

Mathematics

Solution Book Part 1

N

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Standard Completion Times for Level N

Number	Title	SCT	Number	Title	SCT
1 – 10	Arithmetic Sequences	10 – 20	101 – 110	Limits of Functions I	15 – 30
11 – 20	Geometric Sequences	10 – 20	111 – 120	Limits of Functions II	15 – 30
21 – 30	Various Sequences I	15 – 30	121 – 130	Limits of Trigonometric Functions	15 – 30
31 – 40	Various Sequences II	15 – 30	131 – 140	Continuous and Discontinuous Functions	20 – 40
41 – 50	Recurrence Relations	20 – 40	141 – 150	Differentiation I	15 – 30
51 – 60	Mathematical Induction	20 – 40	151 – 160	Differentiation II	15 – 30
61 – 70	Infinite Sequences	10 – 20	161 – 170	Differentiation of Trigonometric Functions	15 – 30
71 – 80	Infinite Geometric Sequences	15 – 30	171 – 180	Differentiation of Logarithmic and Exponential Functions	15 – 30
81 – 90	Infinite Geometric Series	15 – 30	181 – 190	Differentiation of Various Functions and Higher Order Derivatives	20 – 40
91 – 100	Infinite Series	15 – 30	191 – 200	Various Properties of Derivatives	25 – 50

Notes for Level N Worksheets

- Have students utilise the Solution Books as described below:
 - When students can solve the problems without referring to the Solution Books, have them confirm their answers with the Solution Books.
 - In case students cannot identify the steps to solve the problems by themselves, have them read through the Solution Books once and keep it closed while re-working on the problems. It is important not to complete their study just by copying the Solution Books.
 - When doing repetitions, have students work on the problems by themselves without using the Solution Books.
- In N61-140, where students learn limits of sequences and functions, have them check the answers by substituting the values or drawing the graphs when they are unsure if their answers are correct.
- Students study the basics of differentiation in N141-180. It is important for them to develop the solid skills for these basics as they work through the problems.

A succession of numbers which are arranged in a specific order according to a rule is called a **sequence**. Each number in a sequence is called a **term** and the n^{th} number from the beginning is called the n^{th} **term**. (For example, the initial term is called the 1st **term**.)

1. Fill in the blanks with the appropriate number for each given sequence.

(1) 2, 5, 8, 11, 14, 17, ...

(2) 2, 4, 8, 16, 32, 64, ...

(3) $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, $\frac{5}{6}$, $\frac{6}{7}$, ...

(4) 4, 2, 0, -2, -4, -6, ...

(5) 1, 4, 9, 16, 25, 36, ...

2. Find the terms for each given sequence where the n^{th} term is expressed as follows by substituting $n=1, 2, 3, \dots, 6$.

	1 st term	2 nd term	3 rd term	4 th term	5 th term	6 th term
$3n-1$	2	5	8	11	14	17
(1) 2^n	2	4	8	16	32	64
(2) $\frac{n}{n+1}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{5}{6}$	$\frac{6}{7}$
(3) $6-2n$	4	2	0	-2	-4	-6
(4) n^2	1	4	9	16	25	36
(5) $(-1)^n$	-1	1	-1	1	-1	1

Generally, a sequence is expressed as follows:

$$\begin{array}{ccccccc} a_1, & a_2, & a_3, & \dots, & a_n, & \dots \\ \uparrow & \uparrow & \uparrow & & \uparrow & \\ 1^{\text{st}} \text{ term} & 2^{\text{nd}} \text{ term} & 3^{\text{rd}} \text{ term} & \dots & n^{\text{th}} \text{ term} & \dots \end{array}$$

This sequence is also expressed as $\{a_n\}$.

When the n^{th} term a_n of a sequence $\{a_n\}$ is expressed in terms of n , it is called the **general term** of $\{a_n\}$.

3. Choose the general term of each given sequence $\{a_n\}$ from (A)~(H) and find the 10th term.

Ex. 3, 5, 7, 9, 11, ...

[Sol] a_n : (A) $a_{10} = 21$

(1) 7, 4, 1, -2, -5, ...

[Sol] a_n : (E) $a_{10} = -20$

(2) 2, 4, 8, 16, 32, ...

[Sol] a_n : (C) $a_{10} = 1024$

(3) 1, -1, 1, -1, 1, ...

[Sol] a_n : (F) $a_{10} = -1$

(4) $\frac{1}{4}, \frac{2}{5}, \frac{3}{6}, \frac{4}{7}, \frac{5}{8}, \dots$

[Sol] a_n : (H) $a_{10} = \frac{10}{13}$

(A) $2n+1$

(B) $-2n+9$

(C) 2^n

(D) $(-1)^n$

(E) $-3n+10$

(F) $-(-1)^n$

(G) $\frac{n}{2n+2}$

(H) $\frac{n}{n+3}$

A sequence whose terms are found by successively adding a fixed number d to the 1st term a is called an **arithmetic sequence** (or **arithmetic progression**). d is called the **common difference** of the arithmetic sequence.

1. Find the 10th term of the following arithmetic sequence $\{a_n\}$.
10, 13, 16, 19, 22, ...

[Sol] $a_1 = 10$

$$a_2 = 10 + 1 \cdot 3 = 13$$

$$a_3 = 10 + 2 \cdot 3 = 16$$

$$a_4 = 10 + 3 \cdot 3 = 19$$

$\vdots$

$$a_{10} = 10 + 9 \cdot 3 = 37$$

2. Find the general term of the following arithmetic sequence $\{a_n\}$.
8, 4, 0, -4, -8, ...

[Sol] $a_1 = 8$

$$a_2 = 8 + 1 \cdot (-4) = 4$$

$$a_3 = 8 + 2 \cdot (-4) = 0$$

$$a_4 = 8 + 3 \cdot (-4) = -4$$

$\vdots$

$$a_{n-1} = 8 + (n-2) \cdot (-4) = -4n + 16$$

$$a_n = 8 + (n-1) \cdot (-4) = -4n + 12$$

3. Find the general term of the following arithmetic sequence $\{a_n\}$ with 1st term a and common difference d .

[Sol] $a_1 = a$

$$a_2 = a + 1d$$

$$a_3 = a + 2d$$

$$a_4 = a + 3d$$

$\vdots$

$$a_n = a + (n-1)d$$

$$\begin{array}{c} a_1, a_2, a_3, \dots, a_{n-1}, a_n \\ \underbrace{+d +d +d \dots +d +d}_{\text{The number of } +d \text{ is } (n-1).} \end{array}$$

N2b

General Term of an Arithmetic Sequence

The general term of an arithmetic sequence $\{a_n\}$ with 1st term a and common difference d is

$$a_n = a + (n-1)d$$

Ex. Find the 1st term a and the general term of the arithmetic sequence $\{a_n\}$ whose common difference is -2 and 6th term is 8.

[Sol] $a_6 = a + 5 \cdot (-2) = 8$

$$\therefore a = 18$$

Also, $a_n = 18 + (n-1) \cdot (-2) = -2n + 20$

4. Find the 1st term a and the general term of the arithmetic sequence $\{a_n\}$ whose common difference is -3 and 11th term is -27 .

[Sol] $a_{11} = a + 10 \cdot (-3) = -27$

$$\therefore a = 3$$

Also, $a_n = 3 + (n-1) \cdot (-3) = -3n + 6$

5. Find the common difference d and the general term of the arithmetic sequence $\{a_n\}$ whose 1st term is 20 and 15th term is 90.

[Sol] $a_{15} = 20 + 14d = 90$

$$\therefore d = 5$$

Also, $a_n = 20 + (n-1) \cdot 5 = 5n + 15$

6. Given the arithmetic sequence $\{a_n\}$ whose 1st term is -25 and common difference is 5, find the value of n for which the n^{th} term is -5 .

[Sol] $a_n = -25 + (n-1) \cdot 5 = 5n - 30$

Since $a_n = -5$,

$$5n - 30 = -5$$

$$\therefore n = 5$$

Arithmetic Sequences

Ex. Find the general term of the arithmetic sequence $\{a_n\}$ whose 5th term is -5 and 9th term is 11 .

[Sol] Let a be the 1st term and d be the common difference.

$$\begin{cases} a_5 = a + 4d = -5 & \dots \textcircled{1} \\ a_9 = a + 8d = 11 & \dots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}$,

$$a = -21, d = 4$$

$$\therefore a_n = -21 + (n-1) \cdot 4 = 4n - 25$$

1. Find the general term of the arithmetic sequence $\{a_n\}$ whose 5th term is -3 and 14th term is 24 .

[Sol] Let a be the 1st term and d be the common difference.

$$\begin{cases} a_5 = a + 4d = -3 & \dots \textcircled{1} \\ a_{14} = a + 13d = 24 & \dots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}$,

$$a = -15, d = 3$$

$$\therefore a_n = -15 + (n-1) \cdot 3 = 3n - 18$$

2. Find the general term of the arithmetic sequence $\{a_n\}$ whose 3rd term is 7 and 8th term is -8 .

[Sol] Let a be the 1st term and d be the common difference.

$$\begin{cases} a_3 = a + 2d = 7 & \dots \textcircled{1} \\ a_8 = a + 7d = -8 & \dots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}$,

$$a = 13, d = -3$$

$$\therefore a_n = 13 + (n-1) \cdot (-3) = -3n + 16$$

N3b

3. Find the 55th term of the arithmetic sequence $\{a_n\}$ whose 15th term is 77 and 42nd term is 239.

[Sol] Let a be the 1st term and d be the common difference.

$$\begin{cases} a_{15} = a + 14d = 77 & \dots \textcircled{1} \\ a_{42} = a + 41d = 239 & \dots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}$,

$$a = -7, d = 6$$

$$\therefore a_{55} = -7 + 54 \cdot 6 = 317$$

4. Given the arithmetic sequence $\{a_n\}$ whose 53rd term is -47 and 77th term is -95 , find the value of n for which the n^{th} term is -111 .

