)=a\cdot 2+b=2a+b$$

$$\lim_{x\to -1^-} f(x)=\lim_{x\to -1^-}\frac{2}{x}=-2,\quad \lim_{x\to 2^+} f(x)=\lim_{x\to 2^+}\frac{2}{x}=1$$

$$\therefore \begin{cases} -a+b=-2 & \cdots ① \\ 2a+b=1 & \cdots ② \end{cases}$$

From ① and ②, $a=1,\ b=-1$

(2) $f(x)=\begin{cases} 1 & (x \leqq 0) \\ \dfrac{ax^2}{1-\cos x} & (0<x<\pi) \\ b & (x \geqq \pi) \end{cases}$

[Sol] If $f(x)$ is continuous for all x, then $f(x)$ must be continuous at $x=0,\ \pi$.

$$f(0)=1,\ f(\pi)=b$$

$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}\frac{ax^2}{1-\cos x}=\lim_{x\to 0^+}\frac{ax^2(1+\cos x)}{(1-\cos x)(1+\cos x)} \quad \leftarrow \boxed{\text{N124}}$$

$$=\lim_{x\to 0^+}\frac{ax^2(1+\cos x)}{1-\cos^2 x}=\lim_{x\to 0^+}\frac{ax^2(1+\cos x)}{\sin^2 x}$$

$$=\lim_{x\to 0^+}\frac{a(1+\cos x)}{\left(\dfrac{\sin x}{x}\right)^2}=2a \quad \leftarrow \boxed{\lim_{x\to 0}\frac{\sin x}{x}=1}$$

$$\lim_{x\to \pi^-} f(x)=\lim_{x\to \pi^-}\frac{ax^2}{1-\cos x}=\frac{1}{2}\pi^2 a$$

$$\therefore \begin{cases} 1=2a & \cdots ① \\ b=\dfrac{1}{2}\pi^2 a & \cdots ② \end{cases}$$

From ① and ②, $a=\dfrac{1}{2},\ b=\dfrac{\pi^2}{4}$

Continuous and Discontinuous Functions

Name

Date / /

Time : to :

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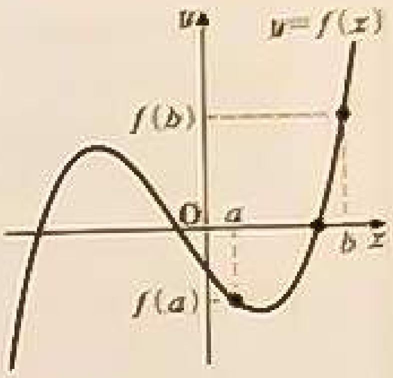
Given that the function $f(x)$ is continuous on the closed interval $[a, b]$, then its graph has no break between points $(a, f(a))$ and $(b, f(b))$.
If $f(a)$ and $f(b)$ have different signs, then the graph intersects with the x-axis between a and b.
Since the x-coordinates of these points are solutions for the equation $f(x)=0$, the following statement is true.

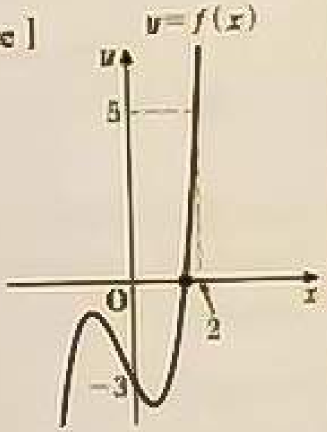


Intermediate Value Theorem

If the function $f(x)$ is continuous on the closed interval $[a, b]$ and $f(a)\cdot f(b)<0$, then the equation $f(x)=0$ has at least one real solution in the interval $a<x<b$.

Ex. Prove that equation $x^3+x^2-2x-3=0$ has at least one real solution in the interval $0<x<2$.

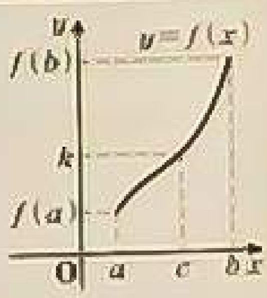
[Reference]



[Sol] Let $f(x)=x^3+x^2-2x-3$.
The function $f(x)$ is continuous on the interval $[0, 2]$.
Also, $f(0)=\boxed{-3}<0$, $f(2)=\boxed{5}>0$
Therefore, from the Intermediate Value Theorem, $f(x)=0$ has at least one real solution in the interval $0<x<2$.

Answers: -3, 5

The interval $a \leqq x \leqq b$ is called a ***closed interval*** and the interval $a<x<b$ is called an ***open interval***. They are expressed as $[a, b]$ and (a, b), respectively.
Generally, the Intermediate Value Theorem is explained as follows:
If the function $f(x)$ is continuous on the closed interval $[a, b]$ and $f(a) \neq f(b)$, then there is at least one value c that satisfies $f(c)=k$ and $a<c<b$ for any value k which lies between $f(a)$ and $f(b)$.



NI37b

1. Prove that equation $x^4-5x+2=0$ has at least one real solution in the interval $0<x<1$.

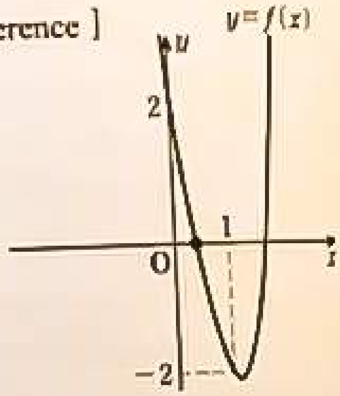
[Sol] Let $f(x)=x^4-5x+2$.

The function $f(x)$ is continuous on the interval $[0, 1]$.

Also, $f(0)=2>0$, $f(1)=-2<0$

Therefore, from the Intermediate Value Theorem, $f(x)=0$ has at least one real solution in the interval $0<x<1$.

[Reference]



2. Prove that equation $\log_2 x+x-2=0$ has at least one real solution in the interval $1<x<2$.

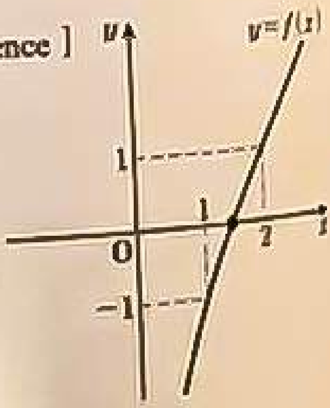
[Sol] Let $f(x)=\log_2 x+x-2$.

The function $f(x)$ is continuous on the interval $[1, 2]$.

Also, $f(1)=-1<0$, $f(2)=1>0$

Therefore, from the Intermediate Value Theorem, $f(x)=0$ has at least one real solution in the interval $1<x<2$.

[Reference]



3. Prove that equation $x=\cos x$ has at least one real solution in the interval $0<x<\pi$.

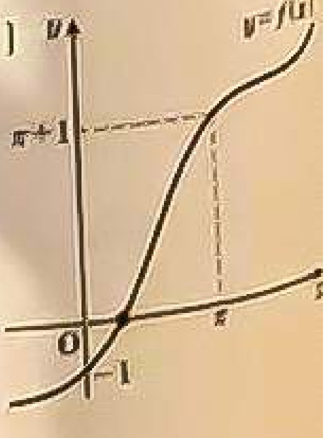
[Sol] Let $f(x)=x-\cos x$.

The function $f(x)$ is continuous on the interval $[0, \pi]$.

Also, $f(0)=-1<0$, $f(\pi)=\pi+1>0$

Therefore, from the Intermediate Value Theorem, $f(x)=0$ has at least one real solution in the interval $0<x<\pi$.

[Reference]



N138a

KUMON

Continuous and Discontinuous Functions

Name ______________________

Date ____ / ____ / ____

Time ____ : ____ to ____ : ____

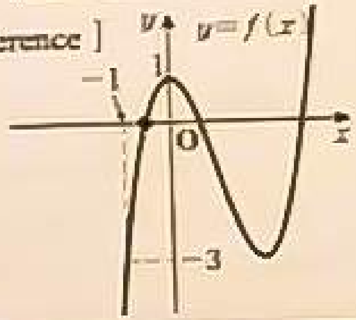
100%	~90%	~80%	~70%	69%~
(mistakes) 0	—	—	—	1~

Ex. Prove that equation $x^3-3x^2+1=0$ has at least one negative solution.

[Sol] Let $f(x)=x^3-3x^2+1$. The function $f(x)$ is continuous for all real numbers x.
Also, $f(0)=1>0$, $f(-1)=-3<0$
Therefore, from the Intermediate Value Theorem, $f(x)=0$ has real solutions in the interval $-1<x<0$; that is, at least one negative solution.

Finding negative number x satisfying $f(0)\cdot f(x)<0$

[Reference]

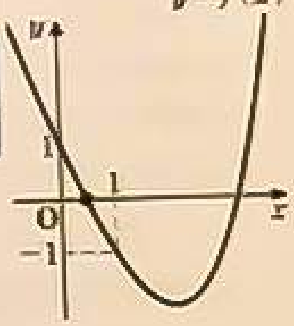


1. Prove that equation $2^x-3x=0$ has at least one positive solution.

[Sol] Let $f(x)=2^x-3x$. The function $f(x)$ is continuous for all real numbers x.
Also, $f(0)=1>0$, $f(1)=-1<0$
Therefore, from the Intermediate Value Theorem, $f(x)=0$ has real solutions in the interval $0<x<1$; that is, at least one positive solution.

Finding positive number x satisfying $f(0)\cdot f(x)<0$

[Reference]



2. Prove that equation $x-2\sin x=k$ $(k>0)$ has at least one positive solution.
$\left(\lim\limits_{x\to\infty}\dfrac{\sin x}{x}=0\right)$

[Sol] Let $f(x)=x-2\sin x-k$ $(k>0)$. The function $f(x)$ is continuous for all real numbers x.

Also, $f(0)=\boxed{-k}<0$ and

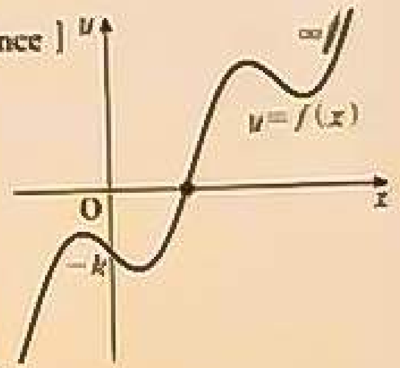
since $f(x)=x\left(\boxed{1-\dfrac{2\sin x}{x}-\dfrac{k}{x}}\right)$,

$\lim\limits_{x\to\infty}f(x)=\boxed{\infty}>0$

Therefore, from the Intermediate Value Theorem, $f(x)=0$ has real solutions in the interval

$0<x<\boxed{\infty}$; that is, at least one positive solution.

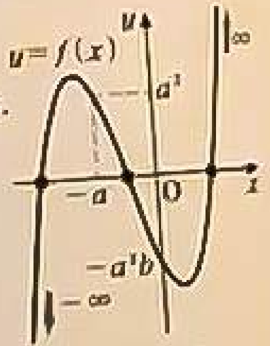
[Reference]



N138b

3. Given $a>0$ and $b>0$, prove that cubic equation $(x+a)^2(x-b)+x^2=0$ has one positive and two negative solutions.
(Use the fact that a cubic equation cannot have more than three solutions.)

[Reference]



[Sol] Let $f(x)=(x+a)^2(x-b)+x^2$.

The function $f(x)$ is continuous for all real numbers x.

Also, $f(0)=-a^2b<0$, $f(-a)=a^2>0$ and

since $f(x)=x^3\left[\left(1+\frac{a}{x}\right)^2\left(1-\frac{b}{x}\right)+\frac{1}{x}\right]$,

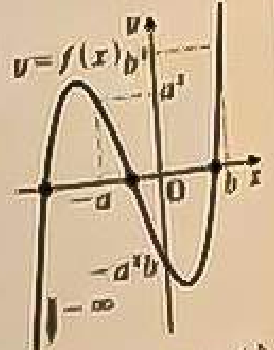
$\lim_{x\to-\infty} f(x)=-\infty$, $\lim_{x\to\infty} f(x)=\infty$

Therefore, from the Intermediate Value Theorem, $f(x)=0$ has real solutions in each interval $-\infty<x<-a$, $-a<x<0$ and $0<x<\infty$.

Since a cubic equation cannot have more than three solutions, $f(x)=0$ has one positive and two negative solutions.

Alternative Solution

[Reference]



Let $f(x)=(x+a)^2(x-b)+x^2$.

The function $f(x)$ is continuous for all real numbers x.

$f(0)=-a^2b<0$, $f(b)=b^2>0$ and

Also $f(-a)=a^2>0$,

since $f(x)=x^3\left[\left(1+\frac{a}{x}\right)^2\left(1-\frac{b}{x}\right)+\frac{1}{x}\right]$,

$\lim_{x\to-\infty} f(x)=-\infty$

Therefore, from the Intermediate Value Theorem, $f(x)=0$ has real solutions in each interval $-\infty<x<-a$, $-a<x<0$ and $0<x<b$.

Since a cubic equation cannot have more than three solutions, $f(x)=0$ has one positive and two negative solutions.

Continuous and Discontinuous Functions

Name

Date / /

Time : to :

100%	-90%	-80%	-70%	69%-
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1. Show that the following infinite series converges for all real numbers x. Then, let $f(x)$ be the sum. Graph the function $y=f(x)$ and find x at which $f(x)$ is discontinuous.

$$x^2+\frac{x^2}{1+x^2}+\frac{x^2}{(1+x^2)^2}+\cdots+\frac{x^2}{(1+x^2)^{n-1}}+\cdots$$

[Sol] This infinite series is the infinite geometric series whose 1st term is x^2 and common ratio is $\frac{1}{1+x^2}$. ← N83

(i) When $x\neq 0$,

$1+x^2>1$; therefore, $\left|\frac{1}{1+x^2}\right|<1$

Thus, this infinite geometric series converges.

The sum is $\dfrac{x^2}{1-\dfrac{1}{1+x^2}}=1+x^2$.

(ii) When $x=0$, all terms become 0.

Therefore, the series converges and the sum is 0.

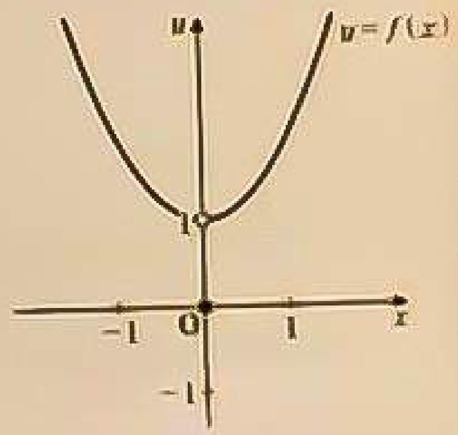
From (i) and (ii), this infinite series converges for all real numbers x.

Thus,

$$f(x)=\begin{cases}1+x^2 & (x\neq 0)\\ 0 & (x=0)\end{cases}$$

Therefore, the graph of $y=f(x)$ is as shown in the diagram.

Also, $f(x)$ is discontinuous at $x=0$.



N139b

2. Solve the following questions.

(1) Determine $f(x)=\lim_{n\to\infty}\dfrac{x^{2n}-x^{2n-1}+ax^2+bx}{x^{2n}+1}$.

[Sol] (i) When $|x|<1$,

$$f(x)=ax^2+bx \quad \leftarrow \boxed{\lim_{n\to\infty}x^{2n}=\lim_{n\to\infty}x^{2n-1}=0}$$

(ii) When $x=1$,

$$f(1)=\frac{a+b}{2} \quad \leftarrow \boxed{\lim_{n\to\infty}x^{2n}=\lim_{n\to\infty}x^{2n-1}=1}$$

(iii) When $x=-1$,

$$f(-1)=\frac{a-b+2}{2} \quad \leftarrow \boxed{\begin{array}{l}\lim_{n\to\infty}x^{2n}=1\\ \lim_{n\to\infty}x^{2n-1}=-1\end{array}}$$

(iv) When $|x|>1$,

$$f(x)=\lim_{n\to\infty}\frac{1-\frac{1}{x}+\frac{a}{x^{2n-2}}+\frac{b}{x^{2n-1}}}{1+\frac{1}{x^{2n}}}=1-\frac{1}{x} \quad \leftarrow \boxed{\begin{array}{l}\lim_{n\to\infty}\frac{1}{x^{2n}}\\ =\lim_{n\to\infty}\frac{1}{x^{2n-2}}\\ =\lim_{n\to\infty}\frac{1}{x^{2n-1}}=0\end{array}}$$

(2) Find the constants a and b for which the function $f(x)$ determined above is continuous for all x.

[Sol] From (1), if $f(x)$ is continuous for all x, then $f(x)$ must be continuous at $x=\pm1$.

$$\lim_{x\to1-}f(x)=\lim_{x\to1-}(ax^2+bx)=a+b$$

$$\lim_{x\to1+}f(x)=\lim_{x\to1+}\left(1-\frac{1}{x}\right)=0$$

$$\lim_{x\to-1-}f(x)=\lim_{x\to-1-}\left(1-\frac{1}{x}\right)=2$$

$$\lim_{x\to-1+}f(x)=\lim_{x\to-1+}(ax^2+bx)=a-b$$

$$\therefore \begin{cases}\dfrac{a+b}{2}=a+b=0 & \cdots① \quad \leftarrow \boxed{f(1)=\lim_{x\to1-}f(x)=\lim_{x\to1+}f(x)}\\[2ex] \dfrac{a-b+2}{2}=2=a-b & \cdots② \quad \leftarrow \boxed{f(-1)=\lim_{x\to-1-}f(x)=\lim_{x\to-1+}f(x)}\end{cases}$$

From ① and ②, $a=1,\ b=-1$

Continuous and Discontinuous Functions

Name ____________

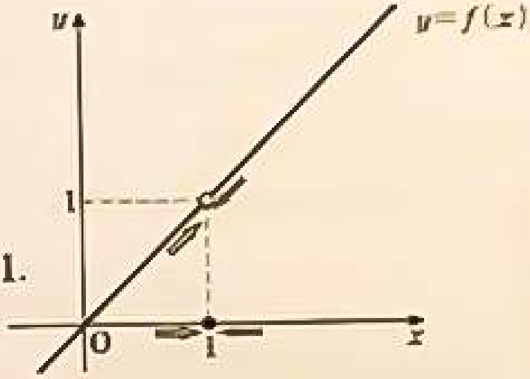
Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
0	—	—	1	2~

1. Examine if the following function is continuous at $x=1$. ➡ N I 31

$$f(x)=\begin{cases} \dfrac{x^2-x}{x-1} & (x \neq 1) \\ 0 & (x=1) \end{cases}$$



[Sol] $\lim_{x\to 1} f(x)=\lim_{x\to 1}\dfrac{x(x-1)}{x-1}=\lim_{x\to 1} x=1$

Also, $f(1)=0$

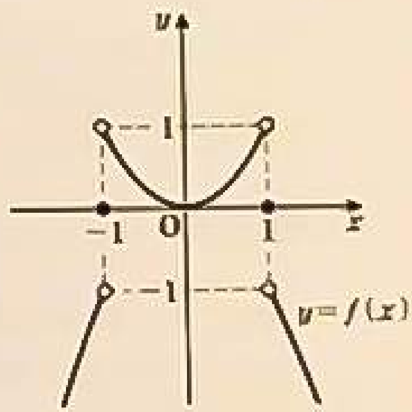
$\therefore \lim_{x\to 1} f(x) \neq f(1)$

Therefore, $f(x)$ is **discontinuous** at $x=1$.

2. Graph the following function and find x at which $f(x)$ is discontinuous.

$$f(x)=\lim_{n\to\infty}\frac{x^2(1-|x|^n)}{1+|x|^n}$$

➡ N I 34



[Sol] (i) When $|x|<1$,

$$f(x)=x^2$$

(ii) When $x=\pm 1$,

$$f(x)=0$$

(iii) When $|x|>1$,

$$f(x)=\lim_{n\to\infty}\frac{x^2\left(\dfrac{1}{|x|^n}-1\right)}{\dfrac{1}{|x|^n}+1}=-x^2$$

From (i) ~ (iii), the graph of $y=f(x)$ is as shown in the diagram.

Also, $f(x)$ is discontinuous at $x=\pm 1$.

3. Find the constants a and b for which the following function $f(x)$ is continuous for all x.

➡ N136

$$f(x)=\begin{cases} x-x^2 & (x<-2,\ 3<x) \\ ax+b & (-2\leqq x\leqq 3) \end{cases}$$

[Sol] If $f(x)$ is continuous for all x, then $f(x)$ must be continuous at $x=-2,\ 3$.

$$f(-2)=a\cdot(-2)+b=-2a+b,\quad f(3)=a\cdot 3+b=3a+b$$

$$\lim_{x\to -2^-} f(x)=\lim_{x\to -2^-}(x-x^2)=-6,\quad \lim_{x\to 3^+} f(x)=\lim_{x\to 3^+}(x-x^2)=-6$$

$$\therefore \begin{cases} -2a+b=-6 & \cdots ① \\ 3a+b=-6 & \cdots ② \end{cases}$$

From ① and ②, $a=0,\ b=-6$

4. Prove that equation $\tan x=\cos x$ has at least one real solution in the interval $0<x<\frac{\pi}{4}$.

➡ N137

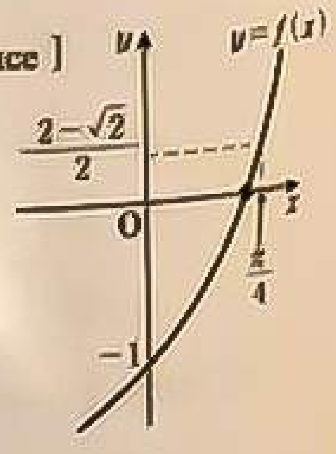
[Sol] Let $f(x)=\tan x-\cos x$.

The function $f(x)$ is continuous on the interval $\left[0,\ \frac{\pi}{4}\right]$.

Also, $f(0)=-1<0,\ f\left(\frac{\pi}{4}\right)=\frac{2-\sqrt{2}}{2}>0$

Therefore, from the Intermediate Value Theorem, $f(x)=0$ has at least one real solution in the interval $0<x<\frac{\pi}{4}$.

[Reference]



N 141 a

KUMON

Differentiation 1

Name

Date / /

Time : to :

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Given the function $f(x)$, if the limit value $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ exists, then it is called the ***derivative value*** of $f(x)$ at $x=a$ and expressed as $f'(a)$.

In this case, $f(x)$ is said to be ***differentiable*** at $x=a$.

For the following functions, find the derivative values at x indicated in the brackets〔 〕.

Ex. $f(x)=x^3-x$ 〔$x=1$〕

[Sol] $f'(1)=\lim_{h\to 0}\frac{[(1+h)^3-(1+h)]-(1^3-1)}{h}$

$=\lim_{h\to 0}\frac{h^3+3h^2+2h}{h}$

$=\lim_{h\to 0}(h^2+3h+2)$

$=2$

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

(1) $f(x)=x^2+x+1$ 〔$x=2$〕

[Sol] $f'(2)=\lim_{h\to 0}\frac{[(2+h)^2+(2+h)+1]-(2^2+2+1)}{h}$

$=\lim_{h\to 0}\frac{h^2+5h}{h}$

$=\lim_{h\to 0}(h+5)$

$=5$

[Reference] L44

Let $a+h=x$. Then, $h=x-a$. As $h\to 0$, $x\to a$. Therefore, $f'(a)$ can also be expressed as

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

N141b

(2) $f(x)=\frac{1}{x}$ $(x=3)$

$$\text{[Sol] } f'(3)=\lim_{h\to 0}\frac{\frac{1}{3+h}-\frac{1}{3}}{h}$$

$$=\lim_{h\to 0}\frac{-h}{3h(3+h)}$$

$$=\lim_{h\to 0}\frac{-1}{3(3+h)}$$

$$=-\frac{1}{9}$$

(3) $f(x)=\frac{1}{x^3}$ $(x=1)$

$$\text{[Sol] } f'(1)=\lim_{h\to 0}\frac{\frac{1}{(1+h)^3}-\frac{1}{1^3}}{h}$$

$$=\lim_{h\to 0}\frac{-(h^3+3h^2+3h)}{h(1+h)^3}$$

$$=\lim_{h\to 0}\frac{-(h^2+3h+3)}{(1+h)^3}$$

$$=-3$$

(4) $f(x)=\sqrt{x}$ $(x=1)$

$$\text{[Sol] } f'(1)=\lim_{h\to 0}\frac{\sqrt{1+h}-\sqrt{1}}{h}$$

$$=\lim_{h\to 0}\frac{(\sqrt{1+h}-1)(\boxed{\sqrt{1+h}+1})}{h(\boxed{\sqrt{1+h}+1})}$$

Muliplying the num
and the denominator
$\sqrt{1+h}+1$

$$\frac{h}{h(\sqrt{1+h}+1)}$$

$$\frac{1}{\sqrt{1+h}+1}$$

KUMON

Differentiation 1

Name

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
0	—	—	1	2~

If the function $f(x)$ is differentiable for each value a on a certain interval, the function corresponding to the derivative value $f'(a)$ on this interval is called the ***derivative*** of $f(x)$ and expressed as $f'(x)$. The process of finding the derivative $f'(x)$ is called ***differentiating*** the function $f(x)$.

Using the definition of the derivative, differentiate the following functions.

Ex. $f(x)=\dfrac{1}{x^2}$

[Sol] $$f'(x)=\lim_{h\to 0}\frac{\dfrac{1}{(x+h)^2}-\dfrac{1}{x^2}}{h} \leftarrow \boxed{f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}}$$

$$=\lim_{h\to 0}\frac{-(2hx+h^2)}{h(x+h)^2x^2}$$

$$=\lim_{h\to 0}\frac{-(2x+h)}{(x+h)^2x^2}$$

$$=-\frac{2}{x^3}$$

(1) $f(x)=\dfrac{1}{x+1}$

[Sol] $$f'(x)=\lim_{h\to 0}\frac{\dfrac{1}{(x+h)+1}-\dfrac{1}{x+1}}{h}$$

$$=\lim_{h\to 0}\frac{-h}{h(x+h+1)(x+1)}$$

$$=\lim_{h\to 0}\frac{-1}{(x+h+1)(x+1)}$$

$$=-\frac{1}{(x+1)^2}$$

The derivative of the function $y=f(x)$ is also expressed by symbols such as y', $[f(x)]'$, $\dfrac{dy}{dx}$ and $\dfrac{d}{dx}f(x)$. $\dfrac{dy}{dx}$ is read as "dy, dx."

N142b

(2) $f(x)=\dfrac{x}{x-1}$

[Sol] $f'(x)=\displaystyle\lim_{h\to 0}\frac{\dfrac{x+h}{(x+h)-1}-\dfrac{x}{x-1}}{h}$

$=\displaystyle\lim_{h\to 0}\frac{-h}{h(x+h-1)(x-1)}$

$=\displaystyle\lim_{h\to 0}\frac{-1}{(x+h-1)(x-1)}$

$=-\dfrac{1}{(x-1)^2}$

(3) $f(x)=\sqrt{x}$

[Sol] $f'(x)=\displaystyle\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}$

$=\displaystyle\lim_{h\to 0}\frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h(\sqrt{x+h}+\sqrt{x})}$

Multiplying the numerator and the denominator by $\sqrt{x+h}+\sqrt{x}$

$=\displaystyle\lim_{h\to 0}\frac{h}{h(\sqrt{x+h}+\sqrt{x})}$

$=\displaystyle\lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}$

$=\dfrac{1}{2\sqrt{x}}$

(4) $f(x)=\sqrt{2x}$

[Sol] $f'(x)=\displaystyle\lim_{h\to 0}\frac{\sqrt{2(x+h)}-\sqrt{2x}}{h}$

$=\displaystyle\lim_{h\to 0}\frac{[\sqrt{2(x+h)}-\sqrt{2x}\,][\sqrt{2(x+h)}+\sqrt{2x}\,]}{h[\sqrt{2(x+h)}+\sqrt{2x}\,]}$

$=\displaystyle\lim_{h\to 0}\frac{2h}{h[\sqrt{2(x+h)}+\sqrt{2x}\,]}$

$=\displaystyle\lim_{h\to 0}\frac{2}{\sqrt{2(x+h)}+\sqrt{2x}}$

$=\dfrac{1}{\sqrt{2x}}$

KUMON

Differentiation 1

Name

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
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Properties of Derivatives

When k is a constant and n is a positive integer,

if $y=x^n$, then $y'=nx^{n-1}$

if $y=kf(x)$, then $y'=kf'(x)$

if $y=f(x)+g(x)$, then $y'=f'(x)+g'(x)$

if $y=f(x)-g(x)$, then $y'=f'(x)-g'(x)$

Differentiate the following functions.

Ex. $y=x^3+3x^2-4x+2$

[Sol] $y'=3x^2+6x-4$

(1) $y=2x^3+5x$

[Sol] $y'=6x^2+5$

(2) $y=x^2-2x-1$

[Sol] $y'=2x-2$

(3) $y=x^4-2x^3+3x-9$

[Sol] $y'=4x^3-6x^2+3$

(4) $y=\frac{2}{3}x^3-x^2+2x-3$

[Sol] $y'=2x^2-2x+2$

(5) $y=4x^3+\frac{2}{3}x^2-\frac{3}{5}x$

[Sol] $y'=12x^2+\frac{4}{3}x-\frac{3}{5}$

(6) $y=-x^3+5x^2-4x+2$

[Sol] $y'=-3x^2+10x-4$

(7) $y=-2x^3+\frac{1}{2}x^2-4x+1$

[Sol] $y'=-6x^2+x-4$

(8) $y=-\frac{3}{4}x^3+\frac{5}{2}x^2+x-\frac{1}{2}$

[Sol] $y'=-\frac{9}{4}x^2+5x+1$

N143b

(9) $y=2(3x-2)$

[Sol] $y=\boxed{6}x-\boxed{4}$

$y'=\boxed{6}$

(10) $y=x(x+1)$

[Sol] $y=x^2+x$

$y'=2x+1$

(11) $y=\frac{2}{3}x^2(x-3)$

[Sol] $y=\frac{2}{3}x^3-2x^2$

$y'=2x^2-4x$

(12) $y=x^3(x^2-4x+1)$

[Sol] $y=x^5-4x^4+x^3$

$y'=5x^4-16x^3+3x^2$

(13) $y=(x-1)(x+2)$

[Sol] $y=x^2+x-2$

$y'=2x+1$

(14) $y=(3x^2-1)(x+2)$

[Sol] $y=3x^3+6x^2-x-2$

$y'=9x^2+12x-1$

Differentiation 1

Name

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
0	—	1	—	2~

Given the two functions $f(x)$ and $g(x)$, prove the following equality using the definition of the derivative.

$$[f(x)g(x)]'=f'(x)g(x)+f(x)g'(x)$$

[Sol] Let $y=f(x)g(x)$.

$$y'=\lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}$$

$$=\lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x+h)+f(x)g(x+h)-f(x)g(x)}{h}$$

$$=\lim_{h\to 0}\frac{[\boxed{f(x+h)-f(x)}]g(x+h)+f(x)[\boxed{g(x+h)-g(x)}]}{h}$$

$$=\lim_{h\to 0}\left[\frac{\boxed{f(x+h)-f(x)}}{h}\cdot g(x+h)+f(x)\cdot\frac{\boxed{g(x+h)-g(x)}}{h}\right]$$

$$=f'(x)g(x)+f(x)g'(x) \leftarrow \boxed{\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=f'(x)}$$

Product Rule

$$[f(x)g(x)]'=f'(x)g(x)+f(x)g'(x)$$

Using the formula above, differentiate the following functions.

Ex. $y=(x^2+1)(x^2-2)$

[Sol] $y'=(x^2+1)'(x^2-2)+(x^2+1)(x^2-2)'$

$=2x(x^2-2)+(x^2+1)\cdot 2x$

$=4x^3-2x$

(1) $y=(3x^2+1)(2x-3)$

[Sol] $y'=(3x^2+1)'(2x-3)+(3x^2+1)(2x-3)'$

$=6x(2x-3)+(3x^2+1)\cdot 2$

$=18x^2-18x+2$ $\quad[=2(9x^2-9x+1)]$

N144b

(2) $y=(2x+4)(-x-3)$

[Sol] $y'=(2x+4)'(-x-3)+(2x+4)(-x-3)'$

$=2(-x-3)+(2x+4)\cdot(-1)$

$=-4x-10$ $[=-2(2x+5)]$

(3) $y=(2x^2+3)(-x^2+5x)$

[Sol] $y'=(2x^2+3)'(-x^2+5x)+(2x^2+3)(-x^2+5x)'$

$=4x(-x^2+5x)+(2x^2+3)(-2x+5)$

$=-8x^3+30x^2-6x+15$

(4) $y=(x^2-2x-3)(x+4)$

[Sol] $y'=(x^2-2x-3)'(x+4)+(x^2-2x-3)(x+4)'$

$=(2x-2)(x+4)+(x^2-2x-3)\cdot 1$

$=3x^2+4x-11$

(5) $y=(x^2-2)(x^3+3x-5)$

[Sol] $y'=(x^2-2)'(x^3+3x-5)+(x^2-2)(x^3+3x-5)'$

$=2x(x^3+3x-5)+(x^2-2)(3x^2+3)$

$=5x^4+3x^2-10x-6$

Differentiation 1

Name

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Time : to :

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1. Differentiate the following functions.

Ex. $y=x(x+1)(2x-3)$

[Sol] $y=(x^2+x)(2x-3)$ ← $x(x+1)=x^2+x$

$y'=(2x+1)(2x-3)+(x^2+x)\cdot 2$ ← $[f(x)g(x)]'=f'(x)g(x)+f(x)g'(x)$

$=6x^2-2x-3$

(1) $y=2x(x-1)(3x-1)$

[Sol] $y=(2x^2-2x)(3x-1)$

$y'=(4x-2)(3x-1)+(2x^2-2x)\cdot 3$

$=18x^2-16x+2$ $[=2(9x^2-8x+1)]$

(2) $y=(x+1)(x+2)(x+3)$

[Sol] $y=(x^2+3x+2)(x+3)$

$y'=(2x+3)(x+3)+(x^2+3x+2)\cdot 1$

$=3x^2+12x+11$

(3) $y=(x-2)(x+1)(2x+1)$

[Sol] $y=(x^2-x-2)(2x+1)$

$y'=(2x-1)(2x+1)+(x^2-x-2)\cdot 2$

$=6x^2-2x-5$

N145b

(4) $y=(3x+2)(x-1)(2x+5)$

[Sol] $y=(3x^2-x-2)(2x+5)$

$$y'=(6x-1)(2x+5)+(3x^2-x-2)\cdot 2$$
$$=18x^2+26x-9$$

(5) $y=(x-3)(x+1)^2(x+3)$

[Sol] $y=(x^2-9)(x^2+2x+1)$

$$y'=2x(x^2+2x+1)+(x^2-9)(2x+2)$$
$$=4x^3+6x^2-16x-18 \quad [=2(x+1)(2x^2+x-9)]$$

2. Show that the derivative of the function $y=f(x)g(x)h(x)$ is given by the following expression.

$$y'=f'(x)g(x)h(x)+f(x)g'(x)h(x)+f(x)g(x)h'(x)$$

[Sol] Since $y=[f(x)g(x)]h(x)$,

$$y'=[f(x)g(x)]'h(x)+[f(x)g(x)]h'(x)$$
$$=[f'(x)g(x)+f(x)g'(x)]h(x)+f(x)g(x)h'(x)$$
$$=f'(x)g(x)h(x)+f(x)g'(x)h(x)+f(x)g(x)h'(x)$$

3. Using the equality shown in question 2, differentiate the following function.

$$y=(x+1)(x-2)(x+3)$$

[Sol] $y'=1\cdot(x-2)(x+3)+(x+1)\cdot 1\cdot(x+3)+(x+1)(x-2)\cdot 1$

$$=3x^2+4x-5$$

Differentiation 1

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Given the two functions $f(x)$ and $g(x)$, prove the following equalities using the definition of the derivative.

[1] $\left[\dfrac{1}{g(x)}\right]' = -\dfrac{g'(x)}{[g(x)]^2}$

[Sol] Let $y = \dfrac{1}{g(x)}$.

$$y' = \lim_{h\to 0}\frac{1}{h}\left[\frac{1}{g(x+h)} - \frac{1}{g(x)}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[-\frac{\boxed{g(x+h)-g(x)}}{g(x+h)g(x)}\right]$$

$$= \lim_{h\to 0}\left[-\frac{1}{g(x+h)g(x)}\cdot\frac{\boxed{g(x+h)-g(x)}}{h}\right]$$

$$= -\frac{1}{[g(x)]^2}\cdot\boxed{g'(x)}$$

$$= -\frac{g'(x)}{[g(x)]^2}$$

$\lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h} = f'(x)$

[2] $\left[\dfrac{f(x)}{g(x)}\right]' = \dfrac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}$ (Use the equality in [1].)

[Sol] $\left[\dfrac{f(x)}{g(x)}\right]' = \left[f(x)\cdot\dfrac{1}{g(x)}\right]'$

$$= f'(x)\cdot\boxed{\frac{1}{g(x)}} + f(x)\cdot\left[\boxed{\frac{1}{g(x)}}\right]'$$

$[f(x)g(x)]' = f'(x)g(x)+f(x)g'(x)$

From [1]

$$= f'(x)\cdot\boxed{\frac{1}{g(x)}} + f(x)\cdot\left\{\boxed{-\frac{g'(x)}{[g(x)]^2}}\right\}$$

$$= \frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}$$

Answers $g(x+h)-g(x)$, $g(x+h)-g(x)$, $g'(x)$, $\dfrac{1}{g(x)}$, $\dfrac{1}{g(x)}$, $\dfrac{1}{g(x)}$, $-\dfrac{g'(x)}{[g(x)]^2}$

N146b

From the results on side a, the following formulas are true.

Quotient Rule

$$\left[\frac{f(x)}{g(x)}\right]'=\frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}, \quad \left[\frac{1}{g(x)}\right]'=-\frac{g'(x)}{[g(x)]^2}$$

Using the formulas above, differentiate the following functions.

Ex.

$$y=\frac{x}{x^2+1}$$

[Sol] $y'=\dfrac{(x)'(x^2+1)-x(x^2+1)'}{(x^2+1)^2}=\dfrac{1\cdot(x^2+1)-x\cdot 2x}{(x^2+1)^2}=-\dfrac{x^2-1}{(x^2+1)^2}$

(1) $y=\dfrac{x}{3x-1}$

[Sol] $y'=\dfrac{(x)'(3x-1)-x(3x-1)'}{(3x-1)^2}=\dfrac{1\cdot(3x-1)-x\cdot 3}{(3x-1)^2}=-\dfrac{1}{(3x-1)^2}$

(2) $y=\dfrac{x-1}{x+1}$

[Sol] $y'=\dfrac{(x-1)'(x+1)-(x-1)(x+1)'}{(x+1)^2}=\dfrac{1\cdot(x+1)-(x-1)\cdot 1}{(x+1)^2}=\dfrac{2}{(x+1)^2}$

(3) $y=\dfrac{1}{4x-1}$

[Sol] $y'=-\dfrac{(4x-1)'}{(4x-1)^2}=-\dfrac{4}{(4x-1)^2}$

(4) $y=\dfrac{3}{x^2+1}$

[Sol] $y'=-\dfrac{3(x^2+1)'}{(x^2+1)^2}=-\dfrac{3\cdot 2x}{(x^2+1)^2}=-\dfrac{6x}{(x^2+1)^2}$

Differentiation 1

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Differentiate the following functions.

(1) $y=\dfrac{x^2}{2x-1}$

[Sol] $y'=\dfrac{2x(\boxed{2x-1})-x^2\cdot\boxed{2}}{(2x-1)^2}$ ← $\left[\dfrac{f(x)}{g(x)}\right]'=\dfrac{f'(x)g(x)-f(x)g'(x)}{\{g(x)\}^2}$

$=\dfrac{2x^2-2x}{(2x-1)^2}$ $\left[=\dfrac{2x(x-1)}{(2x-1)^2}\right]$

(2) $y=\dfrac{x^2-1}{x^2+1}$

[Sol] $y'=\dfrac{2x(x^2+1)-(x^2-1)\cdot 2x}{(x^2+1)^2}$

$=\dfrac{4x}{(x^2+1)^2}$

(3) $y=\dfrac{1}{x^2+2x+3}$

[Sol] $y'=-\dfrac{2x+2}{(x^2+2x+3)^2}$ ← $\left[\dfrac{1}{g(x)}\right]'=-\dfrac{g'(x)}{\{g(x)\}^2}$

$\left[=-\dfrac{2(x+1)}{(x^2+2x+3)^2}\right]$

(4) $y=\dfrac{x}{x^2-x+1}$

[Sol] $y'=\dfrac{1\cdot(x^2-x+1)-x(2x-1)}{(x^2-x+1)^2}$

$=-\dfrac{x^2-1}{(x^2-x+1)^2}$ $\left[=-\dfrac{(x+1)(x-1)}{(x^2-x+1)^2}\right]$

N147b

(5) $y=\dfrac{2x^2-3x+5}{x^2+2}$

[Sol] $y'=\dfrac{(4x-3)(x^2+2)-(2x^2-3x+5)\cdot 2x}{(x^2+2)^2}$

$=\dfrac{3x^2-2x-6}{(x^2+2)^2}$

(6) $y=\dfrac{x^2-x+1}{x^2+x+1}$

[Sol] $y'=\dfrac{(2x-1)(x^2+x+1)-(x^2-x+1)(2x+1)}{(x^2+x+1)^2}$

$=\dfrac{2x^2-2}{(x^2+x+1)^2}\quad\left[=\dfrac{2(x+1)(x-1)}{(x^2+x+1)^2}\right]$

(7) $y=\dfrac{(x^2+1)(x-1)}{x^3}$

[Sol] $[(x^2+1)(x-1)]'=2x(x-1)+(x^2+1)\cdot 1=3x^2-2x+1$

$y'=\dfrac{(3x^2-2x+1)\cdot x^3-(x^2+1)(x-1)\cdot 3x^2}{x^6}$

$=\dfrac{x^2-2x+3}{x^4}$

Differentiation 1

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Prove $(x^n)'=nx^{n-1}$ is true not only when n is a positive integer but also when n is a negative integer or 0.

[Sol] When n is a negative integer, let $n=-m$. Then, m is a positive integer.

$$(x^n)'=(x^{-m})'=\left(\frac{1}{x^m}\right)'=-\frac{(x^m)'}{(x^m)^2}=-\frac{mx^{\boxed{m-1}}}{x^{2m}}=-mx^{\boxed{-m}-1}=nx^{n-1}$$

Therefore, $(x^n)'=nx^{n-1}$ is also true when n is a negative integer.

Also, since $x^n=1$ when $n=0$, $(1)'=0$; therefore, $(x^n)'=nx^{n-1}$ is true.

Answers: $m-1$, $-m$

Derivative of x^n

When n is an integer, $(x^n)'=nx^{n-1}$

Differentiate the following functions.

Ex. $y=\dfrac{1}{x^3}$

[Sol] $y=x^{-3}$

$$y'=-3x^{-4}=-\frac{3}{x^4}$$

(1) $y=\dfrac{1}{x^4}$

[Sol] $y=x^{-4}$

$$y'=-4x^{-5}=-\frac{4}{x^5}$$

(2) $y=\dfrac{1}{3x^3}$

[Sol] $y=\dfrac{1}{3}x^{-3}$

$$y'=\frac{1}{3}\cdot(-3)x^{-4}=-\frac{1}{x^4}$$

(3) $y=x+\dfrac{2}{x}$

[Sol] $y=x+2x^{-1}$

$$y'=1+2\cdot(-1)x^{-2}=1-\frac{2}{x^2}$$

In the case of **Ex.**, $y'=-3x^{-4}$ is not a wrong answer. However, generally, the answer has to be written in the same form as the question. (For **Ex.**, use the fraction form such as $\dfrac{1}{x^3}$.)

N148b

(4) $y=\dfrac{x^2+3}{x}$

[Sol] $y=\boxed{x}+\dfrac{3}{x}=\boxed{x}+3x^{\boxed{-1}}$

$$y'=1+3\cdot(-1)x^{-2}=1-\frac{3}{x^2}\quad\left[=\frac{x^2-3}{x^2}\right]$$

(5) $y=\dfrac{2x^2-x+5}{3x^3}$

[Sol] $y=\dfrac{2}{3x}-\dfrac{1}{3x^2}+\dfrac{5}{3x^3}=\dfrac{2}{3}x^{-1}-\dfrac{1}{3}x^{-2}+\dfrac{5}{3}x^{-3}$

$$y'=\frac{2}{3}\cdot(-1)x^{-2}-\frac{1}{3}\cdot(-2)x^{-3}+\frac{5}{3}\cdot(-3)x^{-4}=-\frac{2}{3x^2}+\frac{2}{3x^3}-\frac{5}{x^4}$$

$$\left[=-\frac{2x^2-2x+15}{3x^4}\right]$$

(6) $y=\dfrac{5x^3+3x^2-4}{2x^2}$

[Sol] $y=\dfrac{5}{2}x+\dfrac{3}{2}-\dfrac{2}{x^2}=\dfrac{5}{2}x+\dfrac{3}{2}-2x^{-2}$

$$y'=\frac{5}{2}-2\cdot(-2)x^{-3}=\frac{5}{2}+\frac{4}{x^3}\quad\left[=\frac{5x^3+8}{2x^3}\right]$$

(7) $y=\dfrac{(3x-1)(2x+3)}{x^3}$

[Sol] $y=\dfrac{6x^2+7x-3}{x^3}=\dfrac{6}{x}+\dfrac{7}{x^2}-\dfrac{3}{x^3}=6x^{-1}+7x^{-2}-3x^{-3}$

$$y'=6\cdot(-1)x^{-2}+7\cdot(-2)x^{-3}-3\cdot(-3)x^{-4}=-\frac{6}{x^2}-\frac{14}{x^3}+\frac{9}{x^4}$$

$$\left[=-\frac{6x^2+14x-9}{x^4}\right]$$

Differentiation 1

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1. Given that $f(x)=\dfrac{ax+b}{x^2-x+1}$ satisfies $f(2)=\dfrac{4}{3}$ and $f'(0)=2$, find the values of constants a and b.

[Sol] $f(2)=\dfrac{a\cdot 2+b}{2^2-2+1}=\dfrac{2a+b}{3}$

Since $f(2)=\dfrac{4}{3}$, $\dfrac{2a+b}{3}=\dfrac{4}{3}$

$\therefore\ 2a+b=4 \quad \cdots①$

$$f'(x)=\frac{a(x^2-x+1)-(ax+b)(2x-1)}{(x^2-x+1)^2}$$

$$=-\frac{ax^2+2bx-(a+b)}{(x^2-x+1)^2}$$

Since $f'(0)=2$, $a+b=2 \quad \cdots②$

From ① and ②, $a=2$, $b=0$

N149b

2. For all natural numbers n, when $x \neq 1$, the following equality is true.

$$1+x+x^2+\cdots+x^n=\frac{x^{n+1}-1}{x-1} \quad \cdots①$$

Using this equality, find the following sums.

(1) $1+2x+3x^2+\cdots+nx^{n-1}$

[Sol] Differentiating both sides of ① with respect to x,

$$1+2x+3x^2+\cdots+nx^{n-1}$$
$$=\frac{(n+1)x^n\cdot(x-1)-(x^{n+1}-1)\cdot 1}{(x-1)^2}$$
$$=\frac{nx^{n+1}-(n+1)x^n+1}{(x-1)^2}$$

(2) $1+\frac{2}{2}+\frac{3}{2^2}+\cdots+\frac{n}{2^{n-1}}$

[Sol] From (1),

$$1+2x+3x^2+\cdots+nx^{n-1}=\frac{nx^{n+1}-(n+1)x^n+1}{(x-1)^2} \quad \cdots②$$

Substituting $x=\frac{1}{2}$ into ②,

$$1+\frac{2}{2}+\frac{3}{2^2}+\cdots+\frac{n}{2^{n-1}}$$
$$=\frac{n\cdot\frac{1}{2^{n+1}}-(n+1)\cdot\frac{1}{2^n}+1}{\left(\frac{1}{2}-1\right)^2}$$
$$=\frac{2^{n+1}-n-2}{2^{n-1}} \quad \left[=4-(n+2)\left(\frac{1}{2}\right)^{n-1}\right]$$

Differentiation 1

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1. Using the definition of the derivative, differentiate $f(x)=\frac{\sqrt{x}}{2}$. ➡ N142

$$[\text{Sol}]\ f'(x)=\lim_{h\to0}\frac{\frac{\sqrt{x+h}}{2}-\frac{\sqrt{x}}{2}}{h}$$

$$=\lim_{h\to0}\frac{\sqrt{x+h}-\sqrt{x}}{2h}$$

$$=\lim_{h\to0}\frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{2h(\sqrt{x+h}+\sqrt{x})}$$

$$=\lim_{h\to0}\frac{h}{2h(\sqrt{x+h}+\sqrt{x})}$$

$$=\lim_{h\to0}\frac{1}{2(\sqrt{x+h}+\sqrt{x})}$$

$$=\frac{1}{4\sqrt{x}}$$

2. Differentiate the following functions.

(1) $y=(x^2+x-3)(2x^2-5)$ ➡ N144

[Sol] $y'=(2x+1)(2x^2-5)+(x^2+x-3)\cdot4x$

$=8x^3+6x^2-22x-5$

(2) $y=(x+3)^2(x-1)$ ➡ N145

[Sol] $y=(x^2+6x+9)(x-1)$

$y'=(2x+6)(x-1)+(x^2+6x+9)\cdot1$

$=3x^2+10x+3$ $\left[=(x+3)(3x+1)\right]$

Alternative Solution

$y'=1\cdot(x+3)(x-1)+(x+3)\cdot1\cdot(x-1)+(x+3)^2\cdot1=3x^2+10x+3$ $\left[=(x+3)(3x+1)\right]$

N150b

(3) $y=\dfrac{1}{x^2-5x+3}$ ➡ N146

[Sol] $y'=-\dfrac{2x-5}{(x^2-5x+3)^2}$

(4) $y=\dfrac{2x+1}{x^2+3x-1}$ ➡ N146

[Sol] $y'=\dfrac{2(x^2+3x-1)-(2x+1)(2x+3)}{(x^2+3x-1)^2}$

$$=-\frac{2x^2+2x+5}{(x^2+3x-1)^2}$$

(5) $y=x^2-2x+\dfrac{3}{x^2}$ ➡ N148

[Sol] $y=x^2-2x+3x^{-2}$

$$y'=2x-2+3\cdot(-2)x^{-3}=2x-2-\frac{6}{x^3}$$

[Alternative Solution
$y'=2x-2-\dfrac{3\cdot 2x}{x^4}=2x-2-\dfrac{6}{x^3}$]

(6) $y=\dfrac{3x^2-2x+5}{x}$ ➡ N148

[Sol] $y=3x-2+\dfrac{5}{x}=3x-2+5x^{-1}$

$$y'=3+5\cdot(-1)x^{-2}=3-\frac{5}{x^2} \quad \left[=\frac{3x^2-5}{x^2}\right]$$

[Alternative Solution
$y'=\dfrac{(6x-2)\cdot x-(3x^2-2x+5)\cdot 1}{x^2}=\dfrac{3x^2-5}{x^2}$]

Differentiation 2

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Given the two functions $f(x)$ and $g(x)$ where the range of $f(x)$ is within the domain of $g(x)$, let $u=f(x)$ and $y=g(u)$. Then, $y=g(u)=g(f(x))$ can be derived.

The function $g(f(x))$ is called the ***composite function*** of $f(x)$ and $g(x)$.

1. For each given pair of functions, find the composite functions $g(f(x))$ and $f(g(x))$.

Ex. $f(x)=x^2-1,\ g(x)=2x+1$

[Sol] $g(f(x))=g(x^2-1)$
$=2(x^2-1)+1$
$=2x^2-1$

$f(g(x))=f(2x+1)$
$=(2x+1)^2-1$
$=4x^2+4x$

(1) $f(x)=x^2+1,\ g(x)=3x-2$

[Sol] $g(f(x))=g(x^2+1)$
$=3(x^2+1)-2$
$=3x^2+1$

$f(g(x))=f(3x-2)$
$=(3x-2)^2+1$
$=9x^2-12x+5$

(2) $f(x)=x^2,\ g(x)=\sqrt{x+1}$

[Sol] $g(f(x))=g(x^2)$
$=\sqrt{x^2+1}$

$f(g(x))=f(\sqrt{x+1})$
$=(\sqrt{x+1})^2$
$=x+1$

N151b

Given that the composite function $y=f(g(x))$ can be differentiated when both $y=f(u)$ and $u=g(x)$ are differentiable, prove that the following equation is true using the definition of the derivative.

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx} \longleftarrow$$

For $\frac{d\bullet}{d\blacktriangle}$, differentiating $\bullet$ with respect to $\blacktriangle$

[Sol] $\displaystyle \frac{dy}{dx}=\lim_{h\to 0}\frac{f(g(x+h))-f(g(x))}{h}$

$$=\lim_{h\to 0}\left[\frac{f(g(x+h))-f(g(x))}{g(x+h)-g(x)}\cdot\frac{g(x+h)-g(x)}{h}\right]$$

Let $g(x+h)-g(x)=k$. $g(x+h)=g(x)+\boxed{k}=u+\boxed{k}$

Also, as $h\to 0$, $k\to\boxed{0}$

$$\therefore \frac{dy}{dx}=\lim_{h\to 0}\left[\frac{f(u+\boxed{k})-f(u)}{k}\cdot\frac{g(x+h)-g(x)}{h}\right]$$

$$=\lim_{k\to\boxed{0}}\frac{f(u+\boxed{k})-f(u)}{k}\cdot\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$=f'(u)\cdot g'(x)$$

$$=\frac{dy}{du}\cdot\frac{du}{dx}$$

$\displaystyle \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=f'(x)$

Answers: k, k, 0, k, 0, h

Chain Rule I

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$

2. Using the formula above, differentiate $y=(x^2+1)^3$.

[Sol] Let $u=x^2+1$, $y=u^3$

$$\frac{dy}{du}=3u^2,\ \frac{du}{dx}=\boxed{2x}$$

$$\therefore \frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=3u^2\cdot\boxed{2x}=\boxed{6x(x^2+1)^2}$$

Differentiation 2

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For the formula on N151b, since $\dfrac{dy}{du}=f'(u)$ and $\dfrac{du}{dx}=g'(x)$ when $y=f(u)$ and $u=g(x)$, it can also be expressed as follows.

Chain Rule II

$$[f(g(x))]'=f'(g(x))g'(x)$$

Differentiate the following functions.

Ex. $y=(x^2+1)^4$ $\leftarrow$ $y=\blacksquare^4$

[Sol] $y'=4(x^2+1)^3\cdot(x^2+1)'$ $\leftarrow$ $y'=4\blacksquare^3\cdot\blacksquare'$

$=4(x^2+1)^3\cdot 2x$

$=8x(x^2+1)^3$

(1) $y=(x^3+2)^5$

[Sol] $y'=5(x^3+2)^4\cdot(x^3+2)'$

$=5(x^3+2)^4\cdot 3x^2$

$=15x^2(x^3+2)^4$

(2) $y=(2x^2+5)^6$

[Sol] $y'=6(2x^2+5)^5\cdot(2x^2+5)'$

$=6(2x^2+5)^5\cdot 4x$

$=24x(2x^2+5)^5$

(3) $y=(1-x^2)^3$

[Sol] $y'=3(1-x^2)^2\cdot(1-x^2)'$

$=3(1-x^2)^2\cdot(-2x)$

$=-6x(1-x^2)^2$

$\left[=-6x(1+x)^2(1-x)^2\right]$

(4) $y=(1-2x^3)^{10}$

[Sol] $y'=10(1-2x^3)^9\cdot(1-2x^3)'$

$=10(1-2x^3)^9\cdot(-6x^2)$

$=-60x^2(1-2x^3)^9$

(5) $y=(3x^3+x-1)^2$

[Sol] $y'=2(3x^3+x-1)\cdot(3x^3+x-1)'$

$=2(3x^3+x-1)(9x^2+1)$

(6) $y=(x^3-7x+1)^4$

[Sol] $y'=4(x^3-7x+1)^3\cdot(x^3-7x+1)'$

$=4(x^3-7x+1)^3(3x^2-7)$

N152b

(7) $y=(3x^2-5)^{-4}$

[Sol] $y'=-4(3x^2-5)^{-5}\cdot(3x^2-5)'$

$$=-4(3x^2-5)^{-5}\cdot 6x$$

$$=-24x(3x^2-5)^{-5} \quad \left[=-\frac{24x}{(3x^2-5)^5}\right]$$

(8) $y=(2x^3-x+4)^{-3}$

[Sol] $y'=-3(2x^3-x+4)^{-4}\cdot(2x^3-x+4)'$

$$=-3(2x^3-x+4)^{-4}(6x^2-1) \quad \left[=-\frac{3(6x^2-1)}{(2x^3-x+4)^4}\right]$$

(9) $y=\dfrac{1}{(x^2+1)^3}$

[Sol] $y=(x^2+1)^{\boxed{-3}}$

$$y'=-3(x^2+1)^{-4}\cdot(x^2+1)'$$

$$=-3(x^2+1)^{-4}\cdot 2x$$

$$=-\frac{6x}{(x^2+1)^4} \quad \left[=-6x(x^2+1)^{-4}\right]$$

(10) $y=\dfrac{1}{(1-2x^3)^2}$

[Sol] $y=(1-2x^3)^{-2}$

$$y'=-2(1-2x^3)^{-3}\cdot(1-2x^3)'$$

$$=-2(1-2x^3)^{-3}\cdot(-6x^2)$$

$$=\frac{12x^2}{(1-2x^3)^3} \quad \left[=12x^2(1-2x^3)^{-3}\right]$$

Generally, the answer has to be written in the same form as the question. (For (7), use the exponent form, such as $(3x^2-5)^{-4}$. For (9), use the fraction form, such as $\dfrac{1}{(x^2+1)^3}$.)

KUMON

Differentiation 2

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Differentiate the following functions.