[Sol] Let a be the 1st term and d be the common difference.

$$\begin{cases} a_{53} = a + 52d = -47 & \dots \textcircled{1} \\ a_{77} = a + 76d = -95 & \dots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}$,

$$a = 57, d = -2$$

$$\therefore a_n = 57 + (n-1) \cdot (-2) = -2n + 59$$

Since $a_n = -111$,

$$-2n + 59 = -111$$

$$\therefore n = 85$$

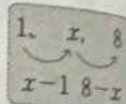
Arithmetic Sequences

Ex. Find the value of x when the sequence $1, x, 8, \dots$ is an arithmetic sequence.

[Sol] Since the sequence $1, x, 8$ is an arithmetic sequence,

$$x - 1 = 8 - x$$

$$\therefore x = \frac{9}{2}$$



1. Find the value of x when the sequence $\frac{5}{6}, x, \frac{5}{2}, \dots$ is an arithmetic sequence.

[Sol] Since the sequence $\frac{5}{6}, x, \frac{5}{2}$ is an arithmetic sequence,

$$x - \frac{5}{6} = \frac{5}{2} - x$$

$$\therefore x = \frac{5}{3}$$

2. Find the value of x when the sequence $x, 6, 2x, \dots$ is an arithmetic sequence.

[Sol] Since the sequence $x, 6, 2x$ is an arithmetic sequence,

$$6 - x = 2x - 6$$

$$\therefore x = 4$$

3. Find the values of x and y when the sequence $1, x, y, -11, \dots$ is an arithmetic sequence.

[Sol] Since the sequence $1, x, y, -11$ is an arithmetic sequence,

$$\begin{cases} x - 1 = y - x & \dots \textcircled{1} \end{cases}$$

$$\begin{cases} y - x = -11 - y & \dots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}$,

$$x = -3, y = -7$$

Ex. Prove that the sequence $\{a_n\}$ where $a_n = 5n + 1$ is an arithmetic sequence. Then, find the 1st term and the common difference.

[Sol] Since $a_n = 5n + 1$, $a_{n+1} = 5(n+1) + 1 = 5n + 6$

$$\therefore a_{n+1} - a_n = (5n + 6) - (5n + 1) = 5$$

Since $a_{n+1} - a_n$ is fixed as 5,

$\{a_n\}$ is an arithmetic sequence.

Also, $a_1 = 5 \cdot 1 + 1 = 6$

Therefore, the 1st term is 6 and the common difference is 5.

If $a_{n+1} - a_n = d$ (fixed), then it is an arithmetic sequence.

4. Prove that the sequence $\{a_n\}$ where $a_n = 4n + 3$ is an arithmetic sequence. Then, find the 1st term and the common difference.

[Sol] Since $a_n = 4n + 3$, $a_{n+1} = 4(n+1) + 3 = 4n + 7$

$$\therefore a_{n+1} - a_n = (4n + 7) - (4n + 3) = 4$$

Since $a_{n+1} - a_n$ is fixed as 4, $\{a_n\}$ is an arithmetic sequence.

Also, $a_1 = 4 \cdot 1 + 3 = 7$

Therefore, the 1st term is 7 and the common difference is 4.

5. Given that p and q are constants, prove that the sequence $\{a_n\}$ where $a_n = pn + q$ is an arithmetic sequence. Then, find the 1st term and the common difference.

[Sol] Since $a_n = pn + q$, $a_{n+1} = p(n+1) + q = pn + p + q$

$$\therefore a_{n+1} - a_n = (pn + p + q) - (pn + q) = p$$

Since $a_{n+1} - a_n$ is fixed as p , $\{a_n\}$ is an arithmetic sequence.

Also, $a_1 = p \cdot 1 + q = p + q$

Therefore, the 1st term is $p + q$ and the common difference is p .

Arithmetic Sequences

Ex. Find the three numbers whose sum is 15 and product is 80, which form an arithmetic sequence.

[Sol] Let c be the middle term and d be the common difference.

Since the three numbers are expressed as $c-d$, c , $c+d$,

$$\begin{cases} (c-d) + c + (c+d) = 15 & \dots \textcircled{1} \\ (c-d)c(c+d) = 80 & \dots \textcircled{2} \end{cases}$$

From ① and ②,

$$c=5, d=\pm 3$$

Therefore, the three numbers are: 2, 5, 8

From ①, $c=5$
Substituting this into ②

2, 5, 8 when $d=3$
and $d=-3$

1. Find the three numbers whose sum is 27 and product is 693, which form an arithmetic sequence.

[Sol] Let c be the middle term and d be the common difference.

Since the three numbers are expressed as $c-d$, c , $c+d$,

$$\begin{cases} (c-d) + c + (c+d) = 27 & \dots \textcircled{1} \\ (c-d)c(c+d) = 693 & \dots \textcircled{2} \end{cases}$$

From ① and ②,

$$c=9, d=\pm 2$$

Therefore, the three numbers are: 7, 9, 11

N5b

2. Find the 1st term and the common difference of the arithmetic sequence $\{a_n\}$, given $a_1 + a_2 + a_3 = -15$ and $a_1 a_2 a_3 = 120$.

[Sol] Let c be a_2 and d be the common difference.

Since the three numbers are expressed as $a_1 = c - d$, $a_2 = c$, $a_3 = c + d$,

$$\begin{cases} (c-d) + c + (c+d) = -15 & \dots \textcircled{1} \\ (c-d)c(c+d) = 120 & \dots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}$,

$$c = -5, d = \pm 7$$

When $d = 7$, $a_1 = -5 - 7 = -12$

When $d = -7$, $a_1 = -5 - (-7) = 2$

← The 1st term varies with d ; therefore, consider the different cases of d .

Therefore, the 1st term is -12 and the common difference is 7 ; or

the 1st term is 2 and the common difference is -7 .

3. Find the 1st term and the common difference of the arithmetic sequence $\{a_n\}$, given $a_1 + a_3 + a_5 = -12$ and $a_1 a_3 a_5 = 80$.

[Sol] Let c be a_3 and d be the common difference.

Since the three numbers are expressed as $a_1 = c - 2d$, $a_3 = c$, $a_5 = c + 2d$,

$$\begin{cases} (c-2d) + c + (c+2d) = -12 & \dots \textcircled{1} \\ (c-2d)c(c+2d) = 80 & \dots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}$,

$$c = -4, d = \pm 3$$

When $d = 3$, $a_1 = -4 - 2 \cdot 3 = -10$

When $d = -3$, $a_1 = -4 - 2 \cdot (-3) = 2$

Therefore, the 1st term is -10 and the common difference is 3 ; or

the 1st term is 2 and the common difference is -3 .

Ex. Find the sum S_6 of the arithmetic sequence whose 1st term is 1, common difference is 2 and number of terms is 6.

[Sol] $S_6 = 1 + 3 + 5 + 7 + 9 + 11$

+ $S_6 = 11 + 9 + 7 + 5 + 3 + 1$

$2S_6 = 12 + 12 + 12 + 12 + 12 + 12$

$\therefore 2S_6 = 12 \cdot 6$

$\therefore S_6 = 36$

$S_6 = a_1 + a_2 + a_3 + \dots + a_6$

$S_6 = a_6 + a_5 + a_4 + \dots + a_1$

S_6 will be the same even if the order of terms is reversed.

1. Find the sum S_8 of the arithmetic sequence whose 1st term is 1, common difference is 4 and number of terms is 8.

[Sol] $S_8 = 1 + 5 + 9 + 13 + 17 + 21 + 25 + 29$

+ $S_8 = 29 + 25 + 21 + 17 + 13 + 9 + 5 + 1$

$2S_8 = 30 + 30 + 30 + 30 + 30 + 30 + 30 + 30$

$\therefore 2S_8 = 30 \cdot 8$

$\therefore S_8 = 120$

Find the sum S_n of the arithmetic sequence with 1st term a , common difference d , last term l and number of terms n .

[Sol] $S_n = a + (a+d) + (a+2d) + \dots + (l-d) + l$

+ $S_n = l + (l-d) + (l-2d) + \dots + (a+d) + a$

$2S_n = (a+l) + (a+l) + (a+l) + \dots + (a+l) + (a+l)$

The number of $(a+l)$ is n .

$\therefore 2S_n = n(a+l)$

$\therefore S_n = \frac{1}{2} n(a+l)$

Also, since l is the n^{th} term, $l = a + (n-1)d$

Sum of an Arithmetic Sequence

Let S_n be the sum of an arithmetic sequence with 1st term a , common difference d , last term l and number of terms n .

$$S_n = \frac{1}{2}n(a + l) = \frac{1}{2}n[2a + (n-1)d]$$

2. Using the formula above, find the sum of each given arithmetic sequence.

Ex. 1st term: 3, last term: 21,
number of terms: 10

[Sol] $\frac{1}{2} \cdot 10(3 + 21) = 120$

1st term: 5, common difference: 4,
number of terms: 12

[Sol] $\frac{1}{2} \cdot 12[2 \cdot 5 + (12-1) \cdot 4] = 324$

(1) 1st term: 2, last term: 34,
number of terms: 13

[Sol] $\frac{1}{2} \cdot 13(2 + 34) = 234$

(3) 1st term: 4, common difference: 3,
number of terms: 15

[Sol] $\frac{1}{2} \cdot 15[2 \cdot 4 + (15-1) \cdot 3] = 375$

(2) 1st term: -3, last term: -42,
number of terms: 20

[Sol] $\frac{1}{2} \cdot 20[-3 + (-42)] = -450$

(4) 1st term: -2, common difference: 5,
number of terms: 18

[Sol] $\frac{1}{2} \cdot 18[2 \cdot (-2) + (18-1) \cdot 5] = 729$

Ex. S_7 , the sum of the first 7 terms of the arithmetic sequence 5, 9, 13, 17, ...

[Sol] $S_7 = \frac{1}{2} \cdot 7[2 \cdot 5 + (7-1) \cdot 4] = 119$

1st term: 5, common difference: 4,
number of terms: 7

(5) S_{11} , the sum of the first 11 terms of the arithmetic sequence -9, -3, 3, 9, ...

[Sol] $S_{11} = \frac{1}{2} \cdot 11[2 \cdot (-9) + (11-1) \cdot 6] = 231$

(6) S_{20} , the sum of the first 20 terms of the arithmetic sequence 50, 47, 44, 41, ...

[Sol] $S_{20} = \frac{1}{2} \cdot 20[2 \cdot 50 + (20-1) \cdot (-3)] = 430$

N7a

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Arithmetic Sequences

Ex. Find the general term of the arithmetic sequence $\{a_n\}$ whose sum of the first 4 terms is 80 and whose sum of the first 8 terms is 320.

[Sol] Let a be the 1st term, d be the common difference and S_n be the sum of the first n terms.

$$\begin{cases} S_4 = \frac{1}{2} \cdot 4[2a + (4-1)d] = 80 \quad \dots \textcircled{1} \\ S_8 = \frac{1}{2} \cdot 8[2a + (8-1)d] = 320 \quad \dots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}$,

$$a = 5, d = 10$$

$$\therefore a_n = 5 + (n-1) \cdot 10 = 10n - 5$$

1. Find the general term of the arithmetic sequence $\{a_n\}$ whose sum of the first 5 terms is 20 and whose sum of the first 7 terms is 7.

[Sol] Let a be the 1st term, d be the common difference and S_n be the sum of the first n terms.

$$\begin{cases} S_5 = \frac{1}{2} \cdot 5[2a + (5-1)d] = 20 \quad \dots \textcircled{1} \\ S_7 = \frac{1}{2} \cdot 7[2a + (7-1)d] = 7 \quad \dots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}$,

$$a = 10, d = -3$$

$$\therefore a_n = 10 + (n-1) \cdot (-3) = -3n + 13$$

N7b

2. Find the sum S of the first 30 terms of the arithmetic sequence $\{a_n\}$ whose sum of the first 10 terms is 100 and whose sum of the first 20 terms is 350.

[Sol] Let a be the 1st term, d be the common difference and S_n be the sum of the first n terms.

$$\begin{cases} S_{10} = \frac{1}{2} \cdot 10[2a + (10-1)d] = 100 \quad \dots \textcircled{1} \\ S_{20} = \frac{1}{2} \cdot 20[2a + (20-1)d] = 350 \quad \dots \textcircled{2} \end{cases}$$

From ① and ②,

$$a = \frac{13}{4}, \quad d = \frac{3}{2}$$

$$\therefore S = \frac{1}{2} \cdot 30 \left[2 \cdot \frac{13}{4} + (30-1) \cdot \frac{3}{2} \right] = 750$$

3. Find the sum S from the 11th to the 20th term of the arithmetic sequence $\{a_n\}$ whose 8th term is 37 and 24th term is 117.

[Sol] Let a be the 1st term, d be the common difference and S_n be the sum of the first n terms.

$$\begin{cases} a_8 = a + 7d = 37 \quad \dots \textcircled{1} \\ a_{24} = a + 23d = 117 \quad \dots \textcircled{2} \end{cases}$$

From ① and ②,

$$a = 2, \quad d = 5$$

$$\therefore S_{20} = \frac{1}{2} \cdot 20[2 \cdot 2 + (20-1) \cdot 5] = 990$$

$$S_{10} = \frac{1}{2} \cdot 10[2 \cdot 2 + (10-1) \cdot 5] = 245$$

$$\therefore S = S_{20} - S_{10} = 990 - 245 = 745$$



$\overbrace{a_1 + a_2 + \dots + a_{10} + a_{11} + \dots + a_{20}}^{990}$	
$\underbrace{a_1 + a_2 + \dots + a_{10}}_{245}$	$\underbrace{a_{11} + \dots + a_{20}}_S$

Arithmetic Sequences

Ex. Given the arithmetic sequence $\{a_n\}$ whose 1st term is 53 and common difference is -6 , find the value of m for which the sum of the first m terms is the greatest. Then, find the sum S .

[Sol] $a_n = 53 + (n-1) \cdot (-6) = -6n + 59$

$-6n + 59 > 0$ when

$n < \frac{59}{6} = 9.8\dots$

m is the greatest natural number that satisfies this.

$\therefore m = 9$

Also, $S = \frac{1}{2} \cdot 9[2 \cdot 53 + (9-1) \cdot (-6)] = 261$

In this arithmetic sequence,
 $53, 47, \dots, 5, -1, -7, \dots$
 $a_n > 0$ $a_n < 0$

The sum is the greatest when $53 + 47 + \dots + 5$.

1. Given the arithmetic sequence $\{a_n\}$ whose 1st term is 58 and common difference is -3 , find the value of m for which the sum of the first m terms is the greatest. Then, find the sum S .

[Sol] $a_n = 58 + (n-1) \cdot (-3) = -3n + 61$

$-3n + 61 > 0$ when

$n < \frac{61}{3} = 20.3\dots$

m is the greatest natural number that satisfies this.

$\therefore m = 20$

Also, $S = \frac{1}{2} \cdot 20[2 \cdot 58 + (20-1) \cdot (-3)] = 590$

N8b

2. Given the arithmetic sequence $\{a_n\}$ whose 1st term is -120 and common difference is 7 , find the value of m for which the sum of the first m terms is the smallest. Then, find the sum S .

[Sol] $a_n = -120 + (n-1) \cdot 7 = 7n - 127$

$$7n - 127 < 0 \text{ when}$$

$$n < \frac{127}{7} = 18.1\dots$$

m is the greatest natural number that satisfies this.

$$\therefore m = 18$$

$$\text{Also, } S = \frac{1}{2} \cdot 18 [2 \cdot (-120) + (18-1) \cdot 7] = -1089$$

In this arithmetic sequence,

$$\underbrace{-120, -113, \dots, -1}_{a_n < 0}, \underbrace{6, 13, \dots}_{a_n > 0}$$

The sum is the smallest when $(-120) + (-113) + \dots + (-1)$.

3. Given the arithmetic sequence $\{a_n\}$ whose sum of the first 5 terms is -445 and whose sum of the first 10 terms is -765 , find the value of m for which the sum of the first m terms is the smallest. Then, find the sum S .

[Sol] Let a be the 1st term, d be the common difference and S_n be the sum of the first n terms.

$$\begin{cases} S_5 = \frac{1}{2} \cdot 5 [2a + (5-1)d] = -445 & \dots \textcircled{1} \\ S_{10} = \frac{1}{2} \cdot 10 [2a + (10-1)d] = -765 & \dots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}$,

$$a = -99, d = 5$$

$$\therefore a_n = -99 + (n-1) \cdot 5 = 5n - 104$$

$$5n - 104 < 0 \text{ when}$$

$$n < \frac{104}{5} = 20.8$$

m is the greatest natural number that satisfies this.

$$\therefore m = 20$$

$$\text{Also, } S = \frac{1}{2} \cdot 20 [2 \cdot (-99) + (20-1) \cdot 5] = -1030$$

From Kumon Hub

Arithmetic Sequences

Ex. Find the sum S of all natural numbers from 10 to 100 which are divisible by 3.

[Sol] The general term of the sequence of natural numbers greater than or equal to 10 which are divisible by 3 is:

$$12 + (n-1) \cdot 3 = 3n + 9 \quad \leftarrow 1^{\text{st}} \text{ term: } 12, \text{ common difference: } 3$$

$$3n + 9 \leq 100 \text{ when } n \leq \frac{91}{3} = 30.3\ldots$$

The greatest natural number n that satisfies this is $n = 30$.

$$\therefore S = \frac{1}{2} \cdot 30 [2 \cdot 12 + (30-1) \cdot 3] = 1665$$

1. Find the sum S of all natural numbers from 30 to 100 which are divisible by 6.

[Sol] The general term of the sequence of natural numbers greater than or equal to 30 which are divisible by 6 is:

$$30 + (n-1) \cdot 6 = 6n + 24$$

$$6n + 24 \leq 100 \text{ when } n \leq \frac{76}{6} = 12.6\ldots$$

The greatest natural number n that satisfies this is $n = 12$.

$$\therefore S = \frac{1}{2} \cdot 12 [2 \cdot 30 + (12-1) \cdot 6] = 756$$

2. Find the sum S of all natural numbers from 10 to 200 which leave a remainder of 4 when divided by 7.

[Sol] The general term of the sequence of natural numbers greater than or equal to 10 which leave a remainder of 4 when divided by 7 is:

$$11 + (n-1) \cdot 7 = 7n + 4$$

$$7n + 4 \leq 200 \text{ when } n \leq \frac{196}{7} = 28$$

The greatest natural number n that satisfies this is $n = 28$.

$$\therefore S = \frac{1}{2} \cdot 28 [2 \cdot 11 + (28-1) \cdot 7] = 2954$$

3. Find the
either 3 or

[Sol] The g
to 100
102

$3n + 9$
The g
There
by 3 i
 $\frac{1}{2}$

The g
to 100

N9b

3. Find the sum S of all natural numbers from 100 to 200 which are divisible by either 3 or 5.

[Sol] The general term of the sequence of natural numbers greater than or equal to 100 which are divisible by 3 is:

$$102 + (n-1) \cdot 3 = 3n + 99$$

$$3n + 99 \leq 200 \text{ when } n \leq \frac{101}{3} = 33.6\dots$$

The greatest natural number n that satisfies this is $n = 33$.

Therefore, the sum of natural numbers from 100 to 200 which are divisible by 3 is:

$$\frac{1}{2} \cdot 33 [2 \cdot 102 + (33-1) \cdot 3] = 4950 \dots \textcircled{1}$$

The general term of the sequence of natural numbers greater than or equal to 100 which are divisible by 5 is:

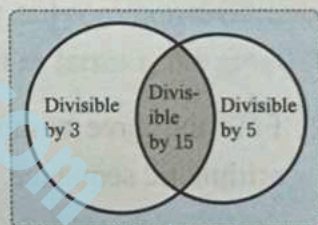
$$100 + (n-1) \cdot 5 = 5n + 95$$

$$5n + 95 \leq 200 \text{ when } n \leq \frac{105}{5} = 21$$

The greatest natural number n that satisfies this is $n = 21$.