Ex. $y=x(4x-1)^3$

$[(4x-1)^3]'=3(4x-1)^2\cdot(4x-1)'$

[Sol] $y'=1\cdot(4x-1)^3+x\cdot 3(4x-1)^2\cdot 4$

$=(4x-1)^2[(4x-1)+12x]$

$=(4x-1)^2(16x-1)$

$[f(x)g(x)]'$
$=f'(x)g(x)+f(x)g'$

(1) $y=(3x-1)^2(x+1)$

[Sol] $y'=2(3x-1)\cdot 3\cdot(x+1)+(3x-1)^2\cdot 1$

$=(3x-1)[6(x+1)+(3x-1)]$

$=(3x-1)(9x+5)$

#1

(2) $y=(2x-1)^2(x+3)^3$

[Sol] $y'=2(2x-1)\cdot 2\cdot(x+3)^3+(2x-1)^2\cdot 3(x+3)^2\cdot 1$

$=(2x-1)(x+3)^2[4(x+3)+3(2x-1)]$

$=(2x-1)(x+3)^2(10x+9)$

(3) $y=(x^2-x+1)^4(3x-1)^2$

[Sol] $y'=4(x^2-x+1)^3(2x-1)\cdot(3x-1)^2+(x^2-x+1)^4\cdot 2(3x-1)\cdot 3$

$=2(x^2-x+1)^3(3x-1)[2(2x-1)(3x-1)+3(x^2-x+1)]$

$=2(x^2-x+1)^3(3x-1)(15x^2-13x+5)$

N153b

(4) $y=\left(\frac{x}{x+1}\right)^3$

[Sol] $y'=3\left(\frac{x}{x+1}\right)^2\cdot\frac{1\cdot(x+1)-x\cdot 1}{(x+1)^2}$ $\longleftarrow$ $\left[\frac{f(x)}{g(x)}\right]'=\frac{f'(x)g(x)-f(x)g'(x)}{\{g(x)\}^2}$

$=\frac{3x^2}{(x+1)^4}$

(5) $y=\left(x+\frac{1}{x}\right)^3$

[Sol] $y'=3\left(x+\frac{1}{x}\right)^2\left(1-\frac{1}{x^2}\right)$

$\left[=3\left(x+\frac{1}{x}\right)^2\left(1+\frac{1}{x}\right)\left(1-\frac{1}{x}\right)\right]$ $\left[=\frac{3(x^2+1)^2(x+1)(x-1)}{x^4}\right]$

(6) $y=\left(\frac{x^2+12}{x+2}\right)^4$

[Sol] $y'=4\left(\frac{x^2+12}{x+2}\right)^3\cdot\frac{2x(x+2)-(x^2+12)\cdot 1}{(x+2)^2}$

$=\frac{4(x^2+12)^3(x^2+4x-12)}{(x+2)^5}$ $\left[=\frac{4(x^2+12)^3(x+6)(x-2)}{(x+2)^5}\right]$

(7) $y=\frac{(2x-1)^3}{(x^2+1)^2}$

[Sol] $y'=\frac{3(2x-1)^2\cdot 2\cdot(x^2+1)^2-(2x-1)^3\cdot 2(x^2+1)\cdot 2x}{(x^2+1)^4}$

$=\frac{2(2x-1)^2\{3(x^2+1)-2x(2x-1)\}}{(x^2+1)^3}$

$=-\frac{2(2x-1)^2(x^2-2x-3)}{(x^2+1)^3}$ $\left[=-\frac{2(2x-1)^2(x-3)(x+1)}{(x^2+1)^3}\right]$

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When the value of y for $y=f(x)$ is determined and then only one corresponding value of x is defined, x is considered to be a function of y. If this function is expressed as $x=g(y)$, then the function $y=g(x)$ where x and y are switched is called the ***inverse function*** of the original function $y=f(x)$.

Find the inverse functions of the following functions.

Ex. $y=2x+1$

[Sol] $x=\frac{1}{2}y-\frac{1}{2}$ ← Solving the equation for x

Switching x and y, $y=\frac{1}{2}x-\frac{1}{2}$

(1) $y=-3x+2$

[Sol] $x=-\frac{1}{3}y+\frac{2}{3}$

Switching x and y, $y=-\frac{1}{3}x+\frac{2}{3}$

(2) $y=3^x$

[Sol] $x=\log_3 y$

When $y=a^x$, $x=\log_a y$

Switching x and y, $y=\log_3 x$

(3) $y=\log_2 x$

[Sol] $x=2^y$

When $y=\log_a x$, $x=a^y$

Switching x and y, $y=2^x$

The inverse function of $f(x)$ is expressed as $f^{-1}(x)$. $f^{-1}(x)$ is read as "f inverse of x."
Therefore, if $y=f^{-1}(x)$, then $x=f(y)$.

N154b

Ex. $y=x^2+1 \quad (x \geqq 1)$

[Sol] The range is $y \geqq 2$. ← Since $x \geqq 1$

Since $x^2=y-1$, $x=\pm\sqrt{y-1}$

Since $x \geqq 1$, $x=\sqrt{y-1}$

Switching x and y, $y=\sqrt{x-1} \quad (x \geqq 2)$ ← For the given function and its inverse function, the domain and the range are switched.

(4) $y=1-x^2 \quad (x \leqq -1)$

[Sol] The range is $y \leqq 0$.

Since $x^2=1-y$, $x=\pm\sqrt{1-y}$

Since $x \leqq -1$, $x=-\sqrt{1-y}$

Switching x and y, $y=-\sqrt{1-x} \quad (x \leqq 0)$

(5) $y=3x-2 \quad (0 \leqq x \leqq 4)$

[Sol] The range is $-2 \leqq y \leqq 10$.

$x=\frac{1}{3}y+\frac{2}{3}$

Switching x and y, $y=\frac{1}{3}x+\frac{2}{3} \quad (-2 \leqq x \leqq 10)$

(6) $y=\sqrt{x+1}$

[Sol] The range is $y \geqq 0$.

Since $y^2=x+1$, $x=y^2-1$

Switching x and y, $y=x^2-1 \quad (x \geqq 0)$

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Prove $\dfrac{dy}{dx}=\dfrac{1}{\dfrac{dx}{dy}}$ if $\dfrac{dx}{dy}\neq 0$.

[Sol] Differentiating both sides of $x=f(y)$ with respect to x,

$$\text{LHS}=\frac{d}{dx}x=1$$

$$\text{RHS}=\frac{d}{dx}f(y)=\frac{d}{dy}f(y)\cdot\frac{dy}{dx}=\boxed{\frac{dx}{dy}}\cdot\frac{dy}{dx}$$

$$\therefore\ \boxed{\frac{dx}{dy}}\cdot\frac{dy}{dx}=1$$

Therefore, if $\dfrac{dx}{dy}\neq 0$, $\dfrac{dy}{dx}=\dfrac{1}{\dfrac{dx}{dy}}$

Answers: All the answers are the same. $\dfrac{dx}{dy}$

Differentiation Formula for Inverse Functions

$$\frac{dy}{dx}=\frac{1}{\dfrac{dx}{dy}}\qquad\left(\frac{dx}{dy}\neq 0\right)$$

Using the formula above, express $\dfrac{dy}{dx}$ in terms of x.

Ex. $y=\sqrt[3]{x}$

[Sol] Since $x=y^3$, $\dfrac{dx}{dy}=3y^2$ ← Rearranging into the form $x=$ ●, then differentiating with respect to y

$$\therefore\ \frac{dy}{dx}=\frac{1}{3y^2}=\frac{1}{3\sqrt[3]{x^2}}$$ ← $\dfrac{dy}{dx}=\dfrac{1}{\dfrac{dx}{dy}}$, then rewriting in terms of x

(1) $y=\sqrt[4]{x}$

[Sol] Since $x=y^4$, $\dfrac{dx}{dy}=4y^3$

$$\therefore\ \frac{dy}{dx}=\frac{1}{4y^3}=\frac{1}{4\sqrt[4]{x^3}}$$

N155b

(2) $y=\sqrt{x}$

[Sol] Since $x=y^2$, $\dfrac{dx}{dy}=2y$

$$\therefore \frac{dy}{dx}=\frac{1}{2y}=\frac{1}{2\sqrt{x}}$$

(3) $y=x^{\frac{1}{6}}$

[Sol] Since $x=y^6$, $\dfrac{dx}{dy}=6y^5$

$$\therefore \frac{dy}{dx}=\frac{1}{6y^5}=\frac{1}{6x^{\frac{5}{6}}} \quad \left[=\frac{1}{6}x^{-\frac{5}{6}}\right]$$

(4) $x=y^5$

[Sol] $\dfrac{dx}{dy}=5y^4$

Since $y^5=x$, $y=x^{\frac{1}{5}}$ ← Solving the equation for y

$$\therefore \frac{dy}{dx}=\frac{1}{5y^4}=\frac{1}{5x^{\frac{4}{5}}} \quad \left[=\frac{1}{5}x^{-\frac{4}{5}}\right]\left[=\frac{1}{5\sqrt[5]{x^4}}\right]$$

(5) $x=y^2+y+1$ $\left(y>-\dfrac{1}{2}\right)$

[Sol] $\dfrac{dx}{dy}=2y+1$

Since $y^2+y+1-x=0$, $y=\dfrac{-1\pm\sqrt{4x-3}}{2}$ ← Considering x as a constant and solving the quadratic equation for y (Quadratic Formula J97)

Since $y>-\dfrac{1}{2}$, $y=\dfrac{-1+\sqrt{4x-3}}{2}$

$$\therefore \frac{dy}{dx}=\frac{1}{2y+1}=\frac{1}{2\cdot\frac{-1+\sqrt{4x-3}}{2}+1}=\frac{1}{\sqrt{4x-3}}$$

N156a

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In N148, it has been proved that $(x^n)' = nx^{n-1}$ is true when n is an integer.
Here, consider if it is also true when n is a rational number.

Differentiate function $y = x^{\frac{m}{n}}$. (m and n ($n > 0$) are integers.)

[Sol] Raising both sides to the n^{th} power, $y^n = x^m$

Differentiating both sides with respect to x,

$$\text{LHS} = \frac{d}{dx}y^n = \frac{d}{dy}y^n \cdot \frac{dy}{dx} = ny^{\boxed{n-1}} \cdot \frac{dy}{dx}$$

$$\text{RHS} = mx^{m-1}$$

$$\therefore\ ny^{\boxed{n-1}} \cdot \frac{dy}{dx} = mx^{m-1}$$

Therefore,

$$\frac{dy}{dx} = \frac{mx^{m-1}}{ny^{\boxed{n-1}}} = \frac{mx^{m-1}}{n\left(x^{\frac{m}{n}}\right)^{\boxed{n-1}}} = \frac{mx^{m-1}}{nx^{\boxed{m}-\boxed{\frac{m}{n}}}} = \frac{m}{n}x^{\boxed{\frac{m}{n}}-1}$$

Answers $n-1$, $n-1$, $n-1$, $n-1$, m, $\frac{m}{n}$, $\frac{m}{n}$

From the above, $(x^n)' = nx^{n-1}$ is also true when n is a rational number.

Derivative of x^p

When p is a rational number, $(x^p)' = px^{p-1}$

Numbers that can be expressed in the fraction form, such as $\frac{m}{n}$, are called ***rational numbe***

N156b

Differentiate the following functions.

Ex. $y=x^{\frac{1}{2}}$

[Sol] $y'=\frac{1}{2}x^{-\frac{1}{2}}$

(1) $y=x^{\frac{2}{3}}$

[Sol] $y'=\frac{2}{3}x^{-\frac{1}{3}}$

(2) $y=x^{\frac{5}{2}}$

[Sol] $y'=\frac{5}{2}x^{\frac{3}{2}}$

(3) $y=x^{-\frac{1}{2}}$

[Sol] $y'=-\frac{1}{2}x^{-\frac{3}{2}}$

(4) $y=x^{-\frac{3}{2}}$

[Sol] $y'=-\frac{3}{2}x^{-\frac{5}{2}}$

Ex. $y=\sqrt[3]{x^2}$

[Sol] $y=x^{\frac{2}{3}}$ ← Rearranging into the form $y=x^{\bullet}$

$$y'=\frac{2}{3}x^{-\frac{1}{3}}=\frac{2}{3\sqrt[3]{x}}$$

(5) $y=\sqrt[5]{x^3}$

[Sol] $y=x^{\frac{3}{5}}$

$$y'=\frac{3}{5}x^{-\frac{2}{5}}=\frac{3}{5\sqrt[5]{x^2}}$$

(6) $y=\frac{1}{\sqrt[3]{x^2}}$

[Sol] $y=x^{-\frac{2}{3}}$

$$y'=-\frac{2}{3}x^{-\frac{5}{3}}=-\frac{2}{3x\sqrt[3]{x^2}}$$

(7) $y=\frac{1}{x\sqrt[4]{x}}$

[Sol] $y=x^{-\frac{5}{4}}$

$$y'=-\frac{5}{4}x^{-\frac{9}{4}}=-\frac{5}{4x^2\sqrt[4]{x}}$$

(8) $y=\sqrt{x\sqrt{x}}$

[Sol] $y=\left[x\cdot\left(x^{\frac{1}{2}}\right)\right]^{\frac{1}{2}}=\left(x^{\boxed{\frac{3}{2}}}\right)^{\frac{1}{2}}=x^{\boxed{\frac{3}{4}}}$

$$y'=\frac{3}{4}x^{-\frac{1}{4}}=\frac{3}{4\sqrt[4]{x}}$$

Differentiation 2

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Differentiate the following functions.

Ex. $y=\sqrt{3x+4}$

[Sol] $y=(3x+4)^{\frac{1}{2}}$ ← $y=\blacksquare^{\frac{1}{2}}$

$y'=\frac{1}{2}(3x+4)^{-\frac{1}{2}}\cdot 3=\frac{3}{2\sqrt{3x+4}}$ ← $y'=\frac{1}{2}\blacksquare^{-\frac{1}{2}}\cdot\blacksquare'$

(1) $y=\sqrt[3]{1-4x}$

[Sol] $y=(1-4x)^{\frac{1}{3}}$

$y'=\frac{1}{3}(1-4x)^{-\frac{2}{3}}\cdot(-4)=-\frac{4}{3\sqrt[3]{(1-4x)^2}}$

(2) $y=\sqrt{2x^2-3x+5}$

[Sol] $y=(2x^2-3x+5)^{\frac{1}{2}}$

$y'=\frac{1}{2}(2x^2-3x+5)^{-\frac{1}{2}}\cdot(4x-3)=\frac{4x-3}{2\sqrt{2x^2-3x+5}}$

(3) $y=\frac{5}{\sqrt[3]{3x-4}}$

[Sol] $y=5(3x-4)^{-\frac{1}{3}}$

$y'=-\frac{5}{3}(3x-4)^{-\frac{4}{3}}\cdot 3=-\frac{5}{(3x-4)\sqrt[3]{3x-4}}$

N157b

(4) $y=(\sqrt{x}+1)^3$

[Sol] $y=\left(x^{\frac{1}{2}}+1\right)^3$

$$y'=3\left(x^{\frac{1}{2}}+1\right)^2\cdot\frac{1}{2}x^{-\frac{1}{2}}=\frac{3(\sqrt{x}+1)^2}{2\sqrt{x}}$$

(5) $y=\sqrt{(4-x^2)^3}$

[Sol] $y=(4-x^2)^{\frac{3}{2}}$

$$y'=\frac{3}{2}(4-x^2)^{\frac{1}{2}}\cdot(-2x)=-3x\sqrt{4-x^2}$$

(6) $y=(x+\sqrt{x^2+1})^4$

[Sol] $y=\left[x+(x^2+1)^{\frac{1}{2}}\right]^4$

$$y'=4\left[x+(x^2+1)^{\frac{1}{2}}\right]^3\cdot\left[1+\frac{1}{2}(x^2+1)^{-\frac{1}{2}}\cdot 2x\right]$$

$$\boxed{\left[x+(x^2+1)^{\frac{1}{2}}\right]' = 1+\frac{1}{2}(x^2+1)^{-\frac{1}{2}}\cdot(x^2+1)'}$$

$$=4(x+\sqrt{x^2+1})^3\cdot\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}$$

$$=\frac{4(x+\sqrt{x^2+1})^4}{\sqrt{x^2+1}}$$

N158a

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Differentiate the following functions.

(1) $y=(3x-1)\sqrt{x+2}$

[Sol] $y=(3x-1)(x+2)^{\frac{1}{2}}$

$$y'=3(x+2)^{\frac{1}{2}}+(3x-1)\cdot\frac{1}{2}(x+2)^{-\frac{1}{2}} \leftarrow$$

$[f(x)g(x)]'$
$=f'(x)g(x)+f(x)g'(x)$

$$=3\sqrt{x+2}+\frac{3x-1}{2\sqrt{x+2}}$$

$$=\frac{9x+11}{2\sqrt{x+2}}$$

(2) $y=(x+4)\sqrt{x^2+4}$

[Sol] $y=(x+4)(x^2+4)^{\frac{1}{2}}$

$$y'=1\cdot(x^2+4)^{\frac{1}{2}}+(x+4)\cdot\frac{1}{2}(x^2+4)^{-\frac{1}{2}}\cdot 2x$$

$$=\sqrt{x^2+4}+\frac{x(x+4)}{\sqrt{x^2+4}}$$

$$=\frac{2x^2+4x+4}{\sqrt{x^2+4}} \quad \left[=\frac{2(x^2+2x+2)}{\sqrt{x^2+4}}\right]$$

(3) $y=(2x+1)\sqrt{2-x^2}$

[Sol] $y=(2x+1)(2-x^2)^{\frac{1}{2}}$

$$y'=2(2-x^2)^{\frac{1}{2}}+(2x+1)\cdot\frac{1}{2}(2-x^2)^{-\frac{1}{2}}\cdot(-2x)$$

$$=2\sqrt{2-x^2}-\frac{x(2x+1)}{\sqrt{2-x^2}}$$

$$=-\frac{4x^2+x-4}{\sqrt{2-x^2}}$$

N158b

(4) $y=\dfrac{x}{\sqrt{x+1}}$

[Sol] $y=x(x+1)^{-\frac{1}{2}}$

$$y'=1\cdot(x+1)^{-\frac{1}{2}}+x\cdot\left(-\frac{1}{2}\right)(x+1)^{-\frac{3}{2}}$$

$$=\frac{1}{\sqrt{x+1}}-\frac{x}{2(x+1)\sqrt{x+1}}$$

$$=\frac{x+2}{2(x+1)\sqrt{x+1}}\quad\left[=\frac{(x+2)\sqrt{x+1}}{2(x+1)^2}\right]$$

Alternative Solution

$$y'=\frac{1\cdot\sqrt{x+1}-x\cdot\frac{1}{2}(x+1)^{-\frac{1}{2}}}{x+1}$$

$$=\frac{2(x+1)-x}{2(x+1)\sqrt{x+1}}$$

$$=\frac{x+2}{2(x+1)\sqrt{x+1}}$$

$$\left[=\frac{(x+2)\sqrt{x+1}}{2(x+1)^2}\right]$$

(5) $y=\dfrac{x}{\sqrt{2x^2+1}}$

[Sol] $y=x(2x^2+1)^{-\frac{1}{2}}$

$$y'=1\cdot(2x^2+1)^{-\frac{1}{2}}+x\cdot\left(-\frac{1}{2}\right)(2x^2+1)^{-\frac{3}{2}}\cdot 4x$$

$$=\frac{1}{\sqrt{2x^2+1}}-\frac{2x^2}{(2x^2+1)\sqrt{2x^2+1}}$$

$$=\frac{1}{(2x^2+1)\sqrt{2x^2+1}}\quad\left[=\frac{\sqrt{2x^2+1}}{(2x^2+1)^2}\right]$$

Alternative Solution

$$y'=\frac{1\cdot\sqrt{2x^2+1}-x\cdot\frac{1}{2}(2x^2+1)^{-\frac{1}{2}}\cdot 4x}{2x^2+1}$$

$$=\frac{(2x^2+1)-2x^2}{(2x^2+1)\sqrt{2x^2+1}}$$

$$=\frac{1}{(2x^2+1)\sqrt{2x^2+1}}$$

$$\left[=\frac{\sqrt{2x^2+1}}{(2x^2+1)^2}\right]$$

(6) $y=\sqrt{\dfrac{x^2+1}{x+1}}$

[Sol] $y=(x^2+1)^{\frac{1}{2}}(x+1)^{-\frac{1}{2}}$

$$y'=\frac{1}{2}(x^2+1)^{-\frac{1}{2}}\cdot 2x\cdot(x+1)^{-\frac{1}{2}}+(x^2+1)^{\frac{1}{2}}\cdot\left(-\frac{1}{2}\right)(x+1)^{-\frac{3}{2}}$$

$$=\frac{x}{\sqrt{x^2+1}\sqrt{x+1}}-\frac{\sqrt{x^2+1}}{2(x+1)\sqrt{x+1}}$$

$$=\frac{x^2+2x-1}{2(x+1)\sqrt{x+1}\sqrt{x^2+1}}\quad\left[=\frac{(x^2+2x-1)\sqrt{x+1}}{2(x+1)^2\sqrt{x^2+1}}\right]$$

Alternative Solution

$$y=\left(\frac{x^2+1}{x+1}\right)^{\frac{1}{2}},\quad y'=\frac{1}{2}\left(\frac{x^2+1}{x+1}\right)^{-\frac{1}{2}}\cdot\frac{2x(x+1)-(x^2+1)\cdot 1}{(x+1)^2}=\frac{1}{2}\sqrt{\frac{x+1}{x^2+1}}\cdot\frac{x^2+2x-1}{(x+1)^2}$$

$$=\frac{(x^2+2x-1)\sqrt{x+1}}{2(x+1)^2\sqrt{x^2+1}}\quad\left[=\frac{x^2+2x-1}{2(x+1)\sqrt{x+1}\sqrt{x^2+1}}\right]$$

Differentiation 2

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1. Given that $f(x)=\sqrt{ax^2-1}$ satisfies $f'(1)=2$, find the value of constan

[Sol] $f(x)=(ax^2-1)^{\frac{1}{2}}$

$$f'(x)=\frac{1}{2}(ax^2-1)^{-\frac{1}{2}}\cdot 2ax$$

$$=\frac{ax}{\sqrt{ax^2-1}}$$

Since $f'(1)=2$, $\dfrac{a}{\sqrt{a-1}}=2$

Squaring both sides and simplifying

$$a^2-4a+4=0$$

$$(a-2)^2=0$$

$$\therefore\ a=2$$

2. Show that $(y-x)y'=y$ when $y=x+\sqrt{x^2+a^2}$.

[Sol] $y=x+(x^2+a^2)^{\frac{1}{2}}$

$$y'=1+\frac{1}{2}(x^2+a^2)^{-\frac{1}{2}}\cdot 2x$$

$$=\frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}}$$

$$\therefore\ (y-x)y'=\sqrt{x^2+a^2}\cdot\frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}}$$

$$=x+\sqrt{x^2+a^2}$$

$$=y$$

$$\therefore\ (y-x)y'=y$$

N159b

3. Given $f(x)=\lim_{t\to x}\dfrac{t\sqrt{x-1}-x\sqrt{t-1}}{t-x}$, solve the following questions. $(1<x)$

(1) Find $f(x)$.

[Sol] $f(x)=\lim_{t\to x}\dfrac{(t\sqrt{x-1}-x\sqrt{t-1})(t\sqrt{x-1}+x\sqrt{t-1})}{(t-x)(t\sqrt{x-1}+x\sqrt{t-1})}$

$$=\lim_{t\to x}\frac{(t-x)(tx-t-x)}{(t-x)(t\sqrt{x-1}+x\sqrt{t-1})}$$

$$=\lim_{t\to x}\frac{tx-t-x}{t\sqrt{x-1}+x\sqrt{t-1}}$$

$$=\frac{x^2-2x}{2x\sqrt{x-1}}$$

$$=\frac{x-2}{2\sqrt{x-1}}$$

$$\begin{aligned}&(t\sqrt{x-1})^2-(x\sqrt{t-1})^2\\&=t^2(x-1)-x^2(t-1)\\&=tx(t-x)-(t+x)(t-x)\\&=(t-x)(tx-t-x)\end{aligned}$$

(2) Find $f'(x)$.

[Sol] From (1), $f(x)=\frac{1}{2}(x-2)(x-1)^{-\frac{1}{2}}$

$$f'(x)=\frac{1}{2}\left[1\cdot(x-1)^{-\frac{1}{2}}+(x-2)\cdot\left(-\frac{1}{2}\right)(x-1)^{-\frac{3}{2}}\right]$$

$$=\frac{1}{2}\left[\frac{1}{\sqrt{x-1}}-\frac{x-2}{2(x-1)\sqrt{x-1}}\right]$$

$$=\frac{x}{4(x-1)\sqrt{x-1}}\quad\left[=\frac{x\sqrt{x-1}}{4(x-1)^2}\right]$$

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Differentiate the following functions.

(1) $y=(3x^2-5)^4$

[Sol] $y'=4(3x^2-5)^3\cdot 6x$

$=24x(3x^2-5)^3$

(2) $y=\dfrac{1}{(2-x)^3}$

[Sol] $y=(2-x)^{-3}$

$y'=-3(2-x)^{-4}\cdot(-1)$

$=\dfrac{3}{(2-x)^4}$

Alternative Solution

$y'=-\dfrac{3(2-x)^2\cdot(-1)}{(2-x)^6}$

$=\dfrac{3}{(2-x)^4}$

(3) $y=\left(\dfrac{x}{3x+1}\right)^4$

[Sol] $y'=4\left(\dfrac{x}{3x+1}\right)^3\cdot\dfrac{1\cdot(3x+1)-x\cdot 3}{(3x+1)^2}$

$=\dfrac{4x^3}{(3x+1)^5}$

N160b

(4) $y=\sqrt[3]{4x+1}$

➡ N157

[Sol] $y=(4x+1)^{\frac{1}{3}}$

$$y'=\frac{1}{3}(4x+1)^{-\frac{2}{3}}\cdot 4$$

$$=\frac{4}{3\sqrt[3]{(4x+1)^2}}$$

(5) $y=x\sqrt{1+x^2}$

➡ N158

[Sol] $y=x(1+x^2)^{\frac{1}{2}}$

$$y'=1\cdot(1+x^2)^{\frac{1}{2}}+x\cdot\frac{1}{2}(1+x^2)^{-\frac{1}{2}}\cdot 2x$$

$$=\sqrt{1+x^2}+\frac{x^2}{\sqrt{1+x^2}}$$

$$=\frac{1+2x^2}{\sqrt{1+x^2}}$$

(6) $y=\frac{x}{\sqrt{x^2+1}}$

➡ N158

[Sol] $y=x(x^2+1)^{-\frac{1}{2}}$

$$y'=1\cdot(x^2+1)^{-\frac{1}{2}}+x\cdot\left(-\frac{1}{2}\right)(x^2+1)^{-\frac{3}{2}}\cdot 2x$$

$$=\frac{1}{\sqrt{x^2+1}}-\frac{x^2}{(x^2+1)\sqrt{x^2+1}}$$

$$=\frac{1}{(x^2+1)\sqrt{x^2+1}}\quad\left[=\frac{\sqrt{x^2+1}}{(x^2+1)^2}\right]$$

[Alternative Solution

$$y'=\frac{1\cdot\sqrt{x^2+1}-x\cdot\frac{1}{2}(x^2+1)^{-\frac{1}{2}}\cdot 2x}{x^2+1}=\frac{(x^2+1)-x^2}{(x^2+1)\sqrt{x^2+1}}=\frac{1}{(x^2+1)\sqrt{x^2+1}}\quad\left[=\frac{\sqrt{x^2+1}}{(x^2+1)^2}\right]$$

]

Differentiation of Trigonometric Functions

Name

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
0	—	—	—	1~

[I] Using the definition of the derivative, differentiate $y = \sin x$.

[Sol] $$y' = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{2\cos\frac{(x+h)+x}{2}\sin\frac{(x+h)-x}{2}}{h}$$

$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$

$$= \lim_{h \to 0} \frac{2\cos\left(x + \boxed{\frac{h}{2}}\right)\sin\frac{h}{2}}{h}$$

$$= \lim_{h \to 0} \cos\left(x + \boxed{\frac{h}{2}}\right) \cdot \frac{\sin\frac{h}{2}}{\frac{h}{2}}$$

$\lim_{x \to 0} \frac{\sin x}{x} = 1$

$$= \boxed{\cos x} \cdot 1$$

$$= \boxed{\cos x}$$

$\frac{h}{2}$, $\frac{h}{2}$, $\cos x$, $\cos x$

The Sum-to-Product Formulas introduced in M177 are often used for the differen… and integration of trigonometric functions.

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

N161b

[II] Using the definition of the derivative, differentiate $y=\cos x$.

[Sol] $$y'=\lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}$$

$$=\lim_{h\to 0}\frac{-2\sin\dfrac{(x+h)+x}{2}\sin\dfrac{(x+h)-x}{2}}{h}$$

$$\cos A-\cos B=-2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$=\lim_{h\to 0}\frac{-2\sin\left(x+\boxed{\dfrac{h}{2}}\right)\sin\dfrac{h}{2}}{h}$$

$$=\lim_{h\to 0}\left[-\sin\left(x+\boxed{\dfrac{h}{2}}\right)\cdot\frac{\sin\dfrac{h}{2}}{\dfrac{h}{2}}\right]$$

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

$$=\boxed{-\sin x}\cdot 1$$

$$=\boxed{-\sin x}$$

Answers: $\frac{h}{2}$, $\frac{h}{2}$, $-\sin x$, $-\sin x$

[III] Differentiate $y=\tan x$.

[Sol] $$y'=\left(\frac{\boxed{\sin x}}{\cos x}\right)'$$

$$\left[\frac{f(x)}{g(x)}\right]'=\frac{f'(x)g(x)-f(x)g'(x)}{\{g(x)\}^2}$$

$$=\frac{(\boxed{\sin x})'\cos x-\boxed{\sin x}(\cos x)'}{\cos^2 x}$$

From [I] and [II]

$$=\frac{\boxed{\cos^2 x+\sin^2 x}}{\cos^2 x}$$

$$=\frac{\boxed{1}}{\cos^2 x}$$

Answers: $\sin x$, $\sin x$, $\sin x$, $\cos^2 x+\sin^2 x$, 1

N162a

KUMON

N

Differentiation of Trigonometric Functions

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From the results in N161, the following formulas are true.

Derivatives of Trigonometric Functions

$$(\sin x)'=\cos x,\quad (\cos x)'=-\sin x,\quad (\tan x)'=\frac{1}{\cos^2 x}$$

Differentiate the following functions.