Therefore, the sum of natural numbers from 100 to 200 which are divisible by 5 is:

$$\frac{1}{2} \cdot 21 [2 \cdot 100 + (21-1) \cdot 5] = 3150 \dots \textcircled{2}$$



The general term of the sequence of natural numbers greater than or equal to 100 which are divisible by 15 is:

$$105 + (n-1) \cdot 15 = 15n + 90$$

$$15n + 90 \leq 200 \text{ when } n \leq \frac{110}{15} = 7.3\dots$$

The greatest natural number n that satisfies this is $n = 7$.

Therefore, the sum of natural numbers from 100 to 200 which are divisible by 15 is:

$$\frac{1}{2} \cdot 7 [2 \cdot 105 + (7-1) \cdot 15] = 1050 \dots \textcircled{3}$$

From $\textcircled{1} \sim \textcircled{3}$,

$$S = 4950 + 3150 - 1050 \\ = 7050$$

Since $\textcircled{3}$ is included in both $\textcircled{1}$ and $\textcircled{2}$, $\textcircled{3}$ has to be subtracted when finding S .

Arithmetic Sequences

1. Find the general term of the arithmetic sequence $\{a_n\}$ whose 3rd term is 10 and 8th term is 50. ➡ N3

[Sol] Let a be the 1st term and d be the common difference.

$$\begin{cases} a_3 = a + 2d = 10 & \dots \textcircled{1} \\ a_8 = a + 7d = 50 & \dots \textcircled{2} \end{cases}$$

From ① and ②,

$$a = -6, d = 8$$

$$\therefore a_n = -6 + (n-1) \cdot 8 = 8n - 14$$

2. Find the three numbers whose sum is 18 and product is 162, which form an arithmetic sequence. ➡ N5 [Sc]

[Sol] Let c be the middle term and d be the common difference.

Since the three numbers are expressed as $c-d$, c , $c+d$,

$$\begin{cases} (c-d) + c + (c+d) = 18 & \dots \textcircled{1} \\ (c-d)c(c+d) = 162 & \dots \textcircled{2} \end{cases}$$

From ① and ②,

$$c = 6, d = \pm 3$$

Therefore, the three numbers are: 3, 6, 9

N10b

3. Find the general term of the arithmetic sequence $\{a_n\}$ whose sum of the first 5 terms is 100 and whose sum of the first 10 terms is 150. $\Rightarrow$ N7

[Sol] Let a be the 1st term, d be the common difference and S_n be the sum of the first n terms.

$$\begin{cases} S_5 = \frac{1}{2} \cdot 5[2a + (5-1)d] = 100 & \dots \textcircled{1} \\ S_{10} = \frac{1}{2} \cdot 10[2a + (10-1)d] = 150 & \dots \textcircled{2} \end{cases}$$

From ① and ②,

$$a = 24, d = -2$$

$$\therefore a_n = 24 + (n-1) \cdot (-2) = -2n + 26$$

4. Given the arithmetic sequence $\{a_n\}$ whose 1st term is 50 and common difference is -3 , find the value of m for which the sum of the first m terms is the greatest. Then, find the sum S . $\Rightarrow$ N8

[Sol] $a_n = 50 + (n-1) \cdot (-3) = -3n + 53$

$$-3n + 53 > 0 \text{ when}$$

$$n < \frac{53}{3} = 17.6\dots$$

m is the greatest natural number that satisfies this.

$$\therefore m = 17$$

$$\text{Also, } S = \frac{1}{2} \cdot 17[2 \cdot 50 + (17-1) \cdot (-3)] = 442$$

Geometric Sequences

A sequence whose terms are found by successively multiplying a fixed number r to the 1st term a is called a **geometric sequence** (or **geometric progression**). r is called the **common ratio** of the geometric sequence.

1. For each given geometric sequence $\{a_n\}$, find the common ratio r and the 5th term.

Ex. 3, 6, 12, 24, ...

[Sol] $r=2$, $a_5=48$

(1) 1, 3, 9, 27, ...

[Sol] $r=3$, $a_5=81$

(2) $-\frac{1}{2}$, 1, -2, 4, ...

[Sol] $r=-2$, $a_5=-8$

(3) 4, -2, 1, $-\frac{1}{2}$, ...

[Sol] $r=-\frac{1}{2}$, $a_5=\frac{1}{4}$

(4) 5, -5, 5, -5, ...

[Sol] $r=-1$, $a_5=5$

(5) -3, -3, -3, -3, ...

[Sol] $r=1$, $a_5=-3$

2. Find the 10th term of the following geometric sequence $\{a_n\}$.

1, 2, 4, 8, 16, ...

[Sol] $a_1 = 1$

$$a_2 = 1 \cdot 2^1 = 2$$

$$a_3 = 1 \cdot 2^2 = 4$$

$$a_4 = 1 \cdot 2^3 = 8$$

$\vdots$

$$a_{10} = 1 \cdot 2^9 = \boxed{512}$$

3. Find the general term of the following geometric sequence $\{a_n\}$.

5, -10, 20, -40, 80, ...

[Sol] $a_1 = 5$

$$a_2 = 5(-2)^1 = -10$$

$$a_3 = 5(-2)^2 = 20$$

$$a_4 = 5(-2)^3 = -40$$

$\vdots$

$$a_{n-1} = 5(-2)^{n-2}$$

$$a_n = 5(-2)^{n-1}$$

4. Find the general term of the following geometric sequence $\{a_n\}$ with 1st term a and common ratio r .

[Sol] $a_1 = a$

$$a_2 = ar^1$$

$$a_3 = ar^2$$

$$a_4 = ar^3$$

$\vdots$

$$a_n = ar^{n-1}$$

←

$$a_1, a_2, a_3, \dots, a_{n-1}, a_n$$

$$\underbrace{\times r \times r \times r \dots \times r \times r}_{(n-1) \text{ times}}$$

The number of $\times r$ is $(n-1)$.

N12a

KUMON

Geometric Sequences

General Term of a Geometric Sequence

The general term of a geometric sequence $\{a_n\}$ with 1st term a and common ratio r is

$$a_n = ar^{n-1}$$

1. Find the general term and the 6th term of each given geometric sequence $\{a_n\}$.

Ex. 3, 6, 12, 24, ...

[Sol] Since the 1st term is 3 and the common ratio is 2,

$$a_n = 3 \cdot 2^{n-1}$$

$$a_6 = 3 \cdot 2^5 = 96$$

(1) 2, 6, 18, 54, ...

[Sol] Since the 1st term is 2 and the common ratio is 3,

$$a_n = 2 \cdot 3^{n-1}$$

$$a_6 = 2 \cdot 3^5 = 486$$

(2) -1, -4, -16, -64, ...

[Sol] Since the 1st term is -1 and the common ratio is 4,

$$a_n = -1 \cdot 4^{n-1} = -4^{n-1}$$

$$a_6 = -4^5 = -1024$$

(3) -2, 4, -8, 16, ...

[Sol] Since the 1st term is -2 and the common ratio is -2,

$$a_n = -2(-2)^{n-1} = (-2)^n$$

$$a_6 = (-2)^6 = 64$$

N12b

2. Find the 1st term a of the geometric sequence $\{a_n\}$ whose common ratio is $-\frac{1}{3}$ and 3rd term is 6.

[Sol] $a_3 = a \left(-\frac{1}{3}\right)^2 = 6$

$\therefore a = 54$

3. Find the common ratio r of the geometric sequence $\{a_n\}$ whose 1st term is 3 and 3rd term is 48.

[Sol] $a_3 = 3r^2 = 48$

$r^2 = 16$

$\therefore r = \pm 4$

4. Find the common ratio r of the geometric sequence $\{a_n\}$ whose 1st term is 5 and 4th term is 135. (r is a real number.)

[Sol] $a_4 = 5r^3 = 135$

$r^3 = 27$

$\therefore r = 3$

$r^3 - 27 = 0$

$(r-3)(r^2+3r+9) = 0$

Since r is a real number, $r = 3$

5. Given the geometric sequence $\{a_n\}$ whose 1st term is 16 and common ratio is $\frac{3}{2}$, find the value of n for which the n^{th} term is 81.

[Sol] $a_n = 16 \left(\frac{3}{2}\right)^{n-1}$

Since $a_n = 81$,

$16 \left(\frac{3}{2}\right)^{n-1} = 81$

$\left(\frac{3}{2}\right)^{n-1} = \left(\frac{3}{2}\right)^4$

$\therefore n = 5$

Rewriting with base $\frac{3}{2}$

(K192)

Ex. Find the general term of the geometric sequence $\{a_n\}$ whose 4th term is -24 and 6th term is -96 .

[Sol] Let a be the 1st term and r be the common ratio.

$$\begin{cases} a_4 = ar^3 = -24 \quad \dots \textcircled{1} \\ a_6 = ar^5 = -96 \quad \dots \textcircled{2} \end{cases}$$

From ① and ②, $-24r^2 = -96$

$$r^2 = 4$$

$$\therefore r = \pm 2$$

(i) When $r = 2$, from ①, $a = -3$

$$\therefore a_n = -3 \cdot 2^{n-1}$$

(ii) When $r = -2$, from ①, $a = 3$

$$\therefore a_n = 3(-2)^{n-1}$$

From (i) and (ii), $a_n = -3 \cdot 2^{n-1}$ or $a_n = 3(-2)^{n-1}$

From ②, $ar^3 \cdot r^2 = -96$
Substituting ① into this

1. Find the general term of the geometric sequence $\{a_n\}$ whose 3rd term is 36 and 5th term is 324 .

[Sol] Let a be the 1st term and r be the common ratio.

$$\begin{cases} a_3 = ar^2 = 36 \quad \dots \textcircled{1} \\ a_5 = ar^4 = 324 \quad \dots \textcircled{2} \end{cases}$$

From ① and ②, $36r^2 = 324$

$$r^2 = 9$$

$$\therefore r = \pm 3$$

(i) When $r = 3$, from ①, $a = 4$

$$\therefore a_n = 4 \cdot 3^{n-1}$$

(ii) When $r = -3$, from ①, $a = 4$

$$\therefore a_n = 4(-3)^{n-1}$$

From (i) and (ii), $a_n = 4 \cdot 3^{n-1}$ or $a_n = 4(-3)^{n-1}$

2. Find the general term of the geometric sequence $\{a_n\}$ whose 2nd term is 6 and 6th term is $\frac{3}{8}$. (The common ratio is a real number.)

[Sol] Let a be the 1st term and r be the common ratio.

$$\begin{cases} a_2 = ar = 6 & \dots \textcircled{1} \\ a_6 = ar^5 = \frac{3}{8} & \dots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}$, $6r^4 = \frac{3}{8}$

$$r^4 = \frac{1}{16}$$

$$\therefore r = \pm \frac{1}{2}$$

$$r^4 - \frac{1}{16} = 0$$

$$\left(r^2 + \frac{1}{4}\right)\left(r^2 - \frac{1}{4}\right) = 0$$

$$\left(r^2 + \frac{1}{4}\right)\left(r + \frac{1}{2}\right)\left(r - \frac{1}{2}\right) = 0$$

Since r is a real number, $r = \pm \frac{1}{2}$

(i) When $r = \frac{1}{2}$, from $\textcircled{1}$, $a = 12$

$$\therefore a_n = 12\left(\frac{1}{2}\right)^{n-1}$$

(ii) When $r = -\frac{1}{2}$, from $\textcircled{1}$, $a = -12$

$$\therefore a_n = -12\left(-\frac{1}{2}\right)^{n-1}$$

From (i) and (ii), $a_n = 12\left(\frac{1}{2}\right)^{n-1}$ or $a_n = -12\left(-\frac{1}{2}\right)^{n-1}$

3. Given the geometric sequence $\{a_n\}$ whose 4th term is $6\sqrt{3}$ and 7th term is 54, find the value of n for which the n th term is 162. (The common ratio is a real number.)

[Sol] Let a be the 1st term and r be the common ratio.

$$\begin{cases} a_4 = ar^3 = 6\sqrt{3} & \dots \textcircled{1} \\ a_7 = ar^6 = 54 & \dots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}$, $6\sqrt{3}r^3 = 54$

$$r^3 = 3\sqrt{3}$$

$$\therefore r = \sqrt{3}$$

From $\textcircled{1}$, $a = 2$

$$\therefore a_n = 2(\sqrt{3})^{n-1}$$

Since $a_n = 162$,

$$2(\sqrt{3})^{n-1} = 162$$

$$3^{\frac{1}{2}(n-1)} = 3^4$$

$$\therefore n = 9$$

$$r^3 - 3\sqrt{3} = 0$$

$$(r - \sqrt{3})(r^2 + \sqrt{3}r + 3) = 0$$

Since r is a real number, $r = \sqrt{3}$

Rewriting with base 3
(K192)

$$\frac{1}{2}(n-1) = 4$$

Geometric Sequences

Ex. Find the 1st term a and the common ratio r of the geometric sequence $\{a_n\}$ whose 2nd term is 2 and whose sum of the first 3 terms is 7.

[Sol]
$$\begin{cases} ar=2 & \dots \textcircled{1} \\ a+ar+ar^2=7 & \dots \textcircled{2} \end{cases}$$

From $\textcircled{2}$, $a(1+r+r^2)=7$

Multiplying both sides by r ,

$$ar(1+r+r^2)=7r \quad \dots \textcircled{3}$$

From $\textcircled{1}$ and $\textcircled{3}$, $2(1+r+r^2)=7r$

$$2r^2-5r+2=0$$

$$(2r-1)(r-2)=0$$

$$\therefore r=\frac{1}{2}, 2$$

(i) When $r=\frac{1}{2}$, from $\textcircled{1}$, $a=4$

(ii) When $r=2$, from $\textcircled{1}$, $a=1$

From (i) and (ii), $a=4$, $r=\frac{1}{2}$ or $a=1$, $r=2$

1. Find the 1st term a and the common ratio r of the geometric sequence $\{a_n\}$ whose 2nd term is 3 and whose sum of the first 3 terms is 13.

[Sol]
$$\begin{cases} ar=3 & \dots \textcircled{1} \\ a+ar+ar^2=13 & \dots \textcircled{2} \end{cases}$$

From $\textcircled{2}$, $a(1+r+r^2)=13$

Multiplying both sides by r ,

$$ar(1+r+r^2)=13r \quad \dots \textcircled{3}$$

From $\textcircled{1}$ and $\textcircled{3}$, $3(1+r+r^2)=13r$

$$3r^2-10r+3=0$$

$$(3r-1)(r-3)=0$$

$$\therefore r=\frac{1}{3}, 3$$

(i) When $r=\frac{1}{3}$, from $\textcircled{1}$, $a=9$

(ii) When $r=3$, from $\textcircled{1}$, $a=1$

From (i) and (ii), $a=9$, $r=\frac{1}{3}$ or $a=1$, $r=3$

2. Find the 1st term a and the common ratio r of the geometric sequence $\{a_n\}$ whose 3rd term is 12 and whose sum of the first 3 terms is 21.

$$[\text{Sol}] \begin{cases} ar^2 = 12 & \dots \textcircled{1} \\ a + ar + ar^2 = 21 & \dots \textcircled{2} \end{cases}$$

From $\textcircled{2}$, $a(1+r+r^2) = 21$

Multiplying both sides by r^2 ,

$$ar^2(1+r+r^2) = 21r^2 \dots \textcircled{3}$$

From $\textcircled{1}$ and $\textcircled{3}$, $12(1+r+r^2) = 21r^2$

$$3r^2 - 4r - 4 = 0$$

$$(3r+2)(r-2) = 0$$

$$\therefore r = -\frac{2}{3}, 2$$

(i) When $r = -\frac{2}{3}$, from $\textcircled{1}$, $a = 27$

(ii) When $r = 2$, from $\textcircled{1}$, $a = 3$

From (i) and (ii), $a = 27$, $r = -\frac{2}{3}$ or $a = 3$, $r = 2$

3. Find the 1st term a and the common ratio r of the geometric sequence $\{a_n\}$ whose sum of the first 3 terms is 35 and whose sum from the 3rd to the 5th term is 140.

$$[\text{Sol}] \begin{cases} a + ar + ar^2 = 35 & \dots \textcircled{1} \\ ar^2 + ar^3 + ar^4 = 140 & \dots \textcircled{2} \end{cases}$$

From $\textcircled{2}$, $r^2(a + ar + ar^2) = 140 \dots \textcircled{3}$

From $\textcircled{1}$ and $\textcircled{3}$, $35r^2 = 140$

$$r^2 = 4$$

$$\therefore r = \pm 2$$

(i) When $r = 2$, from $\textcircled{1}$, $a = 5$

(ii) When $r = -2$, from $\textcircled{1}$, $a = \frac{35}{3}$

From (i) and (ii), $a = 5$, $r = 2$ or $a = \frac{35}{3}$, $r = -2$

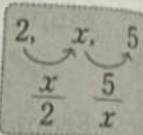
Ex. Find the value of x when the sequence 2, x , 5, ... is a geometric sequence.

[Sol] Since the sequence 2, x , 5 is a geometric sequence,

$$\frac{x}{2} = \frac{5}{x}$$

$$\therefore x^2 = 10$$

$$\therefore x = \pm\sqrt{10}$$



1. Find the value of x when the sequence 3, x , 9, ... is a geometric sequence.

[Sol] Since the sequence 3, x , 9 is a geometric sequence,

$$\frac{x}{3} = \frac{9}{x}$$

$$\therefore x^2 = 27$$

$$\therefore x = \pm 3\sqrt{3}$$

2. Find the values of x and y when the sequence x , -5 , x , y , ... is a geometric sequence.

[Sol] Since the sequence x , -5 , x , y is a geometric sequence,

$$\begin{cases} \frac{-5}{x} = \frac{x}{-5} \dots \textcircled{1} \\ \frac{x}{-5} = \frac{y}{x} \dots \textcircled{2} \end{cases}$$

$$\text{From } \textcircled{1}, x^2 = 25 \dots \textcircled{3}$$

$$\therefore x = \pm 5$$

$$\text{From } \textcircled{2}, x^2 = -5y \dots \textcircled{4}$$

$$\text{From } \textcircled{3} \text{ and } \textcircled{4}, 25 = -5y$$

$$\therefore y = -5$$

Therefore, $x = 5, y = -5$ or $x = -5, y = -5$

N15b

3. Find the values of x and y when the sequence $x, y, -4$ is an arithmetic sequence and the sequence $y, x, -4$ is a geometric sequence.