Ex. $y=x\sin x$

[Sol] $y'=(x)'\sin x+x(\sin x)'$ ← $\{f(x)g(x)\}'=f'(x)g(x)+f(x)g'(x)$

$=1\cdot\sin x+x\cos x$

$=\sin x+x\cos x$

(1) $y=x\cos x$

[Sol] $y'=(x)'\cos x+x(\cos x)'$

$=1\cdot\cos x-x\sin x$

$=\cos x-x\sin x$

(2) $y=x\tan x$

[Sol] $y'=(x)'\tan x+x(\tan x)'$

$=1\cdot\tan x+x\cdot\dfrac{1}{\cos^2 x}$

$=\tan x+\dfrac{x}{\cos^2 x}$ $\left[=\dfrac{\sin x\cos x+x}{\cos^2 x}\right]$

(3) $y=(3x-2)\sin x$

[Sol] $y'=(3x-2)'\sin x+(3x-2)(\sin x)'$

$=3\sin x+(3x-2)\cos x$

N162b

(4) $y=(2x^2+1)\cos x$

[Sol] $y'=(2x^2+1)'\cos x+(2x^2+1)(\cos x)'$

$=4x\cos x-(2x^2+1)\sin x$

(5) $y=\sin x\cos x$

[Sol] $y'=(\sin x)'\cos x+\sin x(\cos x)'$

$=\cos^2 x-\sin^2 x$

$\left[=2\cos^2 x-1\right]$

$\left[=1-2\sin^2 x\right]$

$\left[=\cos 2x\right]$

(6) $y=\sin x\tan x$

[Sol] $y'=(\sin x)'\tan x+\sin x(\tan x)'$

$=\cos x\cdot\dfrac{\sin x}{\cos x}+\sin x\cdot\dfrac{1}{\cos^2 x}$

$=\sin x+\dfrac{\sin x}{\cos^2 x}\quad\left[=\dfrac{\sin x(\cos^2 x+1)}{\cos^2 x}\right]$

N163a

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Differentiation of Trigonometric Functions

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Differentiate the following functions.

Ex. $y=\cos 2x$

[Sol] $y'=-\sin 2x\cdot(2x)'$ ← $[f(g(x))]'=f'(g(x))g'(x)$

$=-2\sin 2x$

(1) $y=\sin 5x$

[Sol] $y'=\cos 5x\cdot(5x)'$

$=5\cos 5x$

(2) $y=\tan 3x+\cos 4x$

[Sol] $y'=\dfrac{1}{\cos^2 3x}\cdot(3x)'-\sin 4x\cdot(4x)'$

$=\dfrac{3}{\cos^2 3x}-4\sin 4x$

(3) $y=\sin(2x-3)$

[Sol] $y'=\cos(2x-3)\cdot(2x-3)'$

$=2\cos(2x-3)$

(4) $y=\cos(3-2x^2)$

[Sol] $y'=-\sin(3-2x^2)\cdot(3-2x^2)'$

$=4x\sin(3-2x^2)$

N163b

(5) $y=\tan(x^3+1)$

[Sol] $y'=\dfrac{1}{\cos^2(x^3+1)}\cdot(x^3+1)'$

$$=\frac{3x^2}{\cos^2(x^3+1)}$$

(6) $y=\cos\left(\dfrac{x}{2}+\dfrac{\pi}{6}\right)$

[Sol] $y'=-\sin\left(\dfrac{x}{2}+\dfrac{\pi}{6}\right)\cdot\left(\dfrac{x}{2}+\dfrac{\pi}{6}\right)'$

$$=-\frac{1}{2}\sin\left(\frac{x}{2}+\frac{\pi}{6}\right)$$

(7) $y=\tan\left(\dfrac{\pi}{3}-\pi x\right)$

[Sol] $y'=\dfrac{1}{\cos^2\left(\dfrac{\pi}{3}-\pi x\right)}\cdot\left(\dfrac{\pi}{3}-\pi x\right)'$

$$=-\frac{\pi}{\cos^2\left(\frac{\pi}{3}-\pi x\right)}$$

(8) $y=x\sin(3x+5)$

[Sol] $y'=(x)'\sin(3x+5)+x[\sin(3x+5)]'$

$$=1\cdot\sin(3x+5)+x\cos(3x+5)\cdot(3x+5)'$$

$$=\sin(3x+5)+3x\cos(3x+5)$$

N164a KUMON

Differentiation of Trigonometric Functions

100%	~90%	~80%	~70%	69%~
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Name

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Time : to :

Differentiate the following functions.

(1) $y=\sin\frac{1}{x}$

[Sol] $y'=\cos\frac{1}{x}\cdot\left(\frac{1}{x}\right)'$

$$=-\frac{1}{x^2}\cos\frac{1}{x}$$

(2) $y=\cos\frac{1}{x^2}$

[Sol] $y'=-\sin\frac{1}{x^2}\cdot\left(\frac{1}{x^2}\right)'$

$$=\frac{2}{x^3}\sin\frac{1}{x^2}$$

$$\left(\frac{1}{x^2}\right)'=(x^{-2})'=-2x^{-3}=-\frac{2}{x^3}$$

(3) $y=\sin\frac{x-1}{x+1}$

[Sol] $y'=\cos\frac{x-1}{x+1}\cdot\left(\frac{x-1}{x+1}\right)'$

$$=\cos\frac{x-1}{x+1}\cdot\frac{1\cdot(x+1)-(x-1)\cdot 1}{(x+1)^2}$$

$$=\frac{2}{(x+1)^2}\cos\frac{x-1}{x+1}$$

N164b

(4) $y=\cos\sqrt{x}$

[Sol] $y'=-\sin\sqrt{x}\cdot(\sqrt{x})'$

$$=-\sin\sqrt{x}\cdot\frac{1}{\boxed{2}}x^{\boxed{-\frac{1}{2}}}$$

$(\sqrt{x})'=\left(x^{\frac{1}{2}}\right)'$

$$=-\frac{\sin\sqrt{x}}{2\sqrt{x}}$$

(5) $y=\sin\sqrt{x^2+x+1}$

[Sol] $y'=\cos\sqrt{x^2+x+1}\cdot(\sqrt{x^2+x+1})'$

$$=\cos\sqrt{x^2+x+1}\cdot\frac{1}{2}(x^2+x+1)^{-\frac{1}{2}}\cdot(2x+1)$$

$$=\frac{(2x+1)\cos\sqrt{x^2+x+1}}{2\sqrt{x^2+x+1}}$$

$(\sqrt{x^2+x+1})'=\left[(x^2+x+1)^{\frac{1}{2}}\right]'$

$=\frac{1}{2}(x^2+x+1)^{-\frac{1}{2}}\cdot(x^2+x+1)$

(6) $y=\tan(\sin x)$

[Sol] $y'=\frac{1}{\cos^2(\sin x)}\cdot(\sin x)'$

$$=\frac{\cos x}{\cos^2(\sin x)}$$

N I 65a

KUMON

Differentiation of Trigonometric Functions

Name

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Time : to :

100%	~90%	~80%	~70%	69%~
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Differentiate the following functions.

Ex. $y=\sin^3 x$ ← $y=\blacksquare^3$

[Sol] $y'=3\sin^2 x\cdot(\sin x)'$ ← $y'=3\blacksquare^2\cdot\blacksquare'$

$=3\sin^2 x\cos x$

(1) $y=\cos^3 x$

[Sol] $y'=3\cos^2 x\cdot(\cos x)'$

$=-3\cos^2 x\sin x$

(2) $y=\tan^2 x$

[Sol] $y'=2\tan x\cdot(\tan x)'$

$=2\tan x\cdot\dfrac{1}{\cos^2 x}$

$=\dfrac{2\tan x}{\cos^2 x}\quad\left[=\dfrac{2\sin x}{\cos^3 x}\right]$

(3) $y=(\sin x+\cos x)^3$

[Sol] $y'=3(\sin x+\cos x)^2\cdot(\sin x+\cos x)'$

$=3(\sin x+\cos x)^2(\cos x-\sin x)$

N165b

(4) $y=\sin^4(3x+2)$

[Sol] $y'=4\sin^3(3x+2)\cdot[\sin(3x+2)]'$

$=4\sin^3(3x+2)\cos(3x+2)\cdot(3x+2)'$ $\boxed{[f(g(x))]'=f'(g(x))g'(x)}$

$=12\sin^3(3x+2)\cos(3x+2)$

(5) $y=\cos^3(1-2x^2)$

[Sol] $y'=3\cos^2(1-2x^2)\cdot[\cos(1-2x^2)]'$

$=3\cos^2(1-2x^2)\cdot[-\sin(1-2x^2)]\cdot(1-2x^2)'$

$=12x\cos^2(1-2x^2)\sin(1-2x^2)$

(6) $y=x^2\sin^3(2x-5)$

[Sol] $y'=(x^2)'\sin^3(2x-5)+x^2[\sin^3(2x-5)]'$

$=2x\sin^3(2x-5)+x^2\cdot 3\sin^2(2x-5)\cdot[\sin(2x-5)]'$

$=2x\sin^3(2x-5)+3x^2\sin^2(2x-5)\cos(2x-5)\cdot(2x-5)'$

$=2x\sin^3(2x-5)+6x^2\sin^2(2x-5)\cos(2x-5)$

$\left[=2x\sin^2(2x-5)[\sin(2x-5)+3x\cos(2x-5)]\right]$

Differentiation of Trigonometric Functions

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Differentiate the following functions.

Ex. $y=\sqrt{\cos 3x}$

[Sol] $y=(\cos 3x)^{\frac{1}{2}}$ ← Rearranging into the form $y=\blacksquare^{\bullet}$

$y'=\frac{1}{2}(\cos 3x)^{-\frac{1}{2}}\cdot(\cos 3x)'$ ← $y'=\frac{1}{2}\blacksquare^{-\frac{1}{2}}\cdot\blacksquare'$

$=-\frac{3\sin 3x}{2\sqrt{\cos 3x}}$ ← $(\cos 3x)'=-\sin 3x\cdot(3x)'$

(1) $y=\sqrt{\sin x}$

[Sol] $y=(\sin x)^{\frac{1}{2}}$

$y'=\frac{1}{2}(\sin x)^{-\frac{1}{2}}\cdot(\sin x)'$

$=\frac{\cos x}{2\sqrt{\sin x}}$

(2) $y=\sqrt{1-\sin x}$

[Sol] $y=(1-\sin x)^{\frac{1}{2}}$

$y'=\frac{1}{2}(1-\sin x)^{-\frac{1}{2}}\cdot(1-\sin x)'$

$=-\frac{\cos x}{2\sqrt{1-\sin x}}$

N166b

(3) $y=\sqrt{\cos 2x-x}$

[Sol] $y=(\cos 2x-x)^{\frac{1}{2}}$

$$y'=\frac{1}{2}(\cos 2x-x)^{-\frac{1}{2}}\cdot(\cos 2x-x)'$$

$$=-\frac{2\sin 2x+1}{2\sqrt{\cos 2x-x}}$$

(4) $y=\sqrt{1+\sin^2 x}$

[Sol] $y=(1+\sin^2 x)^{\frac{1}{2}}$

$$y'=\frac{1}{2}(1+\sin^2 x)^{-\frac{1}{2}}\cdot(1+\sin^2 x)'$$

$$=\frac{2\sin x\cdot(\sin x)'}{2\sqrt{1+\sin^2 x}}$$

$$=\frac{\sin x\cos x}{\sqrt{1+\sin^2 x}}$$

(5) $y=\frac{2}{\sqrt{\tan x}}$

[Sol] $y=2(\tan x)^{-\frac{1}{2}}$

$$y'=2\cdot\left(-\frac{1}{2}\right)(\tan x)^{-\frac{3}{2}}\cdot(\tan x)'$$

$$=-\frac{1}{\tan x\sqrt{\tan x}}\cdot\frac{1}{\cos^2 x}$$

$$=-\frac{1}{\sin x\cos x\sqrt{\tan x}}$$

$$\left[=-\frac{1}{\sqrt{\sin^3 x\cos x}}\right]$$

KUMON

Differentiation of Trigonometric Functions

100%	~90%	~80%	~70%	69%~
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Differentiate the following functions.

Ex.

$$y=\frac{1}{\tan x}$$

[Sol] $y'=-\dfrac{(\tan x)'}{\tan^2 x}$ ← $\left[\dfrac{1}{g(x)}\right]'=-\dfrac{g'(x)}{\{g(x)\}^2}$

$$=-\frac{1}{\tan^2 x\cos^2 x}$$

$$=-\frac{1}{\sin^2 x}$$

(1) $y=\dfrac{1}{\cos x}$

[Sol] $y'=-\dfrac{(\cos x)'}{\cos^2 x}$

$$=\frac{\sin x}{\cos^2 x}$$

(2) $y=\dfrac{1}{\sin^2 x}$

[Sol] $y'=-\dfrac{(\sin^2 x)'}{\sin^4 x}$

$$=-\frac{2\sin x\cdot(\sin x)'}{\sin^4 x}$$

$$=-\frac{2\cos x}{\sin^3 x}$$

N167b

(3) $y=\dfrac{1}{\tan 3x}$

[Sol] $y'=-\dfrac{(\tan 3x)'}{\tan^2 3x}$

$$=-\frac{3}{\tan^2 3x\cos^2 3x}$$

$$=-\frac{3}{\sin^2 3x}$$

(4) $y=\dfrac{\cos x}{x}$

[Sol] $y'=\dfrac{(\cos x)'\cdot x-\cos x\cdot(x)'}{x^2}$ ← $\left[\dfrac{f(x)}{g(x)}\right]'=\dfrac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}$

$$=\frac{-x\sin x-\cos x\cdot 1}{x^2}$$

$$=-\frac{x\sin x+\cos x}{x^2}$$

(5) $y=\dfrac{\tan x}{\cos x}$

[Sol] $y'=\dfrac{(\tan x)'\cos x-\tan x(\cos x)'}{\cos^2 x}$

$$=\frac{\dfrac{1}{\cos^2 x}\cdot\cos x+\dfrac{\sin x}{\cos x}\cdot\sin x}{\cos^2 x}$$

$$=\frac{1+\sin^2 x}{\cos^3 x}$$

N168a

KUMON

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Differentiation of Trigonometric Functions

100%	~90%	~80%	~70%	69%~
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Name

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Time : to :

Differentiate the following functions.

(1) $y=\dfrac{\cos x}{1+\cos x}$

[Sol] $y'=\dfrac{(\cos x)'(1+\cos x)-\cos x(1+\cos x)'}{(1+\cos x)^2}$

$=\dfrac{-\sin x(1+\cos x)+\cos x\sin x}{(1+\cos x)^2}$

$=-\dfrac{\sin x}{(1+\cos x)^2}$

(2) $y=\dfrac{\sin x-\cos x}{\sin x+\cos x}$

[Sol] $y'=\dfrac{(\sin x-\cos x)'(\sin x+\cos x)-(\sin x-\cos x)(\sin x+\cos x)'}{(\sin x+\cos x)^2}$

$=\dfrac{(\sin x+\cos x)^2+(\sin x-\cos x)^2}{(\sin x+\cos x)^2}$

$=\dfrac{2}{(\sin x+\cos x)^2}$

(3) $y=\sin x\cos^2 x$

[Sol] $y'=(\sin x)'\cos^2 x+\sin x(\cos^2 x)'$

$=\cos^3 x+\sin x\cdot 2\cos x\cdot(\cos x)'$

$=\cos^3 x-2\sin^2 x\cos x$

$\left[=\cos^3 x-2(1-\cos^2 x)\cos x\right.$

$\left.=3\cos^3 x-2\cos x\right]$

$\left[=\cos x(3\cos^2 x-2)\right]$

$\left[=\cos x(1-3\sin^2 x)\right]$

Alternative Solution

$y=\sin x(1-\sin^2 x)$

$=\sin x-\sin^3 x$

$y'=\cos x-3\sin^2 x\cdot(\sin x$

$=\cos x-3\sin^2 x\cos x$

$\left[=\cos x-3(1-\cos^2 x)\mathrm{c}\right.$

$\left.=3\cos^3 x-2\cos x\right]$

$\left[=\cos x(3\cos^2 x-2)\right]$

$\left[=\cos x(1-3\sin^2 x)\right]$

N168b

(4) $y = \sin 2x \cos 3x$

[Sol] $y' = (\sin 2x)' \cos 3x + \sin 2x (\cos 3x)'$

$$= 2\cos 2x \cos 3x - 3\sin 2x \sin 3x$$

$$\left[\begin{aligned} &= (\cos 5x + \cos x) + \frac{3}{2}(\cos 5x - \cos x) \\ &= \frac{1}{2}(5\cos 5x - \cos x) \end{aligned} \right]$$

$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-$

$\sin\alpha\sin\beta = -\frac{1}{2}[\cos(\alpha+\beta) - \cos(\alpha$

$\cos(-\theta) = \cos\theta$

Alternative Solution

$y = \frac{1}{2}(\sin 5x - \sin x)$ ←

$y' = \frac{1}{2}(5\cos 5x - \cos x)$

$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$

$\sin(-\theta) = -\sin\theta$

(5) $y = \sin 2x \tan x$

[Sol] $y' = (\sin 2x)' \tan x + \sin 2x (\tan x)'$

$$= 2\cos 2x \tan x + 2\sin x \cos x \cdot \frac{1}{\cos^2 x}$$

$\sin 2x = 2\sin x \cos x$

$$= 2\cos 2x \tan x + 2\tan x$$

$$\left[= 2(\cos 2x + 1)\tan x \right]$$

$$\left[\begin{aligned} &= 2 \cdot 2\cos^2 x \cdot \frac{\sin x}{\cos x} \\ &= 4\sin x \cos x \end{aligned} \right]$$

$\cos 2x = 2\cos^2 x - 1$

$$\left[= 2\sin 2x \right]$$

Alternative Solution 1

$y' = (\sin 2x)' \tan x + \sin 2x (\tan x)'$

$$= 2\cos 2x \cdot \frac{\sin x}{\cos x} + \sin 2x \cdot \frac{1}{\cos^2 x}$$

$$= \frac{\cos 2x \sin 2x + \sin 2x}{\cos^2 x} \quad \left[= \frac{\sin 2x(\cos 2x + 1)}{\cos^2 x} \right]$$

$$\left[= \frac{\sin 2x \cdot 2\cos^2 x}{\cos^2 x} = 2\sin 2x \right]$$

Alternative Solution 2

$y = 2\sin x \cos x \cdot \frac{\sin x}{\cos x}$

$= 2\sin^2 x$

$y' = 2 \cdot 2\sin x \cdot (\sin x)'$

$= 4\sin x \cos x$

$[= 2\sin 2x]$

Differentiation of Trigonometric Functions

100%	~90%	~80%	~70%	69%~

Name

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1. Given that x and y satisfy the following equation, express $\frac{dy}{dx}$ in terms of

$$x=\tan y \quad \left(-\frac{\pi}{2}<y<\frac{\pi}{2}\right)$$

[Sol] $\frac{dx}{dy}=\frac{1}{\cos^2 y}$

$=1+\tan^2 y$ $\quad \boxed{\frac{1}{\cos^2\theta}=1+\tan^2\theta}$

$=1+x^2$

$\therefore \frac{dy}{dx}=\frac{1}{1+x^2}$ $\quad \boxed{\frac{dy}{dx}=\frac{1}{\frac{dx}{dy}}}$

2. Given $f(x)=\frac{\sin^2 x}{x}$ $(x\neq 0)$, find $\lim_{x\to 0} f'(x)$.

[Sol] $f'(x)=\frac{(\sin^2 x)'\cdot x-\sin^2 x\cdot(x)'}{x^2}$

$=\frac{2x\sin x\cdot(\sin x)'-\sin^2 x\cdot 1}{x^2}$

$=\frac{2x\sin x\cos x-\sin^2 x}{x^2}$

$=\frac{\sin x}{x}\cdot 2\cos x-\left(\frac{\sin x}{x}\right)^2$

$\therefore \lim_{x\to 0} f'(x)=\lim_{x\to 0}\left[\frac{\sin x}{x}\cdot 2\cos x-\left(\frac{\sin x}{x}\right)^2\right]$ $\quad \boxed{\lim_{x\to 0}\frac{\sin x}{x}=}$

$=1$

N169b

3. Let $\frac{d}{dx}\left(\sin x\cos x\sqrt{2-\sin^2 x}\right)=\frac{a\sin^4 x+b\sin^2 x+c}{\sqrt{2-\sin^2 x}}$. Find the values of constants a, b and c.

[Sol] $\sin x\cos x\sqrt{2-\sin^2 x}=\sin x\cos x(2-\sin^2 x)^{\frac{1}{2}}$

$$\frac{d}{dx}\left(\sin x\cos x\sqrt{2-\sin^2 x}\right)$$

$$=(\sin x\cos x)'(2-\sin^2 x)^{\frac{1}{2}}+\sin x\cos x\left[(2-\sin^2 x)^{\frac{1}{2}}\right]'$$

$$=[(\sin x)'\cos x+\sin x(\cos x)'](2-\sin^2 x)^{\frac{1}{2}}$$
$$+\sin x\cos x\cdot\frac{1}{2}(2-\sin^2 x)^{-\frac{1}{2}}\cdot(2-\sin^2 x)'$$

$$=(\cos^2 x-\sin^2 x)\sqrt{2-\sin^2 x}+\frac{\sin x\cos x}{2\sqrt{2-\sin^2 x}}\cdot(-2)\sin x\cdot(\sin x)'$$

$$=\frac{(\cos^2 x-\sin^2 x)(2-\sin^2 x)-\sin^2 x\cos^2 x}{\sqrt{2-\sin^2 x}}$$

$$=\frac{(1-2\sin^2 x)(2-\sin^2 x)-\sin^2 x(1-\sin^2 x)}{\sqrt{2-\sin^2 x}}$$

$$=\frac{3\sin^4 x-6\sin^2 x+2}{\sqrt{2-\sin^2 x}}$$

$\therefore\ a=3,\ b=-6,\ c=2$

Alternative Solution

It is possible to solve as follows:

$$\frac{d}{dx}\left(\sin x\cos x\sqrt{2-\sin^2 x}\right)$$

N145

$$=(\sin x)'\cos x(2-\sin^2 x)^{\frac{1}{2}}+\sin x(\cos x)'(2-\sin^2 x)^{\frac{1}{2}}+\sin x\cos x\left[(2-\sin^2 x)^{\frac{1}{2}}\right]'$$

Alternatively, $\sin x\cos x=\frac{1}{2}\sin 2x$ can also be used.

KUMON

N 17

Differentiation of Trigonometric Functions

100%	~90%	~80%	~70%	69%~
0	—	1	—	2

Name

Date / /

Time : to :

Differentiate the following functions.

(1) $y=(x^2+1)\sin x$

[Sol] $y'=(x^2+1)'\sin x+(x^2+1)(\sin x)'$

$=2x\sin x+(x^2+1)\cos x$

➡ N

(2) $y=-\cos 2x+\sin 3x$

[Sol] $y'=2\sin 2x+3\cos 3x$

➡ N

(3) $y=\tan^3 x$

[Sol] $y'=3\tan^2 x\cdot(\tan x)'$

$=3\tan^2 x\cdot\dfrac{1}{\cos^2 x}$

$=\dfrac{3\tan^2 x}{\cos^2 x}\quad\left[=\dfrac{3\sin^2 x}{\cos^4 x}\right]$

➡ N

N170b

(4) $y=\sqrt{1-\cos x}$

[Sol] $y=(1-\cos x)^{\frac{1}{2}}$

$$y'=\frac{1}{2}(1-\cos x)^{-\frac{1}{2}}\cdot(1-\cos x)'$$

$$=\frac{\sin x}{2\sqrt{1-\cos x}}$$

(5) $y=\dfrac{1}{\sin x}$

[Sol] $y'=-\dfrac{(\sin x)'}{\sin^2 x}$

$$=-\frac{\cos x}{\sin^2 x}$$

(6) $y=\dfrac{1-\sin x}{1+\cos x}$

[Sol] $y'=\dfrac{(1-\sin x)'(1+\cos x)-(1-\sin x)(1+\cos x)'}{(1+\cos x)^2}$

$$=\frac{-\cos x(1+\cos x)+(1-\sin x)\sin x}{(1+\cos x)^2}$$

$$=\frac{-\cos x+\sin x-(\cos^2 x+\sin^2 x)}{(1+\cos x)^2}$$

$$=-\frac{\cos x-\sin x+1}{(1+\cos x)^2}$$

Differentiation of Logarithmic and Exponential Functions

100%	~90%	~80%	~70%	69%~
0	—	—	1	2

Name

Date / /

Time : to

Using the definition of the derivative, differentiate $y=\log_a x$. $(x>0)$

[Sol] $y'=\lim\limits_{h\to 0}\dfrac{\log_a(x+h)-\log_a x}{h}$

$=\lim\limits_{h\to 0}\dfrac{1}{h}\log_a\dfrac{x+h}{x}$

$=\lim\limits_{h\to 0}\dfrac{1}{h}\log_a\left(1+\dfrac{h}{x}\right)$

$\log_a M-\log_a N=\log_a\dfrac{M}{N}$

Let $\dfrac{h}{x}=k$. As $h\to 0$, $k\to 0$; therefore,

$y'=\lim\limits_{k\to 0}\dfrac{1}{kx}\log_a(1+k)$

$=\lim\limits_{k\to 0}\dfrac{1}{\boxed{x}}\log_a(1+k)^{\frac{1}{k}}$

$n\log_a M=\log_a M^n$

As $k\to 0$, $(1+k)^{\frac{1}{k}}$ is known to approach a constant value, which is expressed as e.

So, $e=\lim\limits_{k\to 0}(1+k)^{\frac{1}{k}}$

Using this constant value e,

$y'=\dfrac{1}{\boxed{x}}\log_a e=\dfrac{\log_e e}{\boxed{x}\log_e a}=\dfrac{1}{\boxed{x}\log_e a}$ ...① ←

$\log_a M=\dfrac{\log ...}{\log ...}$

$\log_a a=1$

All the answers are the same, x

When $a=e$, $\log_e a=\log_e e=1$; therefore, from ①, $y'=\dfrac{1}{x}$

When examining the value of $(1+k)^{\frac{1}{k}}$ by substituting k with a value which is close to 0, it approaches a constant value as shown on the right. e is an irrational number and $e=2.7182818\ldots$

k	$(1+k)^{\frac{1}{k}}$	k	$(1+$
0.1	2.59374...	−0.1	2.86
0.01	2.70481...	−0.01	2.73
0.001	2.71692...	−0.001	2.71
0.0001	2.71814...	−0.0001	2.71
0.00001	2.71828...	−0.00001	2.71

N171b

The logarithm of x to the base e, $\log_e x$, is called the ***natural logarithm***.
For differentiation and integration, it is often expressed as $\ln x$, i.e. $\log_e x = \mathrm{l}$

Derivatives of Logarithmic Functions I

$$(\ln x)' = \frac{1}{x}, \quad (\log_a x)' = \frac{1}{x \ln a}$$

Differentiate the following functions.

Ex.

$y = x \ln x$

[Sol] $y' = (x)' \ln x + x (\ln x)'$

$= 1 \cdot \ln x + x \cdot \frac{1}{x}$

$= \ln x + 1$

$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$

$y = x \log_2 x$

[Sol] $y' = (x)' \log_2 x + x (\log_2 x)'$

$= 1 \cdot \log_2 x + x \cdot \frac{1}{x \ln 2}$

$= \log_2 x + \frac{1}{\ln 2}$

(1) $y = x^2 \ln x$

[Sol] $y' = (x^2)' \ln x + x^2 (\ln x)'$

$= 2x \ln x + x^2 \cdot \frac{1}{x}$

$= 2x \ln x + x$

$[= x(2 \ln x + 1)]$

(2) $y = x^3 \ln x$

[Sol] $y' = (x^3)' \ln x + x^3 (\ln x)'$

$= 3x^2 \ln x + x^3 \cdot \frac{1}{x}$

$= 3x^2 \ln x + x^2$

$[= x^2(3 \ln x + 1)]$

(3) $y = x^2 \log_3 x$

[Sol] $y' = (x^2)' \log_3 x + x^2 (\log_3 x)'$

$= 2x \log_3 x + x^2 \cdot \frac{1}{x \ln 3}$

$= 2x \log_3 x + \frac{x}{\ln 3}$

$\left[= \frac{x(2 \ln x + 1)}{\ln 3} \right]$

$\log_a M = \frac{\log_b}{\log_b}$

(4) $y = x^3 \log_5 x$

[Sol] $y' = (x^3)' \log_5 x + x^3 (\log_5 x)'$

$= 3x^2 \log_5 x + x^3 \cdot \frac{1}{x \ln 5}$

$= 3x^2 \log_5 x + \frac{x^2}{\ln 5}$

$\left[= \frac{x^2(3 \ln x + 1)}{\ln 5} \right]$

Differentiation of Logarithmic and Exponential Functions

100%	~90%	~80%	~70%	69%~
0	1	2	3	4

Name

Date / /

Time : to :

1. Differentiate the following functions.

EX. $y=\ln(3x-1)$

[Sol] $y'=\dfrac{(3x-1)'}{3x-1}$ ← $[f(g(x))]'=f'(g(x))g'(x)$, i.e. $\{\ln g(x)\}'=\dfrac{g'(x)}{g(x)}$

$=\dfrac{3}{3x-1}$

(1) $y=\ln(2x+1)$

[Sol] $y'=\dfrac{(2x+1)'}{2x+1}$

$=\dfrac{2}{2x+1}$

(2) $y=\ln(x^2-x+1)$

[Sol] $y'=\dfrac{(x^2-x+1)'}{x^2-x+1}$

$=\dfrac{2x-1}{x^2-x+1}$

(3) $y=\ln(1-x^2)$

[Sol] $y'=\dfrac{(1-x^2)'}{1-x^2}$

$=-\dfrac{2x}{1-x^2}$

$\left[=-\dfrac{2x}{(1+x)(1-x)}\right]$

(4) $y=\log_2 3x$

[Sol] $y'=\dfrac{(3x)'}{3x\ln 2}$

$=\dfrac{3}{3x\ln 2}$

$=\dfrac{1}{x\ln 2}$

(5) $y=\log_5(4x-1)$

[Sol] $y'=\dfrac{(4x-1)'}{(4x-1)\ln 5}$

$=\dfrac{4}{(4x-1)\ln 5}$

(6) $y=\log_3(2-5x)$

[Sol] $y'=\dfrac{(2-5x)'}{(2-5x)\ln 3}$

$=-\dfrac{5}{(2-5x)\ln 3}$

N172b

In N171, the derivative of $y=\log_a x$ when $x>0$ is determined. Here, consider the derivative of $y=\log_a(-x)$ when $x<0$ by filling in the following blanks.