[Sol] Since the sequence $x, y, -4$ is an arithmetic sequence,

$$y - x = -4 - y \quad \leftarrow \text{N4}$$

$$\therefore x = 2y + 4 \quad \dots \textcircled{1}$$

Since the sequence $y, x, -4$ is a geometric sequence,

$$\frac{x}{y} = \frac{-4}{x}$$

$$\therefore x^2 = -4y \quad \dots \textcircled{2}$$

From $\textcircled{1}$ and $\textcircled{2}$, $(2y + 4)^2 = -4y$

$$y^2 + 5y + 4 = 0$$

$$(y + 4)(y + 1) = 0$$

$$\therefore y = -4, -1$$

(i) When $y = -4$, from $\textcircled{1}$, $x = -4$

(ii) When $y = -1$, from $\textcircled{1}$, $x = 2$

From (i) and (ii), $x = -4, y = -4$ or $x = 2, y = -1$

Geometric Sequences

Ex. Find the sum S_4 of the geometric sequence whose 1st term is 2, common ratio is 3 and number of terms is 4.

$$\begin{array}{r}
 \text{[Sol]} \quad S_4 = 2 + 2 \cdot 3 + 2 \cdot 3^2 + 2 \cdot 3^3 \\
 -) 3S_4 = \quad 2 \cdot 3 + 2 \cdot 3^2 + 2 \cdot 3^3 + 2 \cdot 3^4 \\
 \hline
 S_4 - 3S_4 = 2 \qquad \qquad \qquad -2 \cdot 3^4 \\
 \therefore -2S_4 = 2(1 - 3^4) \\
 \therefore S_4 = 80
 \end{array}$$

Multiplying both sides of S_4 by the common ratio 3

1. Find the sum S_5 of the geometric sequence whose 1st term is 3, common ratio is 2 and number of terms is 5.

$$\begin{array}{r}
 \text{[Sol]} \quad S_5 = 3 + 3 \cdot 2 + 3 \cdot 2^2 + 3 \cdot 2^3 + 3 \cdot 2^4 \\
 -) 2S_5 = \quad 3 \cdot 2 + 3 \cdot 2^2 + 3 \cdot 2^3 + 3 \cdot 2^4 + 3 \cdot 2^5 \\
 \hline
 S_5 - 2S_5 = 3 \qquad \qquad \qquad -3 \cdot 2^5 \\
 \therefore -S_5 = 3(1 - 2^5) \\
 \therefore S_5 = 93
 \end{array}$$

Find the sum S_n of the geometric sequence with 1st term a , common ratio r and number of terms n .

$$\begin{array}{r}
 \text{[Sol]} \quad S_n = a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1} \\
 -) rS_n = \quad ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1} + ar^n \\
 \hline
 S_n - rS_n = a \qquad \qquad \qquad -ar^n \\
 \therefore (1-r)S_n = a(1-r^n)
 \end{array}$$

(i) When $r \neq 1$,

$$S_n = \frac{a(1 - r^n)}{1 - r} \text{ or } S_n = \frac{a(r^n - 1)}{r - 1}$$

(ii) When $r = 1$,

$$S_n = \underbrace{a + a + a + \cdots + a}_{n \text{ terms}} = na$$

Sum of a Geometric Sequence

Let S_n be the sum of a geometric sequence with 1st term a , common ratio r and number of terms n .

$$\text{When } r \neq 1, S_n = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1}$$

$$\text{When } r = 1, S_n = na \quad \leftarrow$$

When $r = 1$, S_n is the sum of n terms of a .

2. Using the formula above, find the sum of each given geometric sequence.

Ex. 1st term: -2 , common ratio: 3 ,
number of terms: 4

$$[\text{Sol}] \frac{-2(3^4-1)}{3-1} = -80$$

1st term: 3 , common ratio: 1 ,
number of terms: 10

$$[\text{Sol}] 10 \cdot 3 = 30$$

(1) 1st term: -4 , common ratio: 2 ,
number of terms: 5

$$[\text{Sol}] \frac{-4(2^5-1)}{2-1} = -124$$

(3) 1st term: -5 , common ratio: $-\frac{1}{2}$,
number of terms: 6

$$[\text{Sol}] \frac{-5 \left[1 - \left(-\frac{1}{2} \right)^6 \right]}{1 - \left(-\frac{1}{2} \right)} = -\frac{105}{32}$$

(2) 1st term: 3 , common ratio: -1 ,
number of terms: 7

$$[\text{Sol}] \frac{3[1-(-1)^7]}{1-(-1)} = 3$$

(4) 1st term: -2 , common ratio: 1 ,
number of terms: 12

$$[\text{Sol}] 12 \cdot (-2) = -24$$

Ex. S_5 , the sum of the first 5 terms of the geometric sequence $1, 3, 9, 27, \dots$

$$[\text{Sol}] S_5 = \frac{1 \cdot (3^5-1)}{3-1} = 121 \quad \leftarrow$$

1st term: 1 , common ratio: 3 ,
number of terms: 5

(5) S_7 , the sum of the first 7 terms of the geometric sequence $\frac{1}{2}, 1, 2, 4, \dots$

$$[\text{Sol}] S_7 = \frac{\frac{1}{2}(2^7-1)}{2-1} = \frac{127}{2}$$

(6) S_8 , the sum of the first 8 terms of the geometric sequence $1, \sqrt{2}, 2, 2\sqrt{2}, \dots$

$$[\text{Sol}] S_8 = \frac{1 \cdot [(\sqrt{2})^8-1]}{\sqrt{2}-1} = \frac{15}{\sqrt{2}-1} = \frac{15(\sqrt{2}+1)}{(\sqrt{2}-1)(\sqrt{2}+1)} = 15(\sqrt{2}+1)$$

1. Given the geometric sequence whose 1st term is 2 and common ratio is -3 , find the value of n for which the sum of the first n terms is 122.

[Sol] Let S_n be the sum of the first n terms.

$$S_n = \frac{2[1 - (-3)^n]}{1 - (-3)} = 122$$

$$(-3)^n = -243$$

$$(-3)^n = (-3)^5$$

$$\therefore n = 5$$

Rewriting with base -3
(K192)

2. Find the 1st term a of the geometric sequence $\{a_n\}$ whose common ratio is $\frac{1}{3}$ and whose sum of the first 4 terms is 40.

[Sol] Let S_n be the sum of the first n terms.

$$S_4 = \frac{a \left[1 - \left(\frac{1}{3} \right)^4 \right]}{1 - \frac{1}{3}} = 40$$

$$\frac{80}{81}a = \frac{80}{3}$$

$$\therefore a = 27$$

3. Find the sum S from the 7th to the 10th term of the geometric sequence $\{a_n\}$ whose 1st term is $\frac{1}{3}$ and common ratio is 2.

[Sol] Let S_n be the sum of the first n terms.

$$S_{10} = \frac{\frac{1}{3}(2^{10} - 1)}{2 - 1} = 341$$

$$S_6 = \frac{\frac{1}{3}(2^6 - 1)}{2 - 1} = 21$$

$$\therefore S = S_{10} - S_6 = 341 - 21 = 320$$

341	
$a_1 + a_2 + \dots + a_6$	$a_7 + \dots + a_{10}$
21	S

N17b

4. Find the sum S from the 6th to the 9th term of the geometric sequence $\{a_n\}$ whose 5th term is -48 and 8th term is 384 . (The common ratio is a real number.)

[Sol] Let a be the 1st term, r be the common ratio and S_n be the sum of the first n terms.

$$\begin{cases} a_5 = ar^4 = -48 \quad \dots \textcircled{1} \\ a_8 = ar^7 = 384 \quad \dots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}$, $-48r^3 = 384$

$$r^3 = -8$$

$$\therefore r = -2$$

From $\textcircled{1}$, $a = -3$

$$\therefore S_9 = \frac{-3[1 - (-2)^9]}{1 - (-2)} = -513$$

$$S_5 = \frac{-3[1 - (-2)^5]}{1 - (-2)} = -33$$

$$\therefore S = S_9 - S_5 = -513 - (-33) = -480$$

5. Given the geometric sequence whose 1st term is 2 and common ratio is 4, find the value of n for which the sum of the first n terms becomes greater than 1000 for the first time.

[Sol] Let S_n be the sum of the first n terms.

$$S_n = \frac{2(4^n - 1)}{4 - 1} = \frac{2}{3}(4^n - 1)$$

Considering $S_n > 1000$,

$$\frac{2}{3}(4^n - 1) > 1000 \text{ when}$$

$$4^n > 1501$$

The smallest natural number n that satisfies this is $n=6$.

Finding the smallest natural number n that satisfies $4^n > 1501$ by substituting some values into n

Geometric Sequences

Ex. Find the 1st term a and the common ratio r of the geometric sequence $\{a_n\}$ whose sum of the first 3 terms is 9 and whose sum of the first 6 terms is -63 . (The common ratio is a real number.)

[Sol] Let S_n be the sum of the first n terms.

(i) When $r=1$, $S_3=3a=9$, $S_6=6a=-63$

No a can satisfy these conditions simultaneously.

(ii) When $r \neq 1$,

$$\begin{cases} S_3 = \frac{a(r^3-1)}{r-1} = 9 & \dots \textcircled{1} \\ S_6 = \frac{a(r^6-1)}{r-1} = -63 & \dots \textcircled{2} \end{cases}$$

$3a=9$, i.e. $a=3$
Substituting this into $6a$,
 $6 \cdot 3 = 18 \neq -63$; therefore,
this is not suitable.

$$S_6 = \frac{a(r^3-1)(r^3+1)}{r-1} = -63$$

From $\textcircled{1}$ and $\textcircled{2}$, $9(r^3+1) = -63$
 $r^3 = -8$

$\therefore r = -2$

From $\textcircled{1}$, $a=3$

From (i) and (ii), $a=3$, $r=-2$

1. Find the 1st term a and the common ratio r of the geometric sequence $\{a_n\}$ whose sum of the first 3 terms is 52 and whose sum of the first 6 terms is 1456 (The common ratio is a real number.)

[Sol] Let S_n be the sum of the first n terms.

(i) When $r=1$, $S_3=3a=52$, $S_6=6a=1456$

No a can satisfy these conditions simultaneously.

(ii) When $r \neq 1$,

$$\begin{cases} S_3 = \frac{a(r^3-1)}{r-1} = 52 & \dots \textcircled{1} \\ S_6 = \frac{a(r^6-1)}{r-1} = 1456 & \dots \textcircled{2} \end{cases}$$

$(r^3)^2 = r^6$

$\frac{a(r^3-1)(r^3+1)}{r-1} = 1456$

From $\textcircled{1}$ and $\textcircled{2}$, $52(r^3+1) = 1456$
 $r^3 = 27$

$\therefore r = 3$

From $\textcircled{1}$, $a=4$

From (i) and (ii), $a=4$, $r=3$

2. Find the sum S from the 31st to the 60th term of the geometric sequence $\{a_n\}$ whose sum of the first 10 terms is 3 and whose sum from the 11th to the 30th term is 60. (The common ratio is a real number.)

[Sol] Let a be the 1st term, r be the common ratio and S_n be the sum of the first n terms.

(i) When $r=1$, $S_{10}=10a=3$, $S_{30}=30a=3+60=63$ ←

No a can satisfy these conditions simultaneously.

$$\begin{array}{c} S_{30} \\ \hline a_1 + a_2 + \dots + a_{10} + a_{11} + \dots + a_{30} \\ \hline \begin{array}{cc} 3 & 60 \end{array} \end{array}$$

(ii) When $r \neq 1$,

$$\begin{cases} S_{10} = \frac{a(r^{10}-1)}{r-1} = 3 \quad \dots \textcircled{1} \\ S_{30} = \frac{a(r^{30}-1)}{r-1} = 63 \quad \dots \textcircled{2} \end{cases}$$

$$S_{30} = \frac{a(r^{10}-1)(r^{20}+r^{10}+1)}{r-1} = 63$$

From ① and ②, $3(r^{20}+r^{10}+1)=63$ ←

$$r^{20}+r^{10}-20=0$$

$$(r^{10}-4)(r^{10}+5)=0$$

$$r^{10}=4 \quad \dots \textcircled{3}$$

$$r^{10}+5 > 0$$

From ② and ③,

$$\begin{aligned} S_{60} &= \frac{a(r^{60}-1)}{r-1} \\ &= \frac{a(r^{30}-1)(r^{30}+1)}{r-1} \\ &= \frac{a(r^{30}-1)}{r-1} [(r^{10})^3+1] \\ &= 63(4^3+1) \end{aligned}$$

$$\begin{array}{l} \text{From ②,} \\ \frac{a(r^{30}-1)}{r-1} = 63 \end{array}$$

$$= 4095$$

From (i) and (ii), $S = S_{60} - S_{30}$

$$= 4095 - 63$$

$$= 4032$$

Geometric Sequences

Ex. You want to save 10000 dollars in 5 years by saving x dollars at the beginning of each year. Given that the annual interest rate is 2%, find x . Use $(1.02)^5 = 1.1$ and round your answer down to the nearest 10 dollars.

[Sol] After 5 years, x dollars from the 1st year become $x \times (1.02)^5$ dollars.

After 4 years, x dollars from the 2nd year become $x \times (1.02)^4$ dollars.

$\vdots$

$\vdots$

After 1 year, x dollars from the 5th year become $x \times 1.02$ dollars.

Let S be the amount of savings at the end of the 5th year.

$$S = 1.02x + (1.02)^2x + \cdots + (1.02)^5x$$



1st term: $1.02x$,
common ratio: 1.02,
number of terms: 5

$$= \frac{1.02x[(1.02)^5 - 1]}{1.02 - 1}$$

$$= \frac{1.02(1.1 - 1)}{0.02}x$$

$$= 5.1x$$

Thus, since $S = 10000$,

$$5.1x = 10000 \quad \therefore x = 1960.7\dots$$

Rounding down to the nearest 10 dollars, 1960

Ans. 1960 dollars

Answers: in order 1.02, 1.02, 1.1

The method of determining the interest on the initial amount and the accumulated interest of previous periods of a deposit or loan is called the **compound interest method**. An interest calculated by the above method is called a **compound interest**.

For example, when you save 100 dollars in the bank with an annual interest rate of 2%, after 1 year, your savings become $100 \times 1.02 = 102$ (dollars).

After 2 years, the annual interest rate of 2% accrues on the interest of 2 dollars; therefore, your savings become $100 \times (1.02)^2 = 104.04$ (dollars).

N19b

1. You want to save 20000 dollars in 10 years by saving x dollars at the beginning of each year. Given that the annual interest rate is 8%, find x . Use $(1.08)^{10} = 2.1$ and round your answer down to the nearest 10 dollars.

[Sol] After 10 years, x dollars from the 1st year become $x \times (1.08)^{10}$ dollars.

After 9 years, x dollars from the 2nd year become $x \times (1.08)^9$ dollars.

⋮

⋮

After 1 year, x dollars from the 10th year become $x \times 1.08$ dollars.

Let S be the amount of savings at the end of the 10th year.

$$S = 1.08x + (1.08)^2x + \dots + (1.08)^{10}x$$

$$= \frac{1.08x[(1.08)^{10} - 1]}{1.08 - 1}$$

$$= \frac{1.08(2.1 - 1)}{0.08}x$$

$$= 14.85x$$

Thus, since $S = 20000$,

$$14.85x = 20000 \quad \therefore x = 1346.8\dots$$

Rounding down to the nearest 10 dollars, 1340

Ans. 1340 dollars

From Kumon Hub

Geometric Sequences

1. Find the general term of the geometric sequence $\{a_n\}$ whose 2nd term is 12 [Sol] and 4th term is 192. $\Rightarrow$ N13

[Sol] Let a be the 1st term and r be the common ratio.

$$\begin{cases} a_2 = ar = 12 & \dots \textcircled{1} \\ a_4 = ar^3 = 192 & \dots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}$, $12r^2 = 192$

$$r^2 = 16$$

$$\therefore r = \pm 4$$

(i) When $r = 4$, from $\textcircled{1}$, $a = 3$

$$\therefore a_n = 3 \cdot 4^{n-1}$$

(ii) When $r = -4$, from $\textcircled{1}$, $a = -3$

$$\therefore a_n = -3(-4)^{n-1}$$

From (i) and (ii), $a_n = 3 \cdot 4^{n-1}$ or $a_n = -3(-4)^{n-1}$

2. Find the 1st term a and the common ratio r of the geometric sequence $\{a_n\}$ whose 2nd term is -3 and whose sum of the first 3 terms is 7. $\Rightarrow$ N14

[Sol] $\begin{cases} ar = -3 & \dots \textcircled{1} \\ a + ar + ar^2 = 7 & \dots \textcircled{2} \end{cases}$

From $\textcircled{2}$, $a(1+r+r^2) = 7$

Multiplying both sides by r ,

$$ar(1+r+r^2) = 7r \dots \textcircled{3}$$

From $\textcircled{1}$ and $\textcircled{3}$, $-3(1+r+r^2) = 7r$

$$3r^2 + 10r + 3 = 0$$

$$(r+3)(3r+1) = 0$$

$$\therefore r = -3, -\frac{1}{3}$$

(i) When $r = -3$, from $\textcircled{1}$, $a = 1$

(ii) When $r = -\frac{1}{3}$, from $\textcircled{1}$, $a = 9$

From (i) and (ii), $a = 1, r = -3$ or $a = 9, r = -\frac{1}{3}$

Alternative Solution

$$\begin{cases} ar = -3 & \dots \textcircled{1} \\ \frac{a(r^3-1)}{r-1} = 7 & \dots \textcircled{2} \end{cases}$$

$$\text{From } \textcircled{2}, a(r^2+r+1) = 7$$

Multiplying both sides by r ,

$$ar(r^2+r+1) = 7r \dots \textcircled{3}$$

From $\textcircled{1}$ and $\textcircled{3}$, $-3(r^2+r+1) = 7r$

$$3r^2 + 10r + 3 = 0$$

$$(r+3)(3r+1) = 0$$

$$\therefore r = -3, -\frac{1}{3}$$

(i) When $r = -3$, from $\textcircled{1}$, $a = 1$

(ii) When $r = -\frac{1}{3}$, from $\textcircled{1}$, $a = 9$

From (i) and (ii),

$$a = 1, r = -3 \text{ or } a = 9, r = -\frac{1}{3}$$

N20b

3. Given the geometric sequence whose 1st term is 6 and common ratio is -2 , find the value of n for which the sum of the first n terms is -30 . ➡ N17

[Sol] Let S_n be the sum of the first n terms.

$$S_n = \frac{6[1 - (-2)^n]}{1 - (-2)} = -30$$

$$(-2)^n = 16$$

$$(-2)^n = (-2)^4$$

$$\therefore n = 4$$

4. Find the 1st term a and the common ratio r of the geometric sequence $\{a_n\}$ whose sum of the first 3 terms is 6 and whose sum of the first 6 terms is -42 . (The common ratio is a real number.) ➡ N18

[Sol] Let S_n be the sum of the first n terms.

(i) When $r = 1$, $S_3 = 3a = 6$, $S_6 = 6a = -42$

No a can satisfy these conditions simultaneously.

(ii) When $r \neq 1$,

$$\begin{cases} S_3 = \frac{a(r^3 - 1)}{r - 1} = 6 & \dots \text{①} \\ S_6 = \frac{a(r^6 - 1)}{r - 1} = -42 & \dots \text{②} \end{cases}$$

From ① and ②, $6(r^3 + 1) = -42$
 $r^3 = -8$

$$\therefore r = -2$$

From ①, $a = 2$

From (i) and (ii), $a = 2$, $r = -2$

Alternative Solution

$$\begin{cases} a + ar + ar^2 = 6 & \dots \text{①} \\ a + ar + ar^2 + ar^3 + ar^4 + ar^5 = -42 & \dots \text{②} \end{cases}$$

From ②,

$$a + ar + ar^2 + r^3(a + ar + ar^2) = -42 \dots \text{③}$$

From ① and ③, $6 + 6r^3 = -42$

$$r^3 = -8$$

$$\therefore r = -2$$

From ①, $a = 2$

$$\therefore a = 2, r = -2$$

Various Sequences I

The summation symbol Σ is used to denote the sum of a sequence.
 For example, $a_3 + a_4 + a_5 + a_6$, the sum from the 3rd to the 6th term of the sequence $\{a_n\}$ is denoted by $\sum_{k=3}^6 a_k$.

Generally, $\sum_{k=p}^q a_k$ denotes the sum from the p^{th} to the q^{th} term of a sequence $\{a_n\}$.

(Note that other letters may be used in place of k .)

1. Let $\{a_n\}$ be the sequence

1, 3, 5, 7, 9, 11, ...

Find the following sums.