[1] $[\log_a(-x)]'=\dfrac{(-x)'}{-x\ln a}=\boxed{\dfrac{1}{x\ln a}}$

[2] $[\ln(-x)]'=\dfrac{(-x)'}{-x}=\boxed{\dfrac{1}{x}}$

Answers: $\dfrac{1}{x\ln a}$, $\dfrac{1}{x}$

From the above, combining the conditions when $x>0$ and $x<0$, the derivatives of logarithmic functions can be expressed as follows.

Derivatives of Logarithmic Functions II

$$(\ln|x|)'=\frac{1}{x},\quad (\log_a|x|)'=\frac{1}{x\ln a}$$

2. Differentiate the following functions.

Ex. $y=\ln|2x-3|$

[Sol] $y'=\dfrac{(2x-3)'}{2x-3}$

$=\dfrac{2}{2x-3}$

$y=\log_{10}|3x|$

[Sol] $y'=\dfrac{(3x)'}{3x\ln 10}$

$=\dfrac{1}{x\ln 10}$

$y=\ln|4x+1|$

$y'=\dfrac{(4x+1)'}{4x+1}$

$=\dfrac{4}{4x+1}$

$y=\ln|x^2-3|$

] $y'=\dfrac{(x^2-3)'}{x^2-3}$

$=\dfrac{2x}{x^2-3}$

(3) $y=\log_3|5x|$

[Sol] $y'=\dfrac{(5x)'}{5x\ln 3}$

$=\dfrac{1}{x\ln 3}$

(4) $y=\log_2|1-2x|$

[Sol] $y'=\dfrac{(1-2x)'}{(1-2x)\ln 2}$

$=-\dfrac{2}{(1-2x)\ln 2}$

N I 73a

KUMON

N I 7

Differentiation of Logarithmic and Exponential Functions

Name

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
(mistakes) 0	—	1	2	3~

Differentiate the following functions.

(1) $y=\ln(\cos x)$

[Sol] $y'=\dfrac{(\cos x)'}{\cos x}$

$=-\dfrac{\sin x}{\cos x}$

$\left[=-\tan x\right]$

(2) $y=\ln|\ln x|$

[Sol] $y'=\dfrac{(\ln x)'}{\ln x}$

$=\dfrac{1}{x\ln x}$

(3) $y=\ln\left(x+\sqrt{x^2+1}\right)$

[Sol] $y'=\dfrac{\left(x+\sqrt{x^2+1}\right)'}{x+\sqrt{x^2+1}}$

$=\dfrac{1+\frac{1}{2}(x^2+1)^{-\frac{1}{2}}\cdot 2x}{x+\sqrt{x^2+1}}$

$=\dfrac{\sqrt{x^2+1}+x}{\left(x+\sqrt{x^2+1}\right)\sqrt{x^2+1}}$

$=\dfrac{1}{\sqrt{x^2+1}}$

$\left(x+\sqrt{x^2+1}\right)'=\left[x+(x^2+1)^{\frac{1}{2}}\right]'$

$=1+\frac{1}{2}(x^2+1)^{-\frac{1}{2}}\cdot(x^2+$

N173b

(4) $y=x\ln 2x$

[Sol] $y'=(x)'\ln 2x+x(\ln 2x)'$ ← $\boxed{[f(x)g(x)]'=f'(x)g(x)+f(x)g'(x)}$

$$=1\cdot\ln 2x+x\cdot\frac{(2x)'}{2x}$$

$$=\ln 2x+1$$

(5) $y=\sqrt{x}\ln x$

[Sol] $y'=(\sqrt{x})'\ln x+\sqrt{x}(\ln x)'$

$$=\frac{1}{2}x^{-\frac{1}{2}}\cdot\ln x+\sqrt{x}\cdot\frac{1}{x}$$

$$=\frac{\ln x+2}{2\sqrt{x}}\quad\left[=\frac{\sqrt{x}(\ln x+2)}{2x}\right]$$

(6) $y=\dfrac{\ln x}{x}$

[Sol] $y'=\dfrac{(\ln x)'\cdot x-\ln x\cdot(x)'}{x^2}$ ← $\boxed{\left[\dfrac{f(x)}{g(x)}\right]'=\dfrac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}}$

$$=\frac{\frac{1}{x}\cdot x-\ln x\cdot 1}{x^2}$$

$$=\frac{1-\ln x}{x^2}$$

(7) $y=\dfrac{\ln x}{\sqrt{x}}$

[Sol] $y'=\dfrac{(\ln x)'\cdot\sqrt{x}-\ln x\cdot(\sqrt{x})'}{x}$

$$=\frac{\frac{1}{x}\cdot\sqrt{x}-\ln x\cdot\frac{1}{2}x^{-\frac{1}{2}}}{x}$$

$$=\frac{2-\ln x}{2x\sqrt{x}}\quad\left[=\frac{\sqrt{x}(2-\ln x)}{2x^2}\right]$$

Differentiation of Logarithmic and Exponential Functions

Name

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
0	—	1	—	2

Differentiate the following functions.

Ex. $y=(\ln x)^3$ ← $y=\blacksquare^3$

[Sol] $y'=3(\ln x)^2\cdot(\ln x)'$ ← $y'=3\blacksquare^2\cdot\blacksquare'$

$=\dfrac{3(\ln x)^2}{x}$

(1) $y=(\ln x)^4$

[Sol] $y'=4(\ln x)^3\cdot(\ln x)'$

$=\dfrac{4(\ln x)^3}{x}$

(2) $y=(\ln 3x)^2$

[Sol] $y'=2\ln 3x\cdot(\ln 3x)'$

$=2\ln 3x\cdot\dfrac{(3x)'}{3x}$

$=\dfrac{2\ln 3x}{x}$

(3) $y=x(\ln x)^2$

[Sol] $y'=(x)'(\ln x)^2+x[(\ln x)^2]'$

$=1\cdot(\ln x)^2+x\cdot 2\ln x\cdot(\ln x)'$

$=(\ln x)^2+2\ln x$

$[=\ln x(\ln x+2)]$

Ex.

$$y=\ln\left|\frac{3x-1}{2x+1}\right|$$

$$\boxed{\ln\frac{M}{N}=\ln M-\ln N}$$

[Sol] $y=\ln|3x-1|-\ln|2x+1|$

$$y'=\frac{(3x-1)'}{3x-1}-\frac{(2x+1)'}{2x+1}$$

$$=\frac{3}{3x-1}-\frac{2}{2x+1}$$

$$=\frac{5}{(3x-1)(2x+1)}$$

(4) $y=\ln\left|\frac{x+1}{x-1}\right|$

[Sol] $y=\ln|x+1|-\ln|x-1|$

$$y'=\frac{(x+1)'}{x+1}-\frac{(x-1)'}{x-1}$$

$$=\frac{1}{x+1}-\frac{1}{x-1}$$

$$=-\frac{2}{(x+1)(x-1)}$$

(5) $y=\ln\frac{1+\sin x}{1-\sin x}$

[Sol] $y=\ln(1+\sin x)-\ln(1-\sin x)$

$$y'=\frac{(1+\sin x)'}{1+\sin x}-\frac{(1-\sin x)'}{1-\sin x}$$

$$=\frac{\cos x}{1+\sin x}+\frac{\cos x}{1-\sin x}$$

$$=\frac{2\cos x}{(1+\sin x)(1-\sin x)}$$

$$=\frac{2\cos x}{1-\sin^2 x}$$

$$=\frac{2}{\cos x}$$

$$\left|\frac{a}{b}\right|=\frac{|a|}{|b|}$$

Differentiation of Logarithmic and Exponential Functions

Name

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~

Ex.

Differentiate $y=\dfrac{(x-2)^2}{x+1}$.

[Sol] Taking the natural logarithm of the absolute values of both sides,

$$\ln|y|=\ln\left|\frac{(x-2)^2}{x+1}\right|$$
$$=2\ln|x-2|-\ln|x+1|$$

$\boxed{\ln\dfrac{M}{N}=\ln M-\ln N \quad \ln M^n=n\ln M}$

Differentiating both sides with respect to x,

$$\frac{y'}{y}=\frac{2}{x-2}-\frac{1}{x+1}$$
$$=\frac{x+4}{(x-2)(x+1)}$$

$\leftarrow \boxed{(\ln|y|)'=\dfrac{y'}{y}}$

$$\therefore\ y'=\frac{x+4}{(x-2)(x+1)}\cdot\frac{(x-2)^2}{x+1}=\frac{(x+4)(x-2)}{(x+1)^2}$$

As shown above, differentiation by taking the natural logarithm of both sides called ***logarithmic differentiation***.

Differentiate the following functions using logarithmic differentiation.

(1) $y=\dfrac{x-1}{(x+3)^2}$

[Sol] Taking the natural logarithm of the absolute values of both sides,

$$\ln|y|=\ln\left|\frac{x-1}{(x+3)^2}\right|$$
$$=\ln|x-1|-2\ln|x+3|$$

Differentiating both sides with respect to x,

$$\frac{y'}{y}=\frac{1}{x-1}-\frac{2}{x+3}$$
$$=-\frac{x-5}{(x-1)(x+3)}$$
$$\therefore\ y'=-\frac{x-5}{(x-1)(x+3)}\cdot\frac{x-1}{(x+3)^2}=-\frac{x-5}{(x+3)^3}$$

$|a^2|=|a|^2$

N I 75b

(2) $y=\dfrac{(x+1)^3}{(x-2)^2(x+3)^4}$

[Sol] Taking the natural logarithm of the absolute values of both sides,

$$\ln|y|=\ln\left|\frac{(x+1)^3}{(x-2)^2(x+3)^4}\right|$$

$$=3\ln|x+1|-(2\ln|x-2|+4\ln|x+3|)$$

$\ln MN$
$=\ln M+\ln N$

Differentiating both sides with respect to x,

$$\frac{y'}{y}=\frac{3}{x+1}-\left(\frac{2}{x-2}+\frac{4}{x+3}\right)$$

$$=-\frac{3x^2+x+16}{(x+1)(x-2)(x+3)}$$

$$\therefore\ y'=-\frac{3x^2+x+16}{(x+1)(x-2)(x+3)}\cdot\frac{(x+1)^3}{(x-2)^2(x+3)^4}$$

$$=-\frac{(3x^2+x+16)(x+1)^2}{(x-2)^3(x+3)^5}$$

(3) $y=\dfrac{x}{\sqrt{2x+1}}$

[Sol] Taking the natural logarithm of the absolute values of both sides,

$$\ln|y|=\ln\left|\frac{x}{\sqrt{2x+1}}\right|$$

$$=\ln|x|-\frac{1}{2}\ln|2x+1|$$

$\ln|\sqrt{2x+1}|=\ln|2x+1|^{\frac{1}{2}}$
$=\frac{1}{2}\ln|2x+1|$

Differentiating both sides with respect to x,

$$\frac{y'}{y}=\frac{1}{x}-\frac{(2x+1)'}{2(2x+1)}$$

$$=\frac{1}{x}-\frac{1}{2x+1}$$

$$=\frac{x+1}{x(2x+1)}$$

$$\therefore\ y'=\frac{x+1}{x(2x+1)}\cdot\frac{x}{\sqrt{2x+1}}=\frac{x+1}{(2x+1)\sqrt{2x+1}}\quad\left[=\frac{(x+1)\sqrt{2x+1}}{(2x+1)^2}\right]$$

$|ab|=|a||b|$

N176a KUMON N 176

Differentiation of Logarithmic and Exponential Functions

Name

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
0	—	—	1	2~

1. Differentiate the following functions using logarithmic differentiation.

(1) $y = x^x \quad (x > 0)$

[Sol] Since $x > 0$, $y > 0$

Taking the natural logarithm of both sides.

Since $x > 0$ and $y > 0$, the absol... value symbol does not have to be included when taking the logarit...

$$\ln y = \ln x^x$$

$$= x \ln x$$

$\boxed{\ln M^n = n \ln M}$

Differentiating both sides with respect to x,

$$\frac{y'}{y} = (x)' \ln x + x(\ln x)'$$

$$= 1 \cdot \ln x + x \cdot \frac{1}{x}$$

$$= \ln x + 1$$

$$\therefore\ y' = (\ln x + 1) \cdot x^x = x^x(\ln x + 1)$$

(2) $y = x^{\ln x}$

[Sol] Since the antilogarithm should be a positive number, $x > 0$ ← $\boxed{L21}$

Since $x > 0$, $y > 0$

Taking the natural logarithm of both sides,

$$\ln y = \ln x^{\ln x}$$

$$= \ln x \cdot \ln x$$

$$= (\ln x)^2$$

Differentiating both sides with respect to x,

$$\frac{y'}{y} = 2 \ln x \cdot (\ln x)'$$

$$= \frac{2 \ln x}{x}$$

$$\therefore\ y' = \frac{2 \ln x}{x} \cdot x^{\ln x} = 2x^{\ln x - 1} \ln x$$

N176b

Differentiate $y=x^{\alpha}$ using logarithmic differentiation. ($x>0$ and α is a real number.)

[Sol] Since $x>0$, $y>0$

Taking the natural logarithm of both sides,

$$\ln y=\ln x^{\alpha}$$
$$=\boxed{\alpha}\ln x$$

Differentiating both sides with respect to x,

$$\frac{y'}{y}=\frac{\boxed{\alpha}}{x}$$

$$\therefore\ y'=\frac{\boxed{\alpha}}{x}\cdot x^{\alpha}=\boxed{\alpha}\,x^{\boxed{\alpha-1}}$$

Answers: α, α, α, α, $\alpha-1$

From the above, $(x^{n})'=nx^{n-1}$ is also true when n is a real number.

Derivative of x^{α}

When α is a real number, $(x^{\alpha})'=\alpha x^{\alpha-1}$

2. Differentiate the following functions.

Ex. $y=x^{\sqrt{2}}$

[Sol] $y'=\sqrt{2}x^{\sqrt{2}-1}$

(1) $y=x^{\sqrt{5}}$

[Sol] $y'=\sqrt{5}x^{\sqrt{5}-1}$

(2) $y=(5x)^{\sqrt{2}}$

[Sol] $y'=\sqrt{2}\cdot(5x)^{\sqrt{2}-1}\cdot(5x)'=5\sqrt{2}(5x)^{\sqrt{2}-1}$ $\left[=\sqrt{2}\cdot5^{\sqrt{2}}x^{\sqrt{2}-1}\right]$

Differentiation of Logarithmic and Exponential Functions

Name

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
0	1	2	3	4~

Differentiate $y=a^x$. ($a>0$, $a\neq 1$)

[Sol] Since $a>0$, $y>0$

Taking the natural logarithm of both sides,

$\ln y=\ln a^x$

$=x\boxed{\ln a}$ $\ln M^n=n\ln M$

Differentiating both sides with respect to x,

$\frac{y'}{y}=\boxed{\ln a}$

$\therefore\ y'=y\boxed{\ln a}$, i.e. $(a^x)'=a^x\boxed{\ln a}$ …①

When $a=e$, $\ln e=\log_e e=1$; therefore, from ①, $(e^x)'=e^x$

Derivatives of Exponential Functions

$(e^x)'=e^x,\quad (a^x)'=a^x\ln a$

Differentiate the following functions. ($a>0$, $a\neq 1$)

Ex. $y=e^{2x}$ $[f(g(x))]'=f'(g(x))g'(x)$

[Sol] $y'=e^{2x}\cdot(2x)'$

$=2e^{2x}$

$y=2^{3x}$

[Sol] $y'=2^{3x}\ln 2\cdot(3x)'$

$=3\cdot 2^{3x}\ln 2$

(1) $y=e^{4x}$

[Sol] $y'=e^{4x}\cdot(4x)'$

$=4e^{4x}$

(2) $y=3^{2x}$

[Sol] $y'=3^{2x}\ln 3\cdot(2x)'$

$=2\cdot 3^{2x}\ln 3$

(3) $y=5^{-x}$

[Sol] $y'=5^{-x}\ln 5\cdot(-x)'$

$=-5^{-x}\ln 5$

(4) $y=a^{3x}$

[Sol] $y'=a^{3x}\ln a\cdot(3x)'$

$=3a^{3x}\ln a$

(5) $y=e^{x^2}$

[Sol] $y'=e^{x^2}\cdot(x^2)'$

$=2xe^{x^2}$

(6) $y=e^{-x^3}$

[Sol] $y'=e^{-x^3}\cdot(-x^3)'$

$=-3x^2e^{-x^3}$

(7) $y=e^{\frac{1}{x}}$

[Sol] $y'=e^{\frac{1}{x}}\cdot\left(\frac{1}{x}\right)'$

$=-\frac{e^{\frac{1}{x}}}{x^2}$

(8) $y=e^{\sqrt{x}}$

[Sol] $y'=e^{\sqrt{x}}\cdot(\sqrt{x})'$

$=e^{\sqrt{x}}\cdot\frac{1}{2}x^{-\frac{1}{2}}$

$=\frac{e^{\sqrt{x}}}{2\sqrt{x}}$

(9) $y=e^{\sin x}$

[Sol] $y'=e^{\sin x}\cdot(\sin x)'$

$=e^{\sin x}\cos x$

(10) $y=2^{\ln x}$

[Sol] $y'=2^{\ln x}\ln 2\cdot(\ln x)'$

$=\frac{2^{\ln x}\ln 2}{x}$

Differentiation of Logarithmic and Exponential Functions

Name

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
0	—	1	2	3~

Differentiate the following functions. ($a > 0$, $a \neq 1$)

Ex. $y = xe^{2x}$

[Sol] $y' = (x)'e^{2x} + x(e^{2x})'$ ← $[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$

$= 1 \cdot e^{2x} + 2xe^{2x}$

$= (2x+1)e^{2x}$

(1) $y = (x+3)e^x$

[Sol] $y' = (x+3)'e^x + (x+3)(e^x)'$

$= 1 \cdot e^x + (x+3)e^x$

$= (x+4)e^x$

(2) $y = xe^{-x^2}$

[Sol] $y' = (x)'e^{-x^2} + x(e^{-x^2})'$

$= 1 \cdot e^{-x^2} - 2x^2e^{-x^2}$

$= (1-2x^2)e^{-x^2}$

(3) $y = \sqrt{x}e^x$

[Sol] $y' = (\sqrt{x})'e^x + \sqrt{x}(e^x)'$

$= \frac{1}{2}x^{-\frac{1}{2}} \cdot e^x + \sqrt{x}e^x$

$= \frac{(2x+1)e^x}{2\sqrt{x}}$

$\left[= \frac{(2x+1)\sqrt{x}e^x}{2x}\right]$

(4) $y = xa^x$

[Sol] $y' = (x)'a^x + x(a^x)'$

$= 1 \cdot a^x + xa^x \ln a$

$= a^x(1 + x\ln a)$

(5) $y = e^x \cos x$

[Sol] $y' = (e^x)'\cos x + e^x(\cos x)'$

$= e^x \cos x - e^x \sin x$

$= e^x(\cos x - \sin x)$

(6) $y = e^x \ln x$

[Sol] $y' = (e^x)'\ln x + e^x(\ln x)'$

$= e^x \ln x + e^x \cdot \frac{1}{x}$

$= e^x\left(\ln x + \frac{1}{x}\right)$

$\left[= \frac{e^x(x\ln x + 1)}{x}\right]$

(7) $y=\dfrac{e^x}{x}$

[Sol] $y'=\dfrac{(e^x)'\cdot x-e^x\cdot(x)'}{x^2}$ ← $\left[\dfrac{f(x)}{g(x)}\right]'=\dfrac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}$

$=\dfrac{xe^x-e^x\cdot 1}{x^2}$

$=\dfrac{(x-1)e^x}{x^2}$

(8) $y=\dfrac{e^x-1}{e^x+1}$

[Sol] $y'=\dfrac{(e^x-1)'(e^x+1)-(e^x-1)(e^x+1)'}{(e^x+1)^2}$

$=\dfrac{e^x(e^x+1)-(e^x-1)e^x}{(e^x+1)^2}$

$=\dfrac{2e^x}{(e^x+1)^2}$

(9) $y=\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$

[Sol] $y'=\dfrac{(e^x-e^{-x})'(e^x+e^{-x})-(e^x-e^{-x})(e^x+e^{-x})'}{(e^x+e^{-x})^2}$

$=\dfrac{(e^x+e^{-x})^2-(e^x-e^{-x})^2}{(e^x+e^{-x})^2}$

$=\dfrac{4}{(e^x+e^{-x})^2}$ $\left[=\dfrac{4e^{2x}}{(e^{2x}+1)^2}\right]$

[Alternative Solution

$y=\dfrac{e^x(e^x-e^{-x})}{e^x(e^x+e^{-x})}=\dfrac{e^{2x}-1}{e^{2x}+1}$, $y'=\dfrac{(e^{2x}-1)'(e^{2x}+1)-(e^{2x}-1)(e^{2x}+1)'}{(e^{2x}+1)^2}$

$=\dfrac{2e^{2x}(e^{2x}+1)-(e^{2x}-1)\cdot 2e^{2x}}{(e^{2x}+1)^2}=\dfrac{4e^{2x}}{(e^{2x}+1)^2}$]

KUMON

N I

Differentiation of Logarithmic and Exponential Functions

Name

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
0	—	—	1	2

1. Differentiate the following functions.

(1) $y=\ln[e^x(1-x)]$

[Sol] $y'=\dfrac{[e^x(1-x)]'}{e^x(1-x)}$

$=\dfrac{(e^x)'(1-x)+e^x(1-x)'}{e^x(1-x)}$

$=\dfrac{e^x(1-x)-e^x}{e^x(1-x)}$

$=-\dfrac{x}{1-x}$

(2) $y=\ln\dfrac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}$

[Sol] $y=\ln(\sqrt{1+e^x}-1)-\ln(\sqrt{1+e^x}+1)$

$y'=\dfrac{(\sqrt{1+e^x}-1)'}{\sqrt{1+e^x}-1}-\dfrac{(\sqrt{1+e^x}+1)'}{\sqrt{1+e^x}+1}$

$=\dfrac{\frac{1}{2}(1+e^x)^{-\frac{1}{2}}\cdot e^x}{\sqrt{1+e^x}-1}-\dfrac{\frac{1}{2}(1+e^x)^{-\frac{1}{2}}\cdot e^x}{\sqrt{1+e^x}+1}$

$(\sqrt{1+e^x})'=\Big[(1+e^x)^{\frac{1}{2}}$

$=\frac{1}{2}(1+e^x)^{-\frac{1}{2}}\cdot(1+e^x$

$=\dfrac{e^x}{2(\sqrt{1+e^x}-1)\sqrt{1+e^x}}-\dfrac{e^x}{2(\sqrt{1+e^x}+1)\sqrt{1+e^x}}$

$=\dfrac{e^x(\sqrt{1+e^x}+1)-e^x(\sqrt{1+e^x}-1)}{2(\sqrt{1+e^x}-1)(\sqrt{1+e^x}+1)\sqrt{1+e^x}}$

$=\dfrac{2e^x}{2e^x\sqrt{1+e^x}}$

$=\dfrac{1}{\sqrt{1+e^x}}$

N179b

2. Find the values of a and b that satisfy the following equation.

$$\frac{d}{dx}\left[x\sqrt{x^2+3}+a\ln\left(x+\sqrt{x^2+3}\right)\right]=b\sqrt{x^2+3}$$

[Sol] $\frac{d}{dx}\left[x\sqrt{x^2+3}+a\ln\left(x+\sqrt{x^2+3}\right)\right]$

$$=1\cdot\sqrt{x^2+3}+x\cdot\frac{1}{2}(x^2+3)^{-\frac{1}{2}}\cdot 2x+a\cdot\frac{\left(x+\sqrt{x^2+3}\right)'}{x+\sqrt{x^2+3}}$$

$$=\sqrt{x^2+3}+\frac{x^2}{\sqrt{x^2+3}}+\frac{a\left[1+\frac{1}{2}(x^2+3)^{-\frac{1}{2}}\cdot 2x\right]}{x+\sqrt{x^2+3}}$$

$$=\frac{2x^2+3}{\sqrt{x^2+3}}+\frac{a\left(\sqrt{x^2+3}+x\right)}{\left(x+\sqrt{x^2+3}\right)\sqrt{x^2+3}}$$

$$=\frac{2x^2+a+3}{\sqrt{x^2+3}}$$

$$\therefore\ \frac{2x^2+a+3}{\sqrt{x^2+3}}=b\sqrt{x^2+3}$$

$(2-b)x^2+(a-3b+3)=0$ ← Identity for x (J185)

$\therefore\ 2-b=0,\ a-3b+3=0$

$\therefore\ a=3,\ b=2$

3. Let $f(x)=(ax^2+bx+c)e^{-x}$ for constants a, b and c.
Given that $f'(x)=f(x)+xe^{-x}$ is satisfied for all real numbers x, find the values of a, b and c.

[Sol] $f'(x)=(ax^2+bx+c)'e^{-x}+(ax^2+bx+c)(e^{-x})'$

$$=(2ax+b)e^{-x}-(ax^2+bx+c)e^{-x}$$

$$=[-ax^2+(2a-b)x+(b-c)]e^{-x}\quad\cdots①$$

$$f(x)+xe^{-x}=(ax^2+bx+c)e^{-x}+xe^{-x}$$

$$=[ax^2+(b+1)x+c]e^{-x}\quad\cdots②$$

From ①, ② and $e^{-x}\neq 0$,

$-a=a,\ 2a-b=b+1,\ b-c=c$ ← Coefficient Comparison Method (J181)

$\therefore\ a=0,\ b=-\frac{1}{2},\ c=-\frac{1}{4}$

Differentiation of Logarithmic and Exponential Functions

100%	~90%	~80%	~70%	69%~
0	—	1	2	3~

Name

Date / /

Time : to :

1. Differentiate the following functions.

(1) $y=x(\ln x-1)$ ➡ N I

[Sol] $y'=(x)'(\ln x-1)+x(\ln x-1)'$

$=1\cdot(\ln x-1)+x\cdot\dfrac{1}{x}$

$=\ln x$

(2) $y=\ln(2x+3)$ ➡ N I

[Sol] $y'=\dfrac{(2x+3)'}{2x+3}$

$=\dfrac{2}{2x+3}$

(3) $y=\log_3|x^2-6|$ ➡ N I

[Sol] $y'=\dfrac{(x^2-6)'}{(x^2-6)\ln 3}$

$=\dfrac{2x}{(x^2-6)\ln 3}$

(4) $y=(\ln x)^2$ ➡ N I

[Sol] $y'=2\ln x\cdot(\ln x)'$

$=\dfrac{2\ln x}{x}$

N I 80b

2. Differentiate the following function using logarithmic differentiation.

$$y=\frac{\sqrt{x+2}}{x+1}$$ ➡ N I 75

[Sol] Taking the natural logarithm of the absolute values of both sides,

$$\ln|y|=\ln\left|\frac{\sqrt{x+2}}{x+1}\right|$$
$$=\frac{1}{2}\ln|x+2|-\ln|x+1|$$

Differentiating both sides with respect to x,

$$\frac{y'}{y}=\frac{1}{2(x+2)}-\frac{1}{x+1}$$
$$=-\frac{x+3}{2(x+2)(x+1)}$$
$$\therefore\ y'=-\frac{x+3}{2(x+2)(x+1)}\cdot\frac{\sqrt{x+2}}{x+1}=-\frac{(x+3)\sqrt{x+2}}{2(x+2)(x+1)^2}$$
$$\left[=-\frac{x+3}{2(x+1)^2\sqrt{x+2}}\right]$$

3. Differentiate the following functions.

(1) $y=e^{3x}+3^{-x}$ ➡ N I 77

[Sol] $y'=e^{3x}\cdot(3x)'+3^{-x}\ln 3\cdot(-x)'$

$$=3e^{3x}-3^{-x}\ln 3$$

(2) $y=(x-2)e^x$ ➡ N I 78

[Sol] $y'=(x-2)'e^x+(x-2)(e^x)'$

$$=1\cdot e^x+(x-2)e^x$$
$$=(x-1)e^x$$

Differentiation of Various Functions and Higher Order Derivatives

100%	~90%	~80%	~70%	69%~
0	—	—	1	2~

Name

Date / /

Time : to :

Given that y is a function of x, express $\dfrac{dy}{dx}$ in terms of x and y for the follow equations.