Ex.

$$\sum_{k=2}^4 a_k = a_2 + a_3 + a_4 = 3 + 5 + 7 = 15$$



$$\sum_{k=2}^4 a_k = a_2 + a_3 + a_4$$

$$(1) \sum_{k=3}^5 a_k = a_3 + a_4 + a_5 = 5 + 7 + 9 = 21$$

$$(2) \sum_{m=3}^6 a_m = a_3 + a_4 + a_5 + a_6 = 5 + 7 + 9 + 11 = 32$$

$$(3) \sum_{n=1}^5 a_n = a_1 + a_2 + a_3 + a_4 + a_5 = 1 + 3 + 5 + 7 + 9 = 25$$

$$(4) \sum_{k=1}^3 a_{k+1} = a_{1+1} + a_{2+1} + a_{3+1} = a_2 + a_3 + a_4 = 3 + 5 + 7 = 15$$

$$(5) \sum_{k=1}^3 a_{2k} = a_{2 \cdot 1} + a_{2 \cdot 2} + a_{2 \cdot 3} = a_2 + a_4 + a_6 = 3 + 7 + 11 = 21$$

$$(6) \sum_{k=1}^2 a_{2k-1} = a_{2 \cdot 1 - 1} + a_{2 \cdot 2 - 1} = a_1 + a_3 = 1 + 5 = 6$$

$$(7) \sum_{k=2}^2 a_k = a_2 = 3$$

Σ is read as "sigma".

N21b

2. Find the following sums.

Ex. $\sum_{k=2}^4 (3k-1) = (3 \cdot 2 - 1) + (3 \cdot 3 - 1) + (3 \cdot 4 - 1) = 5 + 8 + 11 = 24$

(1) $\sum_{k=3}^5 2k^2 = 2 \cdot 3^2 + 2 \cdot 4^2 + 2 \cdot 5^2 = 18 + 32 + 50 = 100$

(2) $\sum_{k=0}^2 5^k = 5^0 + 5^1 + 5^2 = 1 + 5 + 25 = 31$

(3) $\sum_{n=3}^4 (2n-1)^2 = (2 \cdot 3 - 1)^2 + (2 \cdot 4 - 1)^2 = 25 + 49 = 74$

(4) $\sum_{m=2}^3 (3^m - 2^{m-1}) = (3^2 - 2^{2-1}) + (3^3 - 2^{3-1}) = 7 + 23 = 30$

(5) $\sum_{k=4}^6 (k+n) = (\boxed{4} + n) + (\boxed{5} + n) + (\boxed{6} + n) = 3n + \boxed{15}$

(6) $\sum_{k=2}^5 (n-k) = (n-2) + (n-3) + (n-4) + (n-5) = 4n - 14$

3. Rewrite the following sums using Σ .

Ex. $2 + 4 + 6 + 8 + 10 = \sum_{k=1}^5 2k$ $\leftarrow$ Given $\{a_n\}: 2, 4, 6, \dots$, the sum of the first 5 terms of $\{a_n\}$ is $\sum_{k=1}^5 2k$.

(1) $5 + 10 + 15 + 20 = \sum_{k=1}^4 5k$

(2) $1 + 4 + 9 + 16 + 25 + 36 = \sum_{k=1}^6 k^2$

(3) $3 + 5 + 7 + \dots + (2n+1) = \sum_{k=1}^n (\boxed{2k+1})$

(4) $3 + 12 + 27 + \dots + 3n^2 = \sum_{k=1}^n 3k^2$

(5) $3 + 12 + 27 + \dots + 3n^2 + 3(n+1)^2 = \sum_{k=1}^{n+1} \boxed{3k^2}$ $\leftarrow$

Given $\{a_n\}: 3, 12, 27, \dots$, from (4), the sum of the first n terms of $\{a_n\}$ is $\sum_{k=1}^n 3k^2$.
Adding $3(n+1)^2$ that is the $(n+1)^{\text{th}}$ term of $\{a_n\}$,
 $\sum_{k=1}^n 3k^2 + 3(n+1)^2 = \sum_{k=1}^{n+1} 3k^2$

N22a

Various Sequences I

KUMON

N 22

Since the sum S_n of the arithmetic sequence with 1st term a , last term l and number of terms n is

$$S_n = \frac{1}{2}n(a+l), \quad \leftarrow \text{N6}$$

the sum of the natural numbers from 1 to n (1, 2, 3, ..., n) becomes

$$1+2+3+\dots+n = \frac{1}{2}n(n+1)$$

Also, the sum of a sequence, where $a_1 = a_2 = a_3 = \dots = a_n = c$ (c is a constant), is

$$\underbrace{c+c+\dots+c}_{n \text{ terms}} = nc$$

Rewriting the sums of these sequences using Σ , the following formulas are obtained

Summation Formulas I

$$\sum_{k=1}^n k = \frac{1}{2}n(n+1), \quad \sum_{k=1}^n c = nc \quad (c \text{ is a constant})$$

1. Find the following sums.

Ex. $\sum_{k=1}^{10} k = \frac{1}{2} \cdot 10(10+1) = 55$ $\sum_{k=1}^{15} 2 = 15 \cdot 2 = 30$

(1) $\sum_{k=1}^{15} k = \frac{1}{2} \cdot 15(15+1) = 120$

(2) $\sum_{k=1}^{10} 5 = 10 \cdot 5 = 50$

(3) $\sum_{k=1}^{20} k - 3 \sum_{k=1}^8 2 = \frac{1}{2} \cdot 20(20+1) - 3 \cdot 8 \cdot 2 = 162$

(4) $2 \sum_{k=1}^n k - \sum_{k=1}^n 1 = 2 \cdot \frac{1}{2}n(n+1) - n \cdot 1 = n^2$

N22b

Given two sequences

$$\{a_n\}: a_1, a_2, \dots$$

$$\{b_n\}: b_1, b_2, \dots$$

rewrite the following

[1] (a_1+b_1)

$$= (a_1+a_2)$$

[Sol] $\sum_{k=1}^n (a_k + b_k)$

[2] $ca_1 + ca_2$

[Sol] $\sum_{k=1}^n ca_k$

From the above

Summation

$$\sum_{k=1}^n (a_k + b_k)$$

2. Find the following

Ex. $\sum_{k=1}^n (3k^2 - 2k + 1)$

(1) $\sum_{k=1}^n (4k^2 - 3k + 2)$

(2) $\sum_{k=1}^n (k^2 - 2k + 1)$

(3) $\sum_{k=1}^{n-1} (k^2 - 2k + 1)$

N22b

Given two sequences

$$\{a_n\}: a_1, a_2, a_3, \dots, a_n$$

$$\{b_n\}: b_1, b_2, b_3, \dots, b_n,$$

rewrite the following equations using Σ .

$$[1] (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) + \dots + (a_n + b_n)$$

$$= (a_1 + a_2 + a_3 + \dots + a_n) + (b_1 + b_2 + b_3 + \dots + b_n)$$

$$[\text{Sol}] \sum_{k=1}^n (a_k + b_k) = \sum_{k=1}^n a_k + \sum_{k=1}^n b_k$$

$$[2] ca_1 + ca_2 + ca_3 + \dots + ca_n = c(a_1 + a_2 + a_3 + \dots + a_n) \quad (c \text{ is a constant})$$

$$[\text{Sol}] \sum_{k=1}^n ca_k = c \sum_{k=1}^n a_k$$

Answers: in order $\sum_{k=1}^n a_k, \sum_{k=1}^n b_k$

From the above, the following equations are true.

Summation Properties

$$\sum_{k=1}^n (a_k + b_k) = \sum_{k=1}^n a_k + \sum_{k=1}^n b_k, \quad \sum_{k=1}^n ca_k = c \sum_{k=1}^n a_k \quad (c \text{ is a constant})$$

2. Find the following sums.

$$\text{Ex. } \sum_{k=1}^n (3k + 1) = 3 \sum_{k=1}^n k + \sum_{k=1}^n 1 = 3 \cdot \frac{1}{2} n(n+1) + n = \frac{1}{2} n(3n+5)$$

$$(1) \sum_{k=1}^n (4k + 3) = 4 \sum_{k=1}^n k + \sum_{k=1}^n 3 = 4 \cdot \frac{1}{2} n(n+1) + 3n = n(2n+5)$$

$$(2) \sum_{k=1}^n (-2k + 4) = -2 \sum_{k=1}^n k + \sum_{k=1}^n 4 = -2 \cdot \frac{1}{2} n(n+1) + 4n = -n(n-3)$$

$$(3) \sum_{k=1}^{n-1} (3k + 5) = 3 \sum_{k=1}^{n-1} k + \sum_{k=1}^{n-1} 5 = 3 \cdot \frac{1}{2} (n-1)[(n-1)+1] + 5(n-1)$$

$$= \frac{1}{2} (n-1)(3n+10)$$

N23a

Various Sequences I

KUMON

N 23

Find $\sum_{k=1}^n k^2$.[Sol] Use $(k+1)^3 - k^3 = 3k^2 + 3k + 1$.

$$(k+1)^3 = k^3 + 3k^2 + 3k + 1$$

When $k=1$,

$$2^3 - 1^3 = 3 \cdot 1^2 + 3 \cdot 1 + 1$$

When $k=2$,

$$3^3 - 2^3 = 3 \cdot 2^2 + 3 \cdot 2 + 1$$

When $k=3$,

$$4^3 - 3^3 = 3 \cdot 3^2 + 3 \cdot 3 + 1$$

 $\vdots$ When $k=n$,

$$(n+1)^3 - n^3 = 3 \cdot n^2 + 3 \cdot n + 1$$

Adding up the terms on each side from all these n equations,

$$(n+1)^3 - 1^3 = 3(1^2 + 2^2 + \cdots + n^2) + 3(1 + 2 + \cdots + n) + (1 + 1 + \cdots + 1)$$

$$= 3 \sum_{k=1}^n k^2 + 3 \sum_{k=1}^n k + n$$

Therefore,

Rearranging into $3 \sum_{k=1}^n k^2 =$

$$3 \sum_{k=1}^n k^2 = (n+1)^3 - 1 - 3 \sum_{k=1}^n k - n$$

$$= (n+1)^3 - 3 \cdot \frac{1}{2} n(n+1) - (n+1)$$

$$= \frac{1}{2} (n+1) [2(n+1)^2 - 3n - 2]$$

$$= \frac{1}{2} n(n+1)(2n+1)$$

$$\therefore \sum_{k=1}^n k^2 = \