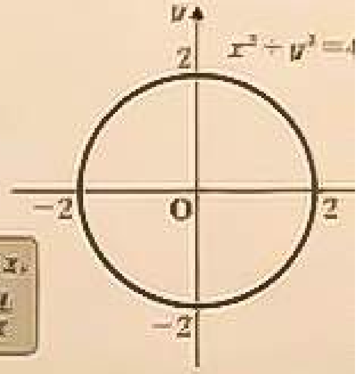
Ex. $x^2+y^2=4$

[Sol] Differentiating both sides with respect to x,

$$2x+2y\cdot\frac{dy}{dx}=0$$

$$\therefore\ \frac{dy}{dx}=-\frac{x}{y}$$

Differentiating y^2 with respect to x, $\dfrac{d}{dx}y^2=\boxed{\dfrac{d}{dy}y^2}\cdot\dfrac{dy}{dx}=\boxed{2y}\cdot\dfrac{dy}{dx}$

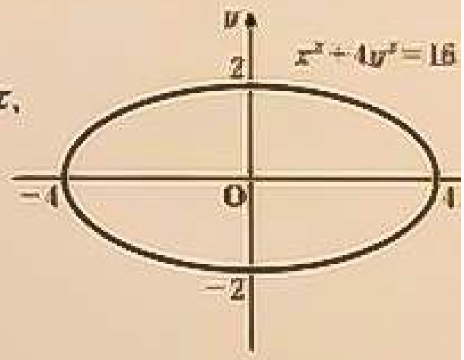


(1) $x^2+4y^2=16$

[Sol] Differentiating both sides with respect to x,

$$2x+8y\cdot\frac{dy}{dx}=0$$

$$\therefore\ \frac{dy}{dx}=-\frac{x}{4y}$$

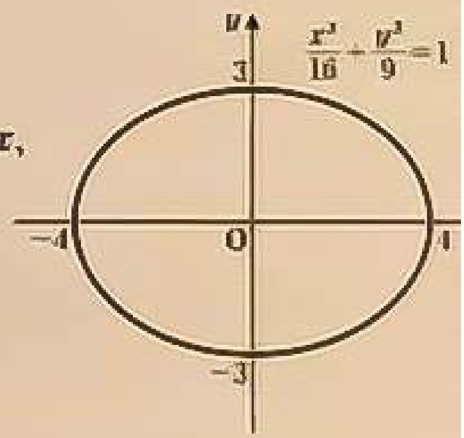


(2) $\dfrac{x^2}{16}+\dfrac{y^2}{9}=1$

[Sol] Differentiating both sides with respect to x,

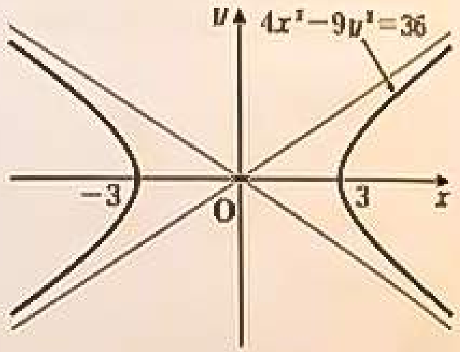
$$\frac{x}{8}+\frac{2y}{9}\cdot\frac{dy}{dx}=0$$

$$\therefore\ \frac{dy}{dx}=-\frac{9x}{16y}$$



N181b

(3) $4x^2-9y^2=36$

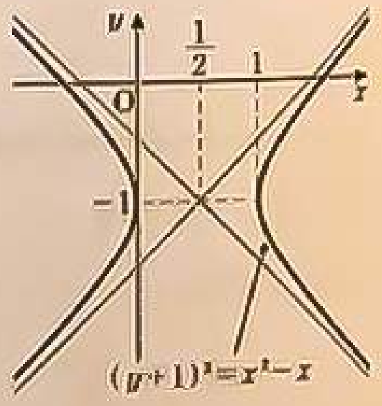


[Sol] Differentiating both sides with respect to x,

$$8x-18y\cdot\frac{dy}{dx}=0$$

$$\therefore\ \frac{dy}{dx}=\frac{4x}{9y}$$

(4) $(y+1)^2=x^2-x$



[Sol] Differentiating both sides with respect to x,

$$2(y+1)\cdot\frac{dy}{dx}=2x-1$$

$$\therefore\ \frac{dy}{dx}=\frac{2x-1}{2(y+1)}$$

The derivatives of these types of functions can also be found by solving the equations for y. In the case of **Ex1**, the answer can be found as follows.

[Sol] Solving the equation for y, $y=\pm\sqrt{4-x^2}$

(i) When $y=\sqrt{4-x^2}$,

$$\frac{dy}{dx}=(\sqrt{4-x^2})'=\frac{1}{2}(4-x^2)^{-\frac{1}{2}}\cdot(-2x)=-\frac{x}{\sqrt{4-x^2}}=-\frac{x}{y}$$

(ii) When $y=-\sqrt{4-x^2}$,

$$\frac{dy}{dx}=(-\sqrt{4-x^2})'=-\frac{1}{2}(4-x^2)^{-\frac{1}{2}}\cdot(-2x)=\frac{x}{\sqrt{4-x^2}}=-\frac{x}{y}$$

From (i) and (ii), $\frac{dy}{dx}=-\frac{x}{y}$

KUMON

Differentiation of Various Functions and Higher Order Derivatives

Name

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
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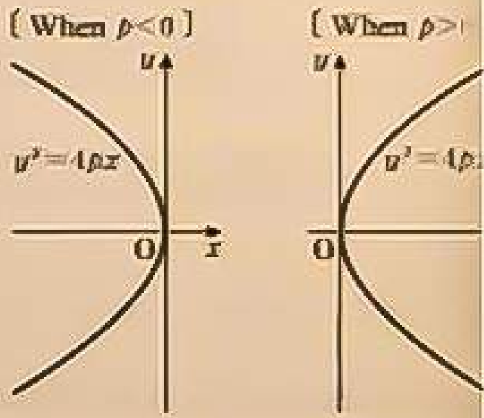
Given that y is a function of x, express $\dfrac{dy}{dx}$ in terms of x and y for the followi[ng] equations.

(1) $y^2=4px$ (p is a constant)

[Sol] Differentiating both sides with respect to x,

$$2y\cdot\frac{dy}{dx}=4p$$

$$\therefore\ \frac{dy}{dx}=\frac{2p}{y}$$

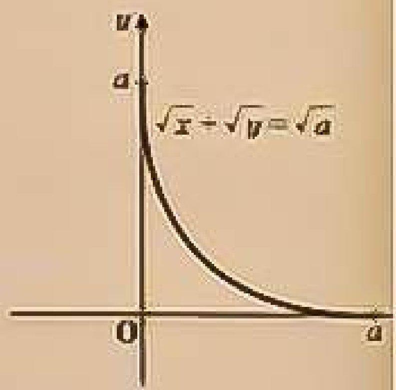


(2) $\sqrt{x}+\sqrt{y}=\sqrt{a}$ (a is a constant)

[Sol] Differentiating both sides with respect to x,

$$\frac{1}{2}x^{-\frac{1}{2}}+\frac{1}{2}y^{-\frac{1}{2}}\cdot\frac{dy}{dx}=0$$

$$\therefore\ \frac{dy}{dx}=-\sqrt{\frac{y}{x}}$$



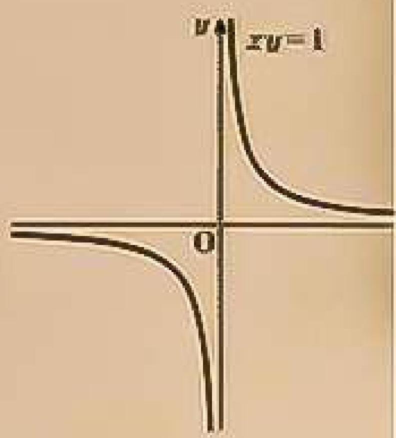
(3) $xy=1$

[Sol] Differentiating both sides with respect to x,

$$y+x\cdot\frac{dy}{dx}=0 \leftarrow$$

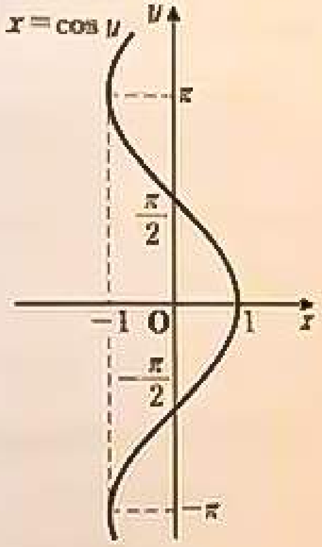
$(x)'y+x(y)'=0$
Since y is not a constant, $(xy)'=y$ is not true.

$$\therefore\ \frac{dy}{dx}=-\frac{y}{x}$$



N182b

(4) $x=\cos y$

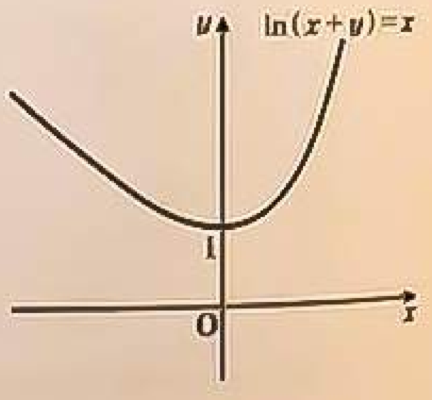


[Sol] Differentiating both sides with respect to x,

$$1=-\sin y\cdot\frac{dy}{dx}$$

$$\therefore\ \frac{dy}{dx}=-\frac{1}{\sin y}$$

(5) $\ln(x+y)=x$



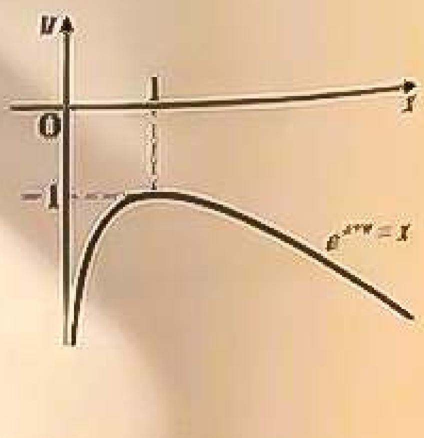
[Sol] Differentiating both sides with respect to x,

$$\frac{(x+y)'}{x+y}=1$$

$$\frac{1+\dfrac{dy}{dx}}{x+y}=1$$

$$\therefore\ \frac{dy}{dx}=x+y-1$$

(6) $e^{x+y}=x$



[Sol] Differentiating both sides with respect to x,

$$e^{x+y}\cdot(x+y)'=1$$

$$e^{x+y}\left(1+\frac{dy}{dx}\right)=1$$

$$\therefore\ \frac{dy}{dx}=\frac{1}{e^{x+y}}-1\quad\left[=\frac{1-e^{x+y}}{e^{x+y}}\right]$$

Differentiation of Various Functions and Higher Order Derivatives

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Name

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Time : to :

Given that y is a function of x and can be expressed as $x=f(t)$ and $y=g(t)$ using the parameter t,

$$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}=\frac{dy}{dt}\cdot\frac{1}{\frac{dx}{dt}} \leftarrow$$

Chain Rule I (N I 51)
Differentiation Formula for Inverse Functions (N I 55)

Therefore, the following is true.

Derivative of Functions Represented by a Parameter

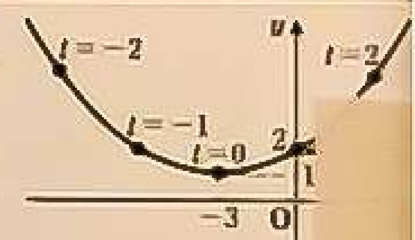
When $x=f(t)$ and $y=g(t)$, $\dfrac{dy}{dx}=\dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}=\dfrac{g'(t)}{f'(t)}$

Given the following functions represented by parameter t, express $\dfrac{dy}{dx}$ in terms of t.

Ex. $x=3(t-1)$, $y=t^2+1$

[Sol] $\dfrac{dx}{dt}=3$, $\dfrac{dy}{dt}=2t$ $\leftarrow$ Differentiating each equation with respect to t

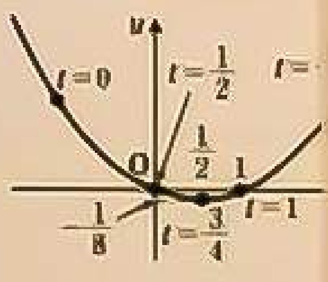
$\therefore\ \dfrac{dy}{dx}=\dfrac{2}{3}t$



(1) $x=2t-1$, $y=2t^2-3t+1$

[Sol] $\dfrac{dx}{dt}=2$, $\dfrac{dy}{dt}=4t-3$

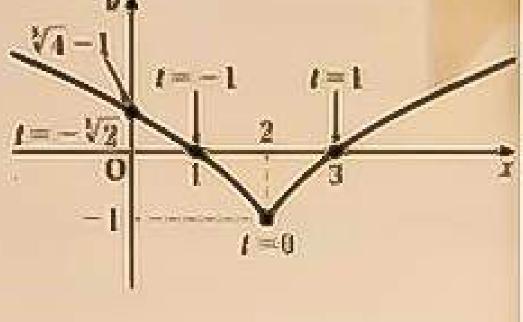
$\therefore\ \dfrac{dy}{dx}=\dfrac{4t-3}{2}$



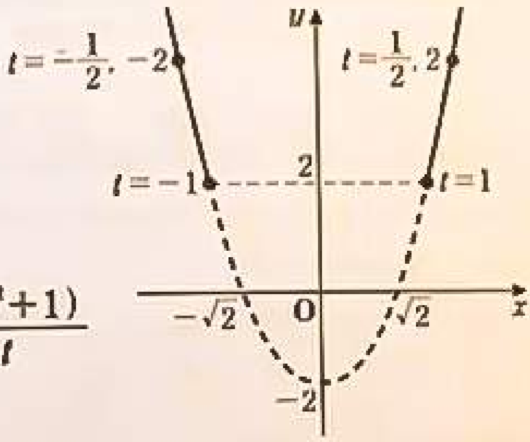
(2) $x=t^3+2$, $y=t^2-1$

[Sol] $\dfrac{dx}{dt}=3t^2$, $\dfrac{dy}{dt}=2t$

$\therefore\ \dfrac{dy}{dx}=\dfrac{2t}{3t^2}=\dfrac{2}{3t}$



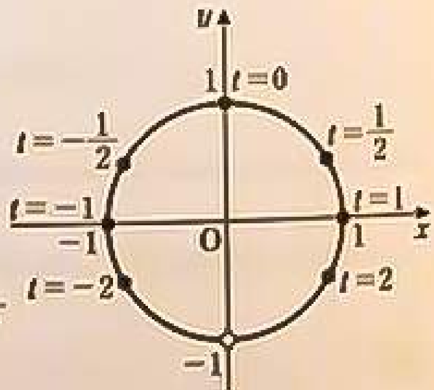
(3) $x=t+\frac{1}{t}$, $y=t^2+\frac{1}{t^2}$



[Sol] $\frac{dx}{dt}=1-\frac{1}{t^2}$, $\frac{dy}{dt}=2t-\frac{2}{t^3}$

$$\therefore\ \frac{dy}{dx}=\frac{2t-\frac{2}{t^3}}{1-\frac{1}{t^2}}=\frac{2(t^4-1)}{t(t^2-1)}=\frac{2(t^2+1)}{t}$$

(4) $x=\frac{2t}{1+t^2}$, $y=\frac{1-t^2}{1+t^2}$



[Sol] $\frac{dx}{dt}=\frac{2(1+t^2)-2t\cdot 2t}{(1+t^2)^2}=\frac{2(1-t^2)}{(1+t^2)^2}$

$$\frac{dy}{dt}=\frac{-2t(1+t^2)-(1-t^2)\cdot 2t}{(1+t^2)^2}=-\frac{4t}{(1+t^2)^2}$$

$$\therefore\ \frac{dy}{dx}=\frac{-\frac{4t}{(1+t^2)^2}}{\frac{2(1-t^2)}{(1+t^2)^2}}=-\frac{2t}{1-t^2}\quad\left[=-\frac{2t}{(1+t)(1-t)}\right]$$

The derivatives of these types of functions can also be found by solving the equations for y. In the case of **Ex**, the answer can be found as follows.

[Sol] Since $x=3(t-1)$, $t=\frac{x+3}{3}$

Substituting into $y=t^2+1$, $y=\left(\frac{x+3}{3}\right)^2+1=\frac{1}{9}(x^2+6x+18)$

Differentiating both sides with respect to x,

$$\frac{dy}{dx}=\frac{1}{9}(2x+6)=\frac{2}{9}(x+3)$$

Since $x=3(t-1)$,

$$\frac{dy}{dx}=\frac{2}{9}\{3(t-1)+3\}=\frac{2}{3}t$$

Differentiation of Various Functions and Higher Order Derivatives

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Name

Date / /

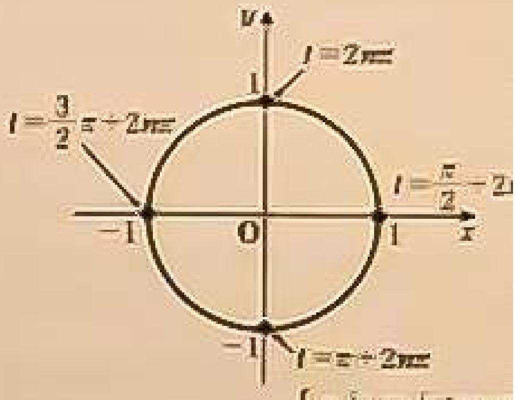
Time : to :

Given the following functions represented by parameter t, express $\dfrac{dy}{dx}$ in terms of t.

(1) $x=\sin t,\ y=\cos t$

[Sol] $\dfrac{dx}{dt}=\cos t,\ \dfrac{dy}{dt}=-\sin t$

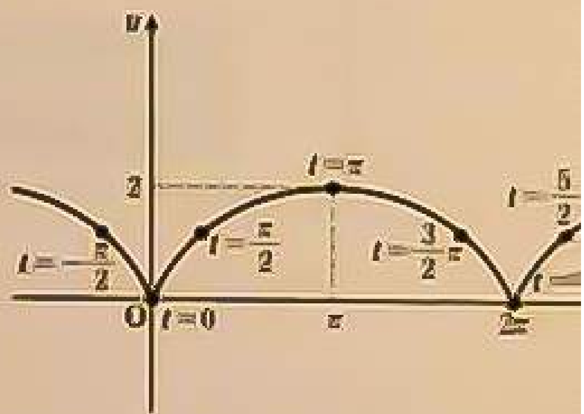
$$\therefore\ \frac{dy}{dx}=-\frac{\sin t}{\cos t}\quad \left[=-\tan t\right]$$



(2) $x=t-\sin t,\ y=1-\cos t$

[Sol] $\dfrac{dx}{dt}=1-\cos t,\ \dfrac{dy}{dt}=\sin t$

$$\therefore\ \frac{dy}{dx}=\frac{\sin t}{1-\cos t}$$

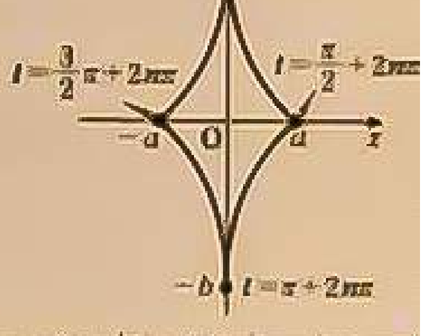


(3) $x=a\sin^3 t,\ y=b\cos^3 t$ (a and b are constants)

[Sol] $\dfrac{dx}{dt}=3a\sin^2 t\cdot(\sin t)'=3a\sin^2 t\cos t$

$\dfrac{dy}{dt}=3b\cos^2 t\cdot(\cos t)'=-3b\cos^2 t\sin t$

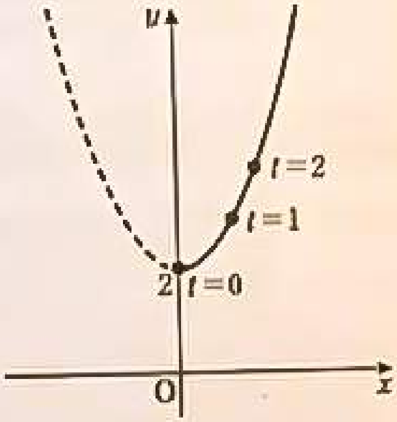
$$\therefore\ \frac{dy}{dx}=\frac{-3b\cos^2 t\sin t}{3a\sin^2 t\cos t}=-\frac{b\cos t}{a\sin t}$$



(When $a>0$ and $b>0$ (n is an integer)

N184b

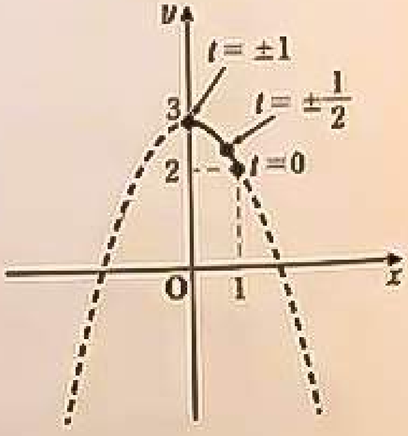
(4) $x=\sqrt{t},\ y=t+2$



[Sol] $\dfrac{dx}{dt}=\dfrac{1}{2}t^{-\frac{1}{2}}=\dfrac{1}{2\sqrt{t}},\quad \dfrac{dy}{dt}=1$

$$\therefore\ \frac{dy}{dx}=\frac{1}{\dfrac{1}{2\sqrt{t}}}=2\sqrt{t}$$

(5) $x=\sqrt{1-t^2},\ y=t^2+2$



[Sol] $\dfrac{dx}{dt}=\dfrac{1}{2}(1-t^2)^{-\frac{1}{2}}\cdot(-2t)=-\dfrac{t}{\sqrt{1-t^2}}$

$$\frac{dy}{dt}=2t$$

$$\therefore\ \frac{dy}{dx}=\frac{2t}{-\dfrac{t}{\sqrt{1-t^2}}}=-2\sqrt{1-t^2}$$

(6) $x=\dfrac{e^t}{1+t^2},\ y=\dfrac{t}{1+t^2}$

[Sol] $\dfrac{dx}{dt}=\dfrac{(e^t)'(1+t^2)-e^t(1+t^2)'}{(1+t^2)^2}=\dfrac{e^t(1+t^2)-e^t\cdot 2t}{(1+t^2)^2}=\dfrac{(1-2t+t^2)e^t}{(1+t^2)^2}$

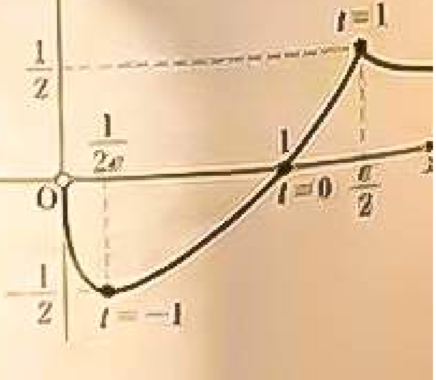
$$=\frac{(1-t)^2e^t}{(1+t^2)^2}$$

$$\frac{dy}{dt}=\frac{(t)'(1+t^2)-t(1+t^2)'}{(1+t^2)^2}=\frac{1\cdot(1+t^2)-t\cdot 2t}{(1+t^2)^2}=\frac{1-t^2}{(1+t^2)^2}$$

$$=\frac{(1+t)(1-t)}{(1+t^2)^2}$$

$$\therefore\ \frac{dy}{dx}=\frac{\dfrac{(1+t)(1-t)}{(1+t^2)^2}}{\dfrac{(1-t)^2e^t}{(1+t^2)^2}}$$

$$=\frac{1+t}{(1-t)e^t}\quad\left[=-\frac{t+1}{(t-1)e^t}\right]$$



Differentiation of Various Functions and Higher Order Derivatives

Name

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The function $[f'(x)]'$ which is derived by differentiating the derivative $f'(x)$ of $y=f(x)$ is called the ***second order derivative*** of $f(x)$, and is expressed as y'' or $f''(x)$. The derivative of the second order derivative $f''(x)$ is called the ***third order derivative*** of $f(x)$ and is expressed as y''' or $f'''(x)$. Likewise, $f'(x)$ is often called the ***first order derivative*** of $f(x)$.

Generally, the function found by differentiating the function $y=f(x)$ for n times is called the ***n^{th} order derivative*** of $f(x)$ and is expressed as $y^{(n)}$ or $f^{(n)}(x)$.

The derivatives of second order and above are called ***higher order derivatives***.

Find the second and third order derivatives of the following functions.

Ex. $y=x^4+2x^3-x+1$

[Sol] $y'=4x^3+6x^2-1$

$\therefore\ y''=12x^2+12x$ — Differentiating y'

$y'''=24x+12$ — Differentiating y''

(1) $y=\frac{1}{2}x^3-x^2+4x$

[Sol] $y'=\frac{3}{2}x^2-2x+4$

$\therefore\ y''=3x-2$

$y'''=3$

(2) $y=x-\frac{1}{x}$

[Sol] $y'=1+\frac{1}{x^2}$

$\therefore\ y''=-\frac{2}{x^3}$

$y'''=\frac{6}{x^4}$

N185b

(3) $y=\sin x$

[Sol] $y'=\cos x$

$\therefore\ y''=-\sin x$

$y'''=-\cos x$

(4) $y=\ln x$

[Sol] $y'=\dfrac{1}{x}$

$\therefore\ y''=-\dfrac{1}{x^2}$

$y'''=\dfrac{2}{x^3}$

(5) $y=e^x$

[Sol] $y'=e^x$

$\therefore\ y''=e^x$

$y'''=e^x$

(6) $y=\sqrt{x}+\dfrac{1}{\sqrt{x}}$

[Sol] $y'=\dfrac{1}{2}x^{-\frac{1}{2}}-\dfrac{1}{2}x^{-\frac{3}{2}}=\dfrac{1}{2\sqrt{x}}-\dfrac{1}{2x\sqrt{x}}$

$\therefore\ y''=-\dfrac{1}{4}x^{-\frac{3}{2}}+\dfrac{3}{4}x^{-\frac{5}{2}}=-\dfrac{1}{4x\sqrt{x}}+\dfrac{3}{4x^2\sqrt{x}}\quad\left[=-\dfrac{x-3}{4x^2\sqrt{x}}\right]$

$y'''=\dfrac{3}{8}x^{-\frac{5}{2}}-\dfrac{15}{8}x^{-\frac{7}{2}}=\dfrac{3}{8x^2\sqrt{x}}-\dfrac{15}{8x^3\sqrt{x}}\quad\left[=\dfrac{3(x-5)}{8x^3\sqrt{x}}\right]$

Differentiation of Various Functions and Higher Order Derivatives

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1. Find the second order derivatives of the following functions.

(1) $y=x\sin x$

[Sol] $y'=(x)'\sin x+x(\sin x)'$

$=1\cdot\sin x+x\cos x$

$=\sin x+x\cos x$

$\therefore\ y''=\cos x+[(x)'\cos x+x(\cos x)']$

$=\cos x+(1\cdot\cos x-x\sin x)$

$=2\cos x-x\sin x$

(2) $y=x^2\ln x$

[Sol] $y'=(x^2)'\ln x+x^2(\ln x)'$

$=2x\ln x+x^2\cdot\frac{1}{x}$

$=x(2\ln x+1)$

$\therefore\ y''=(x)'(2\ln x+1)+x(2\ln x+1)'$

$=1\cdot(2\ln x+1)+x\cdot\frac{2}{x}$

$=2\ln x+3$

(3) $y=\frac{\cos x}{1-\sin x}$

[Sol] $y'=\frac{(\cos x)'(1-\sin x)-\cos x(1-\sin x)'}{(1-\sin x)^2}$

$=\frac{-\sin x(1-\sin x)+\cos^2 x}{(1-\sin x)^2}$

$=\frac{1-\sin x}{(1-\sin x)^2}$

$=\frac{1}{1-\sin x}$

$\therefore\ y''=-\frac{(1-\sin x)'}{(1-\sin x)^2}$

$=\frac{\cos x}{(1-\sin x)^2}$

#2

N186b

2. Find the n^{th} order derivatives of the following functions.

Ex $y=e^{3x}$

[Sol] $y'=e^{3x}\cdot(3x)'=3e^{3x}$

$y''=3e^{3x}\cdot(3x)'=3^2e^{3x}$

$y'''=3^2e^{3x}\cdot(3x)'=3^3e^{3x}$

$\vdots$

$\therefore\ y^{(n)}=3^ne^{3x}$

(1) $y=e^{-2x}$

[Sol] $y'=e^{-2x}\cdot(-2x)'=-2e^{-2x}$

$y''=-2e^{-2x}\cdot(-2x)'=(-2)^2e^{-2x}$

$y'''=(-2)^2e^{-2x}\cdot(-2x)'=(-2)^3e^{-2x}$

$\vdots$

$\therefore\ y^{(n)}=(-2)^ne^{-2x}$

(2) $y=xe^x$

[Sol] $y'=(x)'e^x+x(e^x)'=1\cdot e^x+xe^x=(x+1)e^x$

$y''=(x+1)'e^x+(x+1)(e^x)'=1\cdot e^x+(x+1)e^x=(x+2)e^x$

$y'''=(x+2)'e^x+(x+2)(e^x)'=1\cdot e^x+(x+2)e^x=(x+3)e^x$

$\vdots$

$\therefore\ y^{(n)}=(x+n)e^x$

(3) $y=x^{\alpha}$

[Sol] $y'=\alpha x^{\alpha-1}$

$y''=\alpha(\alpha-1)x^{\alpha-2}$

$y'''=\alpha(\alpha-1)(\alpha-2)x^{\alpha-3}$

$\vdots$

$\therefore\ y^{(n)}=\alpha(\alpha-1)(\alpha-2$

Differentiation of Various Functions and Higher Order Derivatives

Name

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Ex. Prove that function $y=e^{-x}\cos x$ satisfies the following equation.

$y''+2y'+2y=0$

[Sol] $y'=(e^{-x})'\cos x+e^{-x}(\cos x)'$

$=-e^{-x}\cos x-e^{-x}\sin x$

$=-e^{-x}(\cos x+\sin x)$

$y''=-[(e^{-x})'(\cos x+\sin x)+e^{-x}(\cos x+\sin x)']$

$=e^{-x}(\cos x+\sin x)-e^{-x}(-\sin x+\cos x)$

$=2e^{-x}\sin x$

Therefore,

$y''+2y'+2y=2e^{-x}\sin x-2e^{-x}(\cos x+\sin x)+2e^{-x}\cos x$

$=2e^{-x}(\sin x-\cos x-\sin x+\cos x)$

$=0$

1. Prove that function $y=e^{x}\sin x$ satisfies the following equation.

$y''-2y'+2y=0$

[Sol] $y'=(e^{x})'\sin x+e^{x}(\sin x)'$

$=e^{x}\sin x+e^{x}\cos x$

$=e^{x}(\sin x+\cos x)$

$y''=(e^{x})'(\sin x+\cos x)+e^{x}(\sin x+\cos x)'$

$=e^{x}(\sin x+\cos x)+e^{x}(\cos x-\sin x)$

$=2e^{x}\cos x$

Therefore,

$y''-2y'+2y=2e^{x}\cos x-2e^{x}(\sin x+\cos x)+2e^{x}\sin x$

$=2e^{x}(\cos x-\sin x-\cos x+\sin x)$

$=0$

N187b

2. Prove that function $y=x+\sqrt{1+x^2}$ satisfies the following equation.

$$(1+x^2)y''+xy'-y=0$$

[Sol] $y'=1+\frac{1}{2}(1+x^2)^{-\frac{1}{2}}\cdot 2x=1+\frac{x}{\sqrt{1+x^2}}$

$$y''=\frac{(x)'\sqrt{1+x^2}-x(\sqrt{1+x^2})'}{1+x^2}=\frac{1\cdot\sqrt{1+x^2}-x\cdot\frac{1}{2}(1+x^2)^{-\frac{1}{2}}\cdot 2x}{1+x^2}$$

$$=\frac{1}{(1+x^2)\sqrt{1+x^2}}$$

Therefore,

$$(1+x^2)y''+xy'-y=\frac{1+x^2}{(1+x^2)\sqrt{1+x^2}}+x\left(1+\frac{x}{\sqrt{1+x^2}}\right)-(x+\sqrt{1+x^2})$$

$$=\frac{1+x\sqrt{1+x^2}+x^2-x\sqrt{1+x^2}-(1+x^2)}{\sqrt{1+x^2}}$$

$$=0$$

3. Find the values of constants a and b such that $f''(x)=af(x)+bf'(x)$ when $f(x)=e^{3x}\cos x$.

[Sol] $f'(x)=(e^{3x})'\cos x+e^{3x}(\cos x)'$

$$=3e^{3x}\cos x-e^{3x}\sin x$$

$$=e^{3x}(3\cos x-\sin x)$$

$$f''(x)=(e^{3x})'(3\cos x-\sin x)+e^{3x}(3\cos x-\sin x)'$$

$$=3e^{3x}(3\cos x-\sin x)+e^{3x}(-3\sin x-\cos x)$$

$$=e^{3x}(8\cos x-6\sin x)\ \cdots ①$$

$$af(x)+bf'(x)=ae^{3x}\cos x+be^{3x}(3\cos x-\sin x)$$

$$=e^{3x}[(a+3b)\cos x-b\sin x]\ \cdots ②$$

From ①, ② and $e^{3x}\neq 0$,

$a+3b=8,\ b=6$ ← Coefficient Comparison Method (J181)

$\therefore a=-10,\ b=6$

Differentiation of Various Functions and Higher Order Derivatives

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Ex. Using mathematical induction, prove that the n^{th} order derivative of function $y=\sin x$ is represented as follows:

$$y^{(n)}=\sin\left(x+\frac{n\pi}{2}\right)\quad\cdots①$$

[Sol] (i) When $n=1$,

$$\text{LHS}=y^{(1)}=(\sin x)'=\boxed{\cos x} \leftarrow \boxed{y^{(1)}=y'}$$

$$\text{RHS}=\sin\left(x+\frac{1\cdot\pi}{2}\right)=\boxed{\cos x} \leftarrow \boxed{\sin\left(\theta+\frac{\pi}{2}\right)=\cos\theta}$$

Therefore, ① is true when $n=1$.

(ii) If ① is true when $n=k$,

$$y^{(k)}=\sin\left(x+\boxed{\frac{k\pi}{2}}\right)\quad\cdots②$$

When $n=k+1$, differentiating both sides of ② with respect to x,

$$y^{(k+1)}=\cos\left(x+\boxed{\frac{k\pi}{2}}\right)$$

$$=\sin\left(x+\boxed{\frac{k\pi}{2}}+\frac{\pi}{2}\right) \leftarrow \boxed{\cos\theta=\sin\left(\theta+\frac{\pi}{2}\right)}$$

$$=\sin\left[x+\boxed{\frac{(k+1)\pi}{2}}\right]$$

Therefore, ① is also true when $n=k+1$.

From (i) and (ii), ① is true for all natural numbers n.

Answers $\cos x$, $\cos x$, $\frac{k\pi}{2}$, $\frac{k\pi}{2}$, $\frac{k\pi}{2}$, $\frac{(k+1)\pi}{2}$

N188b

1. Using mathematical induction, prove that the n^{th} order derivative of function $y=\ln x$ is represented as follows:

$$y^{(n)}=(-1)^{n-1}\cdot\frac{(n-1)!}{x^n}\quad\cdots①$$

$$((n-1)!=(n-1)(n-2)\cdot\cdots\cdot3\cdot2\cdot1,\ 0!=1)$$

[Sol] (i) When $n=1$,

$$\text{LHS}=y^{(1)}=(\ln x)'=\frac{1}{x}$$

$$\text{RHS}=(-1)^0\cdot\frac{0!}{x^1}=\frac{1}{x}\quad\longleftarrow\ \boxed{a^0=1}$$

Therefore, ① is true when $n=1$.

(ii) If ① is true when $n=k$,

$$y^{(k)}=(-1)^{k-1}\cdot\frac{(k-1)!}{x^k}\quad\cdots②$$

When $n=k+1$, differentiating both sides of ② with respect to x,

$$y^{(k+1)}=(-1)^{k-1}\cdot\left[-\frac{(k-1)!\cdot kx^{k-1}}{x^{2k}}\right]$$

$$=(-1)^{k-1}\cdot(-1)\cdot\frac{(k-1)!\cdot k}{x^{2k-(k-1)}}$$

$$=(-1)^k\cdot\frac{k!}{x^{k+1}}$$

$$\boxed{\begin{aligned}&(k-1)!\cdot k\\&=(k-1)(k-2)\cdot\cdots\cdot3\cdot2\cdot1\cdot k\\&=k!\end{aligned}}$$

$$=(-1)^{(k+1)-1}\cdot\frac{[(k+1)-1]!}{x^{k+1}}$$

Therefore, ① is also true when $n=k+1$.

From (i) and (ii), ① is true for all natural numbers n.

The product of all natural numbers from 1 to n is called the *factorial* of n and is expressed as $n!$. $(0!=1)$

N189a

KUMON

N 1

Differentiation of Various Functions and Higher Order Derivatives

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1. Given that y is a function of x, express $\dfrac{d^2y}{dx^2}$ in terms of x and y for the following equation.

$x^2-y^2=a^2$ (a is a constant)

[Sol] Differentiating both sides with respect to x,

$$2x-2y\cdot\frac{dy}{dx}=0$$

$$\therefore\ \frac{dy}{dx}=\frac{x}{y}$$

Furthermore, differentiating both sides with respect to x,

$$\frac{d^2y}{dx^2}=\frac{(x)'y-x(y)'}{y^2}=\frac{1\cdot y-x\cdot\dfrac{dy}{dx}}{y^2}=\frac{y^2-x^2}{y^3}$$

$$\boxed{\text{Since } \frac{dy}{dx}=\frac{x}{y}}$$

2. Using the equality $\dfrac{d^2y}{dx^2}=\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)=\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)\cdot\dfrac{dt}{dx}$, express $\dfrac{d^2y}{dx^2}$ in terms of t when $x=3t^3$ and $y=9t+1$.

[Sol] $\dfrac{dx}{dt}=9t^2$, $\dfrac{dy}{dt}=9$

$$\therefore\ \frac{dy}{dx}=\frac{9}{9t^2}=\frac{1}{t^2}$$

$$\therefore\ \frac{d^2y}{dx^2}=\frac{d}{dt}\left(\frac{1}{t^2}\right)\cdot\frac{dt}{dx}=-\frac{2}{t^3}\cdot\frac{1}{9t^2}=-\frac{2}{9t^5}$$

$$\boxed{\frac{dt}{dx}=\frac{1}{\dfrac{dx}{dt}}=\frac{1}{9t^2}}$$

The second, third and n^{th} order derivatives are often expressed as

$\dfrac{d^2y}{dx^2}$, $\dfrac{d^3y}{dx^3}$, $\dfrac{d^ny}{dx^n}$ or $\dfrac{d^2}{dx^2}f(x)$, $\dfrac{d^3}{dx^3}f(x)$, $\dfrac{d^n}{dx^n}f(x)$ respectively.

$\dfrac{d^2y}{dx^2}$ is read as "d two y by dx squared."

N189b

3. Given $f(x)=\cos x+1$ and $g(x)=\dfrac{a}{bx^2+cx+1}$, find the values of consta

a, b and c which satisfy $f(0)=g(0)$, $f'(0)=g'(0)$ and $f''(0)=g''(0)$.

[Sol] $f(0)=\cos 0+1=2$, $g(0)=a$

Since $f(0)=g(0)$, $a=2$ ⋯①

From ①, $g(x)=\dfrac{2}{bx^2+cx+1}$

Since $g'(x)=-\dfrac{2(2bx+c)}{(bx^2+cx+1)^2}$, $g'(0)=-2c$

Also, since $f'(x)=-\sin x$, $f'(0)=0$

Since $f'(0)=g'(0)$, $c=0$ ⋯②

From ②, $g'(x)=-\dfrac{4bx}{(bx^2+1)^2}$

$$g''(x)=-\frac{4b(bx^2+1)^2-4bx\cdot 2(bx^2+1)\cdot 2bx}{(bx^2+1)^4}$$

$$=-\frac{4b(bx^2+1-4bx^2)}{(bx^2+1)^3}$$

$$=\frac{4b(3bx^2-1)}{(bx^2+1)^3}$$

$\therefore\ g''(0)=-4b$

Also, since $f''(x)=-\cos x$, $f''(0)=-1$

Since $f''(0)=g''(0)$, $b=\dfrac{1}{4}$

$\therefore\ a=2,\ b=\dfrac{1}{4},\ c=0$

KUMON

Differentiation of Various Functions and Higher Order Derivatives

Name

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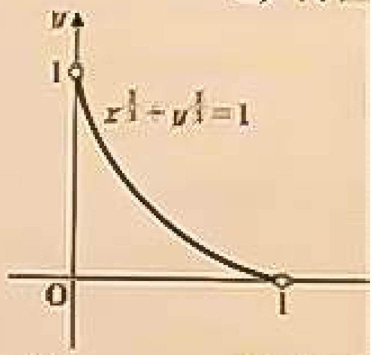
1. Given $x^{\frac{2}{3}}+y^{\frac{2}{3}}=1$, express $\dfrac{dy}{dx}$ in terms of x and y. $(x>0,\ y>0)$

➡ N I

[Sol] Differentiating both sides with respect to x,

$$\frac{2}{3}x^{-\frac{1}{3}}+\frac{2}{3}y^{-\frac{1}{3}}\cdot\frac{dy}{dx}=0$$

$$\therefore\ \frac{dy}{dx}=-\left(\frac{y}{x}\right)^{\frac{1}{3}}\ \left[=-\left(\frac{x}{y}\right)^{-\frac{1}{3}}\right]$$



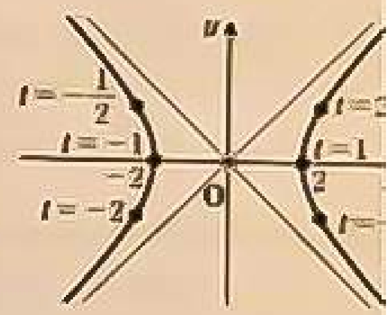
2. Given $x=t+\dfrac{1}{t}$ and $y=t-\dfrac{1}{t}$, express $\dfrac{dy}{dx}$ in terms of t.

➡ N I 8

[Sol] $\dfrac{dx}{dt}=1-\dfrac{1}{t^2}$, $\dfrac{dy}{dt}=1+\dfrac{1}{t^2}$

$$\therefore\ \frac{dy}{dx}=\frac{1+\frac{1}{t^2}}{1-\frac{1}{t^2}}=\frac{t^2+1}{t^2-1}$$

$$\left[=\frac{t^2+1}{(t+1)(t-1)}\right]$$



3. Find the second and third order derivatives of function $y=x^3\ln x$.

➡ N I 8

[Sol] $y'=(x^3)'\ln x+x^3(\ln x)'$

$$=3x^2\ln x+x^3\cdot\frac{1}{x}$$

$$=x^2(3\ln x+1)$$

$$\therefore\ y''=(x^2)'(3\ln x+1)+x^2(3\ln x+1)'$$

$$=2x(3\ln x+1)+x^2\cdot\frac{3}{x}$$

$$=6x\ln x+5x\quad[=x(6\ln x+5)]$$

$$y'''=6[(x)'\ln x+x(\ln x)']+5$$

$$=6\left(1\cdot\ln x+x\cdot\frac{1}{x}\right)+5$$

$$=6\ln x+11$$

4. Prove that function $y=a\cos kx+b\sin kx$ satisfies $y''+k^2y=0$. ➡ N187

[Sol] $y'=-a\sin kx\cdot(kx)'+b\cos kx\cdot(kx)'$

$=-ak\sin kx+bk\cos kx$

$=-k(a\sin kx-b\cos kx)$

$y''=-k[a\cos kx\cdot(kx)'+b\sin kx\cdot(kx)']$

$=-k(ak\cos kx+bk\sin kx)$

$=-k^2(a\cos kx+b\sin kx)$

Therefore,

$$y''+k^2y=-k^2(a\cos kx+b\sin kx)+k^2(a\cos kx+b\sin kx)$$
$$=0$$

5. Using mathematical induction, prove that the n^{th} order derivative of function $y=xe^{-x}$ is represented as follows:

$y^{(n)}=(-1)^n(x-n)e^{-x}$ …① ➡ N188

[Sol] (i) When $n=1$,

LHS $=y^{(1)}=(x)'e^{-x}+x(e^{-x})'=1\cdot e^{-x}-xe^{-x}=-(x-1)e^{-x}$

RHS $=(-1)^1\cdot(x-1)e^{-x}=-(x-1)e^{-x}$

Therefore, ① is true when $n=1$.

(ii) If ① is true when $n=k$,

$y^{(k)}=(-1)^k(x-k)e^{-x}$ …②

When $n=k+1$, differentiating both sides of ② with respect to x,

$y^{(k+1)}=(-1)^k[(x-k)'e^{-x}+(x-k)(e^{-x})']$

$=(-1)^k[1\cdot e^{-x}-(x-k)e^{-x}]$

$=(-1)^k\cdot(-1)\cdot(x-k-1)e^{-x}$

$=(-1)^{k+1}[x-(k+1)]e^{-x}$

Therefore, ① is also true when $n=k+1$.

From (i) and (ii), ① is true for all natural numbers n.

Various Properties of Derivatives

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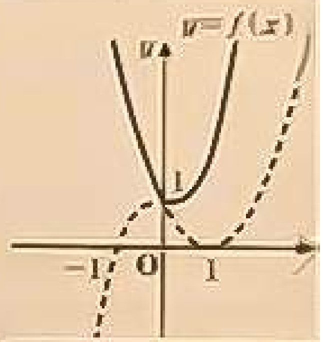
Examine if the following functions are differentiable at the points indicated in the brackets [ ].

Ex.

$$f(x)=\begin{cases} x^3+1 & (x \geqq 0) \\ (x-1)^2 & (x<0) \end{cases} \quad [x=0]$$



[Sol] $\displaystyle\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^+}\frac{[(0+h)^3+1]-1}{h}$

$\displaystyle=\lim_{h\to 0^+}h^2=0$

$\displaystyle\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^-}\frac{[(0+h)-1]^2-1}{h}$

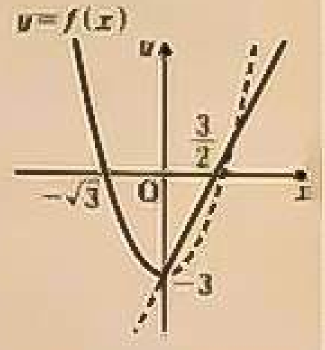
$\displaystyle=\lim_{h\to 0^-}(h-2)=-2$

$\displaystyle\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h} \neq \lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h}$

Therefore, $\displaystyle\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$ does not exist.

Thus, $f(x)$ is not differentiable at $x=0$.

(1) $f(x)=\begin{cases} 2x-3 & (x \geqq 0) \\ x^2-3 & (x<0) \end{cases} \quad [x=0]$



[Sol] $\displaystyle\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^+}\frac{[2(0+h)-3]-(-3)}{h}$

$\displaystyle=\lim_{h\to 0^+}2=2$

$\displaystyle\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^-}\frac{[(0+h)^2-3]-(-3)}{h}$

$\displaystyle=\lim_{h\to 0^-}h=0$

Therefore, $\displaystyle\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$ does not exist.

Thus, $f(x)$ is not differentiable at $x=0$.

[Reference] N141

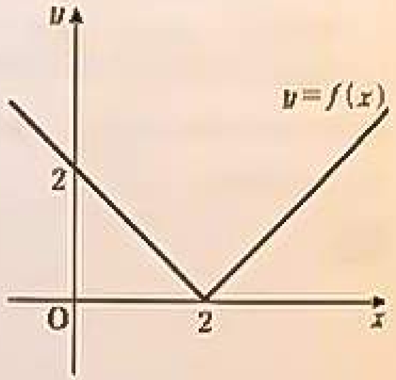
Given the function $f(x)$, if the limit value $\displaystyle\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ exists, then $f(x)$ is differentiable at $x=a$.

N191b

(2) $f(x)=|x-2|$ $(x=2)$

[Sol] When $x \geqslant 2$, $f(x)=x-2$

When $x<2$, $f(x)=-(x-2)$



$$\lim_{h\to 0^+}\frac{f(2+h)-f(2)}{h}=\lim_{h\to 0^+}\frac{[(2+h)-2]-0}{h}$$
$$=\lim_{h\to 0^+}1=1$$

$$\lim_{h\to 0^-}\frac{f(2+h)-f(2)}{h}=\lim_{h\to 0^-}\frac{-[(2+h)-2]-0}{h}$$
$$=\lim_{h\to 0^-}(-1)=-1$$

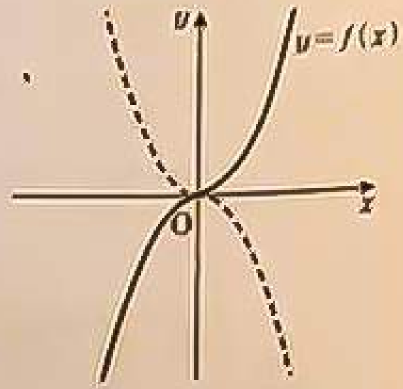
Therefore, $\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$ does not exist.

Thus, $f(x)$ is not differentiable at $x=2$.

(3) $f(x)=x|x|$ $(x=0)$

[Sol] When $x \geqslant 0$, $f(x)=x^2$

When $x<0$, $f(x)=-x^2$



$$\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^+}\frac{(0+h)^2-0}{h}$$
$$=\lim_{h\to 0^+}h=0$$

$$\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^-}\frac{-(0+h)^2-0}{h}$$
$$=\lim_{h\to 0^-}(-h)=0$$

$$\therefore \lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=0 \leftarrow \boxed{\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h}}$$

Therefore, $f(x)$ is differentiable at $x=0$.

Various Properties of Derivatives

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If the function $f(x)$ is differentiable at $x=a$, then $f'(a)$ exists and

$$\lim_{x\to a}[f(x)-f(a)]=\lim_{x\to a}\left[\frac{f(x)-f(a)}{x-a}\cdot(x-a)\right]=f'(a)\cdot 0=0$$

$\therefore\ \lim_{x\to a}f(x)=f(a)$ ← $f(x)$ is continuous at $x=a$.

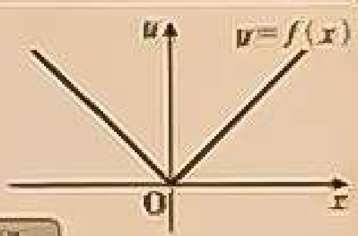
Therefore, the following statement is true.

Differentiability and Continuity

If the function $f(x)$ is differentiable at $x=a$, then it is continuous at $x=a$.

Examine if the following functions are continuous and also differentiable at the points indicated in the brackets ( ).

Ex. $f(x)=|x|\quad(x=0)$



[Sol] When $x \geqq 0$, $f(x)=x$

When $x<0$, $f(x)=-x$

$\lim_{x\to+0}f(x)=\lim_{x\to+0}x=0$

$\lim_{x\to-0}f(x)=\lim_{x\to-0}(-x)=0$ ← Examining if $\lim_{x\to a}f(x)$ exists

Also, since $f(0)=0$, $\lim_{x\to 0}f(x)=f(0)$ ← Examining if $\lim_{x\to a}f(x)=f(a)$ is true

Therefore, $f(x)$ is continuous at $x=0$.

$$\lim_{h\to+0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to+0}\frac{(0+\boxed{h})-0}{h}=\lim_{h\to+0}\boxed{1}=\boxed{1}$$

$$\lim_{h\to-0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to-0}\frac{-(0+\boxed{h})-0}{h}=\lim_{h\to-0}(\boxed{-1})=\boxed{-1}$$

Thus, $\lim_{h\to 0}\dfrac{f(0+h)-f(0)}{h}$ does not exist. ← $\lim_{h\to+0}\dfrac{f(0+h)-f(0)}{h}\neq\lim_{h\to-0}\dfrac{f(0+h)-f(0)}{h}$

Therefore, $f(x)$ is not differentiable at $x=0$.

From **Ex.**, even if the function $f(x)$ is continuous at $x=a$, it is not always differentiable at $x=a$.

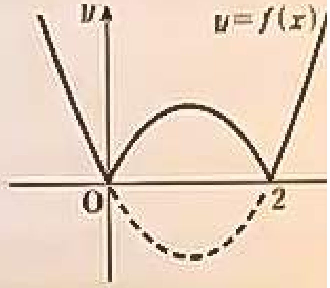
[Reference] N131

N192b

(1) $f(x)=|x(x-2)|$ 〔$x=2$〕

[Sol] When $x(x-2) \geqslant 0$, i.e. $x \leqslant 0,\ 2 \leqslant x$, $f(x)=x(x-2)$

When $x(x-2) < 0$, i.e. $0 < x < 2$, $f(x)=-x(x-2)$



$\lim\limits_{x \to 2^+} f(x)=\lim\limits_{x \to 2^+} x(x-2)=0$

$\lim\limits_{x \to 2^-} f(x)=\lim\limits_{x \to 2^-} [-x(x-2)]=0$

Also, since $f(2)=0$, $\lim\limits_{x \to 2} f(x)=f(2)$

Therefore, $f(x)$ is continuous at $x=2$.

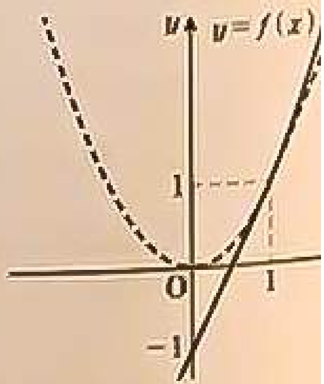
$$\lim_{h \to 0^+} \frac{f(2+h)-f(2)}{h}=\lim_{h \to 0^+} \frac{(2+h)[(2+h)-2]-0}{h}=\lim_{h \to 0^+}(2+h)=2$$

$$\lim_{h \to 0^-} \frac{f(2+h)-f(2)}{h}=\lim_{h \to 0^-} \frac{-(2+h)[(2+h)-2]-0}{h}=\lim_{h \to 0^-}[-(2+h)]=-$$

Thus, $\lim\limits_{h \to 0} \dfrac{f(2+h)-f(2)}{h}$ does not exist.

Therefore, $f(x)$ is not differentiable at $x=2$.

(2) $f(x)=\begin{cases} x^2 & (x \geqslant 1) \\ 2x-1 & (x < 1) \end{cases}$ 〔$x=1$〕



[Sol] $\lim\limits_{x \to 1^+} f(x)=\lim\limits_{x \to 1^+} x^2=1$

$\lim\limits_{x \to 1^-} f(x)=\lim\limits_{x \to 1^-} (2x-1)=1$

Also, since $f(1)=1$, $\lim\limits_{x \to 1} f(x)=f(1)$

Therefore, $f(x)$ is continuous at $x=1$.

$$\lim_{h \to 0^+} \frac{f(1+h)-f(1)}{h}=\lim_{h \to 0^+} \frac{(1+h)^2-1}{h}=\lim_{h \to 0^+}(2+h)=2$$

$$\lim_{h \to 0^-} \frac{f(1+h)-f(1)}{h}=\lim_{h \to 0^-} \frac{[2(1+h)-1]-1}{h}=\lim_{h \to 0^-} 2=2$$

$$\therefore \lim_{h \to 0} \frac{f(1+h)-f(1)}{h}=2$$

Thus, $f(x)$ is differentiable at $x=1$.

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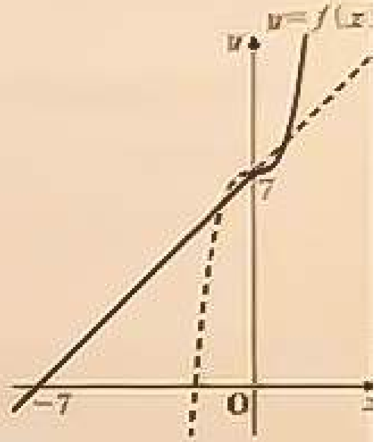
Examine if the following functions are continuous and also differentiable at the points indicated in the brackets [ ].

(1) $f(x)=\begin{cases} x^3+7 & (x \geqq 0) \\ x+7 & (x<0) \end{cases}$ $[x=0]$



[Sol] $\lim\limits_{x\to 0^+} f(x)=\lim\limits_{x\to 0^+}(x^3+7)=7$

$\lim\limits_{x\to 0^-} f(x)=\lim\limits_{x\to 0^-}(x+7)=7$

Also, since $f(0)=7$, $\lim\limits_{x\to 0} f(x)=f(0)$

Therefore, $f(x)$ is **continuous** at $x=0$.

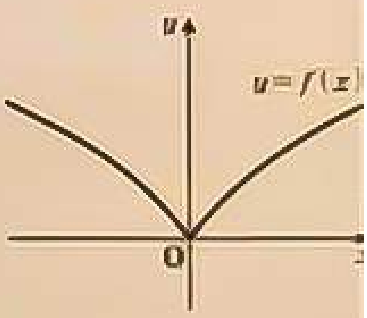
$$\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^+}\frac{[(0+h)^3+7]-7}{h}=\lim_{h\to 0^+}h^2=0$$

$$\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^-}\frac{[(0+h)+7]-7}{h}=\lim_{h\to 0^-}1=1$$

Thus, $\lim\limits_{h\to 0}\dfrac{f(0+h)-f(0)}{h}$ does not exist.

Therefore, $f(x)$ is **not differentiable** at $x=0$.

(2) $f(x)=\sqrt[3]{x^2}$ $[x=0]$



[Sol] $\lim\limits_{x\to 0^+} f(x)=\lim\limits_{x\to 0^+}\sqrt[3]{x^2}=0$

$\lim\limits_{x\to 0^-} f(x)=\lim\limits_{x\to 0^-}\sqrt[3]{x^2}=0$

Also, since $f(0)=0$, $\lim\limits_{x\to 0} f(x)=f(0)$

Therefore, $f(x)$ is **continuous** at $x=0$.

$$\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^+}\frac{\sqrt[3]{(0+h)^2}-0}{h}=\lim_{h\to 0^+}\frac{1}{\sqrt[3]{h}}=\infty$$

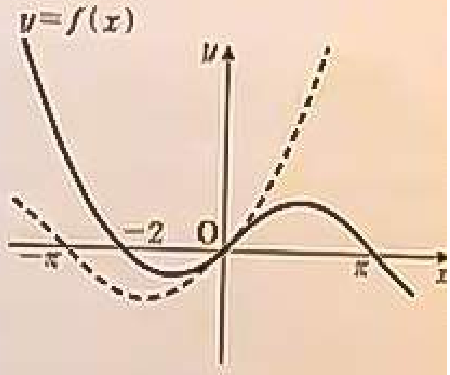
$$\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^-}\frac{\sqrt[3]{(0+h)^2}-0}{h}=\lim_{h\to 0^-}\frac{1}{\sqrt[3]{h}}=-\infty$$

Thus, $\lim\limits_{h\to 0}\dfrac{f(0+h)-f(0)}{h}$ does not exist.

Therefore, $f(x)$ is **not differentiable** at $x=0$.

N193b

(3) $f(x)=\begin{cases} \sin x & (x \geqq 0) \\ \dfrac{1}{2}x^2+x & (x < 0) \end{cases}$ 〔$x=0$〕



[Sol] $\lim\limits_{x\to 0^+} f(x)=\lim\limits_{x\to 0^+} \sin x=0$

$\lim\limits_{x\to 0^-} f(x)=\lim\limits_{x\to 0^-}\left(\dfrac{1}{2}x^2+x\right)=0$

Also, since $f(0)=0$, $\lim\limits_{x\to 0} f(x)=f(0)$

Therefore, $f(x)$ is **continuous** at $x=0$.

$\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1$

$$\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^+}\frac{\sin(0+h)-0}{h}=\lim_{h\to 0^+}\frac{\sin h}{h}=1$$

$$\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^-}\frac{\left[\frac{1}{2}(0+h)^2+(0+h)\right]-0}{h}=\lim_{h\to 0^-}\left(\frac{h}{2}+1\right)=1$$

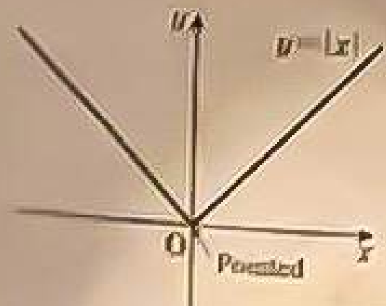
$$\therefore \lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=1$$

Thus, $f(x)$ is **differentiable** at $x=0$.

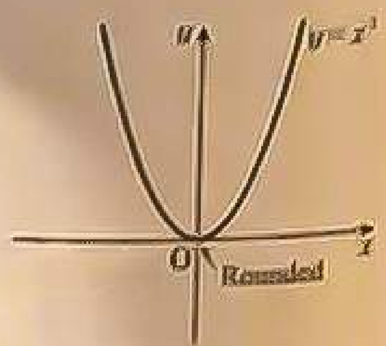
Note Summary

Relationship between continuity and differentiability is as shown below.
(Pay attention to the origin.)

Not continuous and not differentiable
(Not connected)



Continuous but not differentiable
(Connected but not smooth)



Continuous and differentiable
(Connected and smooth)

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Ex. The function $f(x)$ is defined as follows:

$$f(x)=\begin{cases} x^3+ax & (x \geqq 2) \\ bx^2-ax & (x<2) \end{cases}$$

Find the values of a and b such that $f(x)$ is differentiable at $x=2$.

[Sol] If the function $f(x)$ is differentiable at $x=2$,

$f(x)$ is continuous at $x=2$; therefore, $\lim\limits_{x\to 2} f(x)=f(2)$

$\therefore\ \lim\limits_{x\to 2-0} f(x)=\lim\limits_{x\to 2+0} f(x)=f(2)$

$\therefore\ b\cdot 2^2-a\cdot 2=2^3+a\cdot 2$ ←

Since $f(x)=x^3+ax$ when $x \geqq 2$, $\lim\limits_{x\to 2+0} f(x)=f(2)$ is obvious. Therefore, considering $\lim\limits_{x\to 2-0} f(x)=f(2)$

$\therefore\ b-a=2$ ···①

Also,

$$\lim_{h\to +0}\frac{f(2+h)-f(2)}{h}=\lim_{h\to +0}\frac{[(2+h)^3+a(2+h)]-(8+2a)}{h}$$

$$=\lim_{h\to +0}\frac{h^3+6h^2+(\boxed{12+a})h}{h}$$

$$=\lim_{h\to +0}(h^2+6h+\boxed{12+a})$$

$$=\boxed{12+a}$$

$$\lim_{h\to -0}\frac{f(2+h)-f(2)}{h}=\lim_{h\to -0}\frac{[b(2+h)^2-a(2+h)]-(8+2a)}{h}$$

$$=\lim_{h\to -0}\frac{bh^2+(4b-a)h+4(b-a-2)}{h}$$

From ①

$$=\lim_{h\to -0}(bh+4b-a)$$

$$=4b-a$$

$f(x)$ is differentiable at $x=2$ when $\boxed{12+a}=4b-a$ ···②

$\lim\limits_{h\to +0}\dfrac{f(a+h)-f(a)}{h}=\lim\limits_{h\to -0}\dfrac{f(a+h)-f(a)}{h}$

From ① and ②,

$a=\boxed{2}$, $b=\boxed{4}$

Answers 12+a, 12+a, 12+a, 12+a, 2, 4

N194b

1. The function $f(x)$ is defined as follows:

$$f(x)=\begin{cases} \dfrac{ax+b}{x+1} & (x>1) \\ x^2+1 & (x \leq 1) \end{cases}$$

Find the values of a and b such that $f(x)$ is differentiable at $x=1$.

[Sol] If the function $f(x)$ is differentiable at $x=1$,

$f(x)$ is continuous at $x=1$; therefore, $\lim_{x\to 1} f(x)=f(1)$

$$\therefore \lim_{x\to 1^+} f(x)=\lim_{x\to 1^-} f(x)=f(1)$$

$$\therefore \frac{a\cdot 1+b}{1+1}=1^2+1$$

Since $f(x)=x^2+1$ when $x \leq 1$, $\lim_{x\to 1^-} f(x)=f(1)$ is obvious. Therefore, considering $\lim_{x\to 1^+} f(x)=f(1)$

$$\therefore a+b=4 \quad \cdots①$$

Also,

$$\lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^+}\frac{\dfrac{a(1+h)+b}{(1+h)+1}-2}{h}$$

$$=\lim_{h\to 0^+}\frac{a+b-4+(a-2)h}{h(2+h)}$$

From ①

$$=\lim_{h\to 0^+}\frac{a-2}{2+h}$$

$$=\frac{a-2}{2}$$

$$\lim_{h\to 0^-}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^-}\frac{[(1+h)^2+1]-2}{h}$$

$$=\lim_{h\to 0^-}\frac{2h+h^2}{h}$$

$$=\lim_{h\to 0^-}(2+h)$$

$$=2$$

$f(x)$ is differentiable at $x=1$ when $\dfrac{a-2}{2}=2 \quad \cdots②$

From ① and ②,

$a=6,\ b=-2$

Various Properties of Derivatives

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1. Given that the function $f(x)$ is differentiable at $x=a$, express the following limit values in terms of $f'(a)$.

Ex.

$$\lim_{h\to 0}\frac{f(a+3h)-f(a+h)}{h}$$
$$=\lim_{h\to 0}\frac{f(a+3h)\boxed{-f(a)+f(a)}-f(a+h)}{h}$$
$$=\lim_{h\to 0}\left[\frac{f(a+3h)-f(a)}{3h}\cdot 3-\frac{f(a+h)-f(a)}{h}\right]$$
$$=3f'(a)-f'(a)$$
$$=2f'(a)$$

Rearranging into the form $\lim_{h\to 0}\frac{f(a+\bullet h)-f(a)}{\bullet h}$

(1) $\lim_{h\to 0}\frac{f(a+3h)-f(a+2h)}{h}$

$$=\lim_{h\to 0}\frac{f(a+3h)-f(a)+f(a)-f(a+2h)}{h}$$
$$=\lim_{h\to 0}\left[\frac{f(a+3h)-f(a)}{3h}\cdot 3-\frac{f(a+2h)-f(a)}{2h}\cdot 2\right]$$
$$=3f'(a)-2f'(a)$$
$$=f'(a)$$

(2) $\lim_{h\to 0}\frac{f(a+2h)-f(a-h)}{h}$

$$=\lim_{h\to 0}\frac{f(a+2h)-f(a)+f(a)-f(a-h)}{h}$$
$$=\lim_{h\to 0}\left[\frac{f(a+2h)-f(a)}{2h}\cdot 2-\frac{f(a-h)-f(a)}{-h}\cdot(-1)\right]$$
$$=2f'(a)+f'(a)$$
$$=3f'(a)$$

N195b

2. Given that the function $f(x)$ is differentiable at $x=a$, express the followin limit values in terms of a, $f(a)$ and $f'(a)$.

Ex.

$$\lim_{x\to a}\frac{af(x)-xf(a)}{x-a}$$
$$=\lim_{x\to a}\frac{af(x)-af(a)+af(a)-xf(a)}{x-a}$$
$$=\lim_{x\to a}\left\{\frac{a[f(x)-f(a)]}{x-a}-\frac{(x-a)f(a)}{x-a}\right\}$$
$$=af'(a)-f(a)$$

Rearranging into the form $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$

(1) $$\lim_{x\to a}\frac{x^2f(x)-a^2f(a)}{x-a}$$
$$=\lim_{x\to a}\frac{x^2f(x)-x^2f(a)+x^2f(a)-a^2f(a)}{x-a}$$
$$=\lim_{x\to a}\left\{\frac{x^2[f(x)-f(a)]}{x-a}+\frac{(x+a)(x-a)f(a)}{x-a}\right\}$$
$$=\lim_{x\to a}\left\{\frac{x^2[f(x)-f(a)]}{x-a}+(x+a)f(a)\right\}$$
$$=a^2f'(a)+2af(a)\quad [=a[af'(a)+2f(a)]]$$

(2) $$\lim_{x\to a}\frac{[af(x)]^2-[xf(a)]^2}{x-a}$$
$$=\lim_{x\to a}\frac{[af(x)]^2-[af(a)]^2+[af(a)]^2-[xf(a)]^2}{x-a}$$
$$=\lim_{x\to a}\left\{\frac{a^2[f(x)+f(a)][f(x)-f(a)]}{x-a}-\frac{(x+a)(x-a)[f(a)]^2}{x-a}\right\}$$
$$=\lim_{x\to a}\left\{\frac{a^2[f(x)+f(a)][f(x)-f(a)]}{x-a}-(x+a)[f(a)]^2\right\}$$
$$=2a^2f(a)f'(a)-2a[f(a)]^2\quad [=2af(a)[af'(a)-f(a)]]$$

Alternative Solution

$$\lim_{x\to a}\frac{[af(x)]^2-[xf(a)]^2}{x-a}=\lim_{x\to a}\frac{[af(x)+xf(a)][af(x)-xf(a)]}{x-a}$$

$\lim_{x\to a}[af(x)+xf(a)]=2af(a)$, $\lim_{x\to a}\frac{af(x)-xf(a)}{x-a}=af'(a)-f(a)$ ← From Ex.

$$\therefore \lim_{x\to a}\frac{[af(x)]^2-[xf(a)]^2}{x-a}=2af(a)[af'(a)-f(a)]$$
$$[=2a^2f(a)f'(a)-2a[f(a)]^2]$$

N196a KUMON N19

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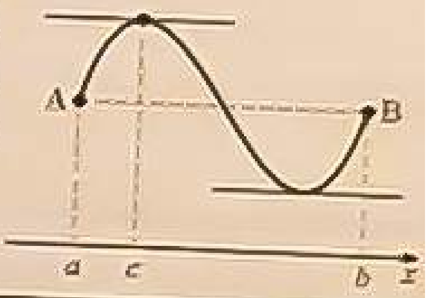
Rolle's Theorem

If the function $f(x)$ is continuous on the closed interval $[a, b]$, differentiable on the open interval (a, b) and $f(a)=f(b)$, then there exists at least one value c such that $f'(c)=0$ and $a<c<b$.



This theorem states that, if $f(a)=f(b)$, then there exists at least on[e] point between A and B on the curve at which the slope of the tangent is 0, i.e. $f'(c)=0$, as shown in the diagram above.

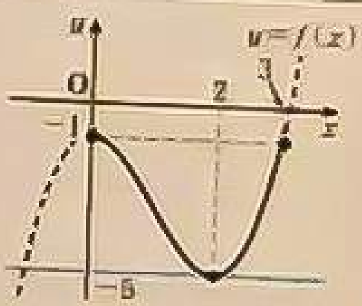
Find the value of c that satisfies Rolle's Theorem for each given function an[d] closed interval.

Ex. $f(x)=x^3-3x^2-1 \quad [0, 3]$

[Sol] $f'(x)=3x^2-6x=3x(x-2)$

$f'(c)=3c(c-2)=0$

Since $0<c<3$, $c=2$ ← $c=0$ is not appropriate.

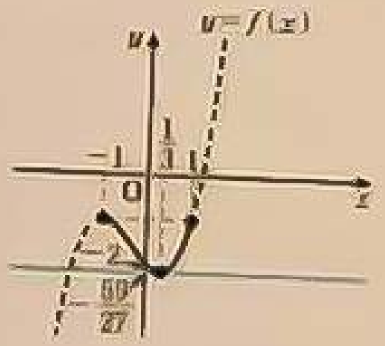


(1) $f(x)=x^3+x^2-x-2 \quad [-1, 1]$

[Sol] $f'(x)=3x^2+2x-1=(x+1)(3x-1)$

$f'(c)=(c+1)(3c-1)=0$

Since $-1<c<1$, $c=\frac{1}{3}$

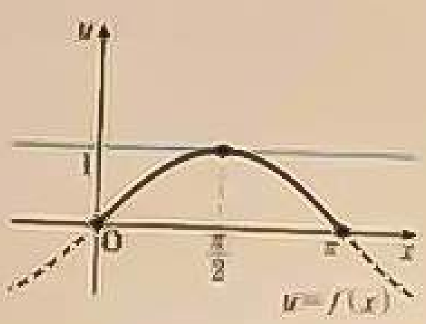


(2) $f(x)=\sin x \quad [0, \pi]$

[Sol] $f'(x)=\cos x$

$f'(c)=\cos c=0$

Since $0<c<\pi$, $c=\frac{\pi}{2}$



N196b

Using Rolle's Theorem, prove the following statement.

If the function $f(x)$ is continuous on the closed interval $[a, b]$ and differentiab on the open interval (a, b), then there exists at least one value c such that $\dfrac{f(b)-f(a)}{b-a}=f'(c)$ and $a<c<b$.

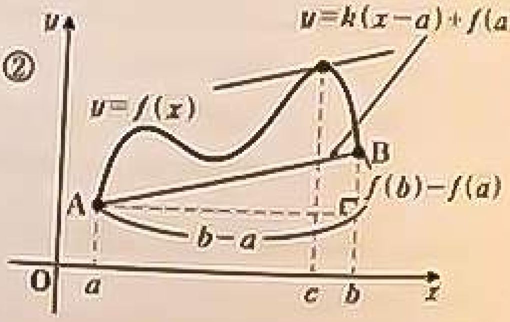
[Sol] Let $\dfrac{f(b)-f(a)}{b-a}=k$. Then show that $f'(c)=k$.

Rearranging $\dfrac{f(b)-f(a)}{b-a}=k$, $f(b)-f(a)-k(b-a)=0$ ···①

Consider the function

$F(x)=f(x)-f(a)-k(x-a)$ ···②

which is derived by changing b of the LHS of ① to x.



From ②, $F(a)=\boxed{0}$

From ①, $F(b)=\boxed{0}$

$\therefore F(a)=F(b)$

Also, differentiating both sides of ② with respect to x,

$F'(x)=f'(x)-\boxed{k}$ ···③

Therefore, since the function $F(x)$ satisfies the conditions of Rolle's Theorem, there exists at least one value c such that $F'(c)=0$ and $a<c<b$.

From ③, $F'(c)=f'(c)-\boxed{k}$; therefore, $f'(c)=\boxed{k}$.

Thus, there exists at least one value c such that $\dfrac{f(b)-f(a)}{b-a}=f'(c)$ and $a<c<b$.

Answers: 0, 0, k, k, k

KUMON

Various Properties of Derivatives

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Mean Value Theorem

If the function $f(x)$ is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists at least one value c such that $\dfrac{f(b)-f(a)}{b-a}=f'(c)$ and $a<c<b$.

Find the value of c that satisfies the Mean Value Theorem for each given function and closed interval.

Ex. $f(x)=x^3-3x^2+2x \quad [0, 3]$

[Sol] $\dfrac{f(3)-f(0)}{3-0}=\dfrac{6-0}{3}=2$ ← $a=0,\ b=3$

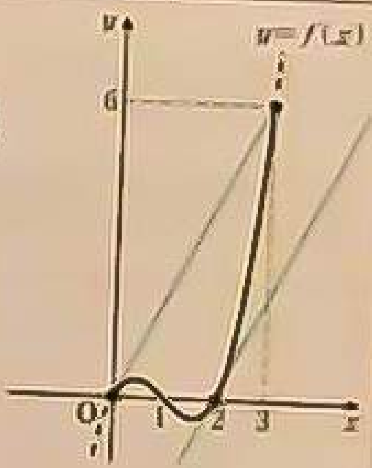
$f'(x)=3x^2-6x+2$

From the Mean Value Theorem, there exists at least one value c such that

$2=3c^2-6c+2$ and $0<c<3$.

$3c(c-2)=0$

Since $0<c<3$, $c=2$ ← $c=0$ is not appropriate.



(1) $f(x)=\dfrac{1}{x} \quad [1, 2]$

[Sol] $\dfrac{f(2)-f(1)}{2-1}=\dfrac{1}{2}-1=-\dfrac{1}{2}$

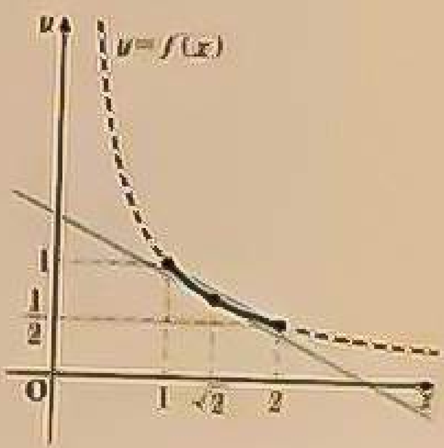
$f'(x)=-\dfrac{1}{x^2}$

From the Mean Value Theorem, there exists at least one value c such that

$-\dfrac{1}{2}=-\dfrac{1}{c^2}$ and $1<c<2$.

$c^2=2$

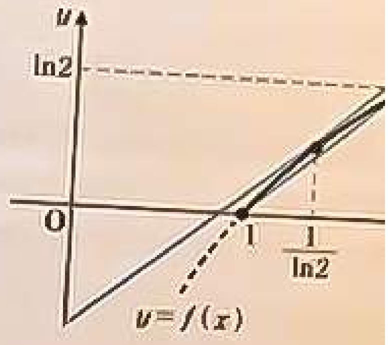
Since $1<c<2$, $c=\sqrt{2}$



N197b

(2) $f(x)=\ln x$ $[1, 2]$ $(\ln 2 \approx 0.69)$

[Sol] $\dfrac{f(2)-f(1)}{2-1}=\ln 2-\ln 1=\ln 2$



$f'(x)=\dfrac{1}{x}$

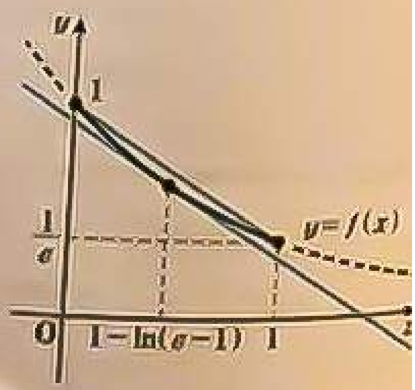
From the Mean Value Theorem, there exists at least one value c such that

$\ln 2=\dfrac{1}{c}$ and $1<c<2$. ← $\ln 2 \approx 0.69$

$\therefore\ c=\dfrac{1}{\ln 2}$ ← $1<c<2$ is satisfied.

(3) $f(x)=e^{-x}$ $[0, 1]$

[Sol] $\dfrac{f(1)-f(0)}{1-0}=\dfrac{1}{e}-1$



$f'(x)=-e^{-x}$

From the Mean Value Theorem, there exists at least one value c such that

$\dfrac{1}{e}-1=-e^{-c}$ and $0<c<1$.

$e^{-c}=\dfrac{e-1}{e}$

$-c=\ln\dfrac{e-1}{e}$ ← When $a^y=x$, $y=\log_a x$

$=\ln(e-1)-1$ ← $\ln\dfrac{M}{N}=\ln M-\ln N$, $\ln e=1$

$\therefore\ c=1-\ln(e-1)$ ← Since $0<\ln(e-1)<1$, $0<c<1$ is satisfied.

N198a

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Various Properties of Derivatives

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Using the Mean Value Theorem, prove the following inequalities.

Ex.

When $0 < a < b$, $\frac{1}{b} < \frac{\ln b - \ln a}{b-a} < \frac{1}{a}$

[Sol] Let $f(x) = \ln x$.

$f(x)$ is differentiable at $x > 0$ and $f'(x) = \frac{1}{x}$.

Using the Mean Value Theorem on the interval $[a, b]$, there exists at least one value c such that $\frac{\ln b - \ln a}{b-a} = \frac{1}{c}$ and $a < c < b$.

$\frac{f(b)-f(a)}{b-a} = f'(c), a < c < b$

Since $0 < a < c < b$, $\frac{1}{b} < \frac{1}{c} < \frac{1}{a}$

When $0 < a < b < c$, $\frac{1}{a} > \frac{1}{b} > \frac{1}{c}$

$\therefore \frac{1}{b} < \frac{\ln b - \ln a}{b-a} < \frac{1}{a}$

(1) When $a > 0$, $\frac{1}{a+1} < \ln(a+1) - \ln a < \frac{1}{a}$

[Sol] Let $f(x) = \ln x$.

$f(x)$ is differentiable at $x > 0$ and $f'(x) = \frac{1}{x}$.

Using the Mean Value Theorem on the interval $[a, a+1]$, there exists at least one value c such that $\frac{\ln(a+1) - \ln a}{(a+1)-a} = \frac{1}{c}$ and $a < c < a+1$.

Since $0 < a < c < a+1$, $\frac{1}{a+1} < \frac{1}{c} < \frac{1}{a}$

$\therefore \frac{1}{a+1} < \ln(a+1) - \ln a < \frac{1}{a}$

N198b

(2) When $0 \leqslant a < b$ and $n \geqslant 2$, $b^n - a^n < nb^{n-1}(b-a)$

[Sol] Let $f(x) = x^n$.

$f(x)$ is differentiable for all real numbers x and $f'(x) = nx^{n-1}$.

Using the Mean Value Theorem on the interval $[a, b]$, there exists at least one value c such that $\dfrac{b^n - a^n}{b-a} = nc^{n-1}$ and $a < c < b$.

Since $0 \leqslant a < c < b$ and $n \geqslant 2$, $nc^{n-1} < nb^{n-1}$

$\therefore \dfrac{b^n - a^n}{b-a} < nb^{n-1}$

$\therefore b^n - a^n < nb^{n-1}(b-a)$

(3) When $0 < \alpha < \beta < \dfrac{\pi}{2}$, $\sin\beta - \sin\alpha < \beta - \alpha$

[Sol] Let $f(x) = \sin x$.

$f(x)$ is differentiable for all real numbers x and $f'(x) = \cos x$.

Using the Mean Value Theorem on the interval $[\alpha, \beta]$, there exists at least one value c such that $\dfrac{\sin\beta - \sin\alpha}{\beta - \alpha} = \cos c$ and $\alpha < c < \beta$.

Since $0 < \alpha < c < \beta < \dfrac{\pi}{2}$, $\cos c < 1$ ←

When $0 < \theta < \dfrac{\pi}{2}$,

$1 = \cos 0 > \cos\theta > \cos\dfrac{\pi}{2} = 0$

$\therefore \dfrac{\sin\beta - \sin\alpha}{\beta - \alpha} < 1$

$\therefore \sin\beta - \sin\alpha < \beta - \alpha$

Various Properties of Derivatives

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Name

Date / /

Time : to :

1. Given that $f(x)$ is a differentiable function that satisfies $f(-x)=f(x)+2x$, $f'(1)=1$ and $f(1)=0$, solve the following questions.

(1) Find the value of $f'(-1)$.

[Sol] Differentiating both sides of $f(-x)=f(x)+2x$ with respect to x,

$$-f'(-x)=f'(x)+2 \quad \cdots①$$

$\boxed{\{f(-x)\}' = f'(-x)\cdot(-x)'}$

Substituting $x=1$ into ①,

$$-f'(-1)=f'(1)+2$$

$$=3$$

$\boxed{f'(1)=1}$

$$\therefore\ f'(-1)=-3$$

(2) Find the value of $\lim_{x\to1}\dfrac{f(x)+f(-x)-2}{x-1}$.

[Sol] $\lim_{x\to1}\dfrac{f(x)+f(-x)-2}{x-1}=\lim_{x\to1}\dfrac{f(x)+\{f(x)+2x\}-2}{x-1}$ ← $\boxed{f(-x)=f(x)+2x}$

$$=\lim_{x\to1}\frac{2f(x)+2(x-1)}{x-1}$$

$$=\lim_{x\to1}\left\{\frac{2\{f(x)-f(1)\}}{x-1}+2\right\}$$

$\boxed{f(1)=0}$

$$=2f'(1)+2$$

$$=4$$

$\boxed{f'(a)=\lim_{x\to a}\dfrac{f(x)-f(a)}{x-a}}$

Alternative Solution

Let $F(x)=f(x)+f(-x)-2$.

$F(1)=f(1)+f(-1)-2=0$ ← $\boxed{\text{Substituting } x=1 \text{ into } f(-x)=f(x)+2x,\ f(-1)=f(1)+2\cdot1=2}$

$F'(x)=f'(x)-f'(-x)$

$$\therefore\ \lim_{x\to1}\frac{f(x)+f(-x)-2}{x-1}=\lim_{x\to1}\frac{F(x)-F(1)}{x-1}=F'(1)=f'(1)-f'(-1)=4$$

N199b

2. Using the Mean Value Theorem, find $\lim_{x\to 0}\frac{\sin x-\sin x^2}{x-x^2}$.

[Sol] Let $f(x)=\sin x$.

$f(x)$ is differentiable for all real numbers x and $f'(x)=\cos x$.

(i) When $x<0$, $x<x^2$; therefore, using the Mean Value Theorem on the interval $[x, x^2]$, there exists at least one value c_1 such that $\frac{\sin x^2-\sin x}{x^2-x}=\cos c_1$ and $x<c_1<x^2$.

$\boxed{x^2>0}$

As $x\to 0^-$, $x^2\to 0$; therefore, $c_1\to 0$

$\boxed{\text{Limits of Functions and Their Relationships (N121)}}$

$$\therefore \lim_{x\to 0^-}\frac{\sin x-\sin x^2}{x-x^2}=\lim_{x\to 0^-}\frac{\sin x^2-\sin x}{x^2-x}$$
$$=\lim_{c_1\to 0}\cos c_1$$
$$=1$$

$\boxed{\text{Since } x\to 0^+\text{, consider } 0<x<1.}$

(ii) When $x>0$, consider $0<x<1$. Then, $0<x^2<x$; therefore, using the Mean Value Theorem on the interval $[x^2, x]$, there exists at least one value c_2 such that $\frac{\sin x-\sin x^2}{x-x^2}=\cos c_2$ and $x^2<c_2<x$.

As $x\to 0^+$, $x^2\to 0$; therefore, $c_2\to 0$

$\boxed{\text{Limits of Functions and Their Relationships (N121)}}$

$$\therefore \lim_{x\to 0^+}\frac{\sin x-\sin x^2}{x-x^2}=\lim_{c_2\to 0}\cos c_2$$
$$=1$$

From (i) and (ii), $\lim_{x\to 0}\frac{\sin x-\sin x^2}{x-x^2}=1$

$\boxed{\lim_{x\to 0^-}\frac{\sin x-\sin x^2}{x-x^2}=\lim_{x\to 0^+}\frac{\sin x-\sin x^2}{x-x^2}}$

N200a

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N 200

Various Properties of Derivatives

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Name

Date / /

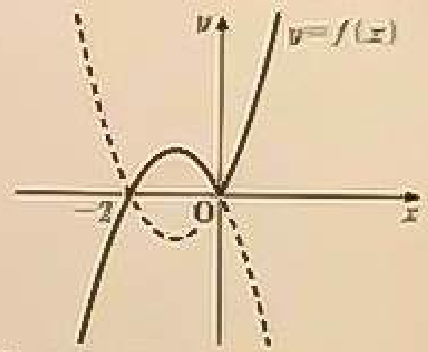
Time : to :

1. Examine if the following function is continuous and also differentiable at $x=0$.

$f(x)=|x|(x+2)$ ➡ N192

[Sol] When $x \geqq 0$, $f(x)=x(x+2)$

When $x<0$, $f(x)=-x(x+2)$



$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+} x(x+2)=0$

$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}[-x(x+2)]=0$

Also, since $f(0)=0$, $\lim_{x\to 0} f(x)=f(0)$

Therefore, $f(x)$ is **continuous** at $x=0$.

$$\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^+}\frac{(0+h)[(0+h)+2]-0}{h}=\lim_{h\to 0^+}(h+2)=2$$

$$\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^-}\frac{-(0+h)[(0+h)+2]-0}{h}=\lim_{h\to 0^-}[-(h+2)]=-2$$

Thus, $\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$ does not exist.

Therefore, $f(x)$ is **not differentiable** at $x=0$.

2. Given that the function $f(x)$ is differentiable at $x=a$, express the following limit value in terms of $f'(a)$. ➡ N195

$$\lim_{h\to 0}\frac{f(a+2h)-f(a+h)}{h}$$

$$=\lim_{h\to 0}\frac{f(a+2h)-f(a)+f(a)-f(a+h)}{h}$$

$$=\lim_{h\to 0}\left[\frac{f(a+2h)-f(a)}{2h}\cdot 2-\frac{f(a+h)-f(a)}{h}\right]$$

$$=2f'(a)-f'(a)$$

$$=f'(a)$$

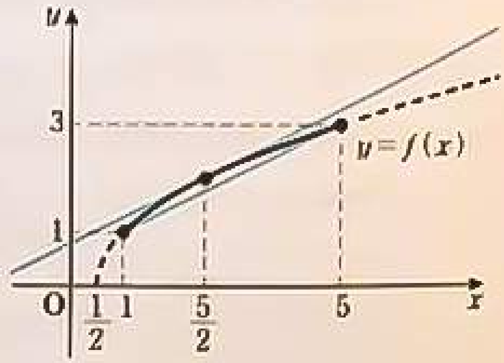
N200b

3. Find the value of c that satisfies the Mean Value Theorem for the following function and closed interval. ➡ N197

$f(x)=\sqrt{2x-1} \quad [1, 5]$

[Sol] $\dfrac{f(5)-f(1)}{5-1}=\dfrac{3-1}{4}=\dfrac{1}{2}$

$f'(x)=\dfrac{1}{2}(2x-1)^{-\frac{1}{2}}\cdot 2=\dfrac{1}{\sqrt{2x-1}}$



From the Mean Value Theorem, there exists at least one value c such that

$\dfrac{1}{2}=\dfrac{1}{\sqrt{2c-1}}$ and $1 < c < 5$.

$2c-1=4$

$\therefore\ c=\dfrac{5}{2}$

4. Using the Mean Value Theorem, prove the following inequality.

When $a < b$, $e^a < \dfrac{e^b-e^a}{b-a} < e^b$ ➡ N198

[Sol] Let $f(x)=e^x$.

$f(x)$ is differentiable for all real numbers x and $f'(x)=e^x$.

Using the Mean Value Theorem on the interval $[a, b]$, there exists at least one value c such that $\dfrac{e^b-e^a}{b-a}=e^c$ and $a < c < b$.

Since $a < c < b$, $e^a < e^c < e^b$

$\therefore\ e^a < \dfrac{e^b-e^a}{b-a} < e^b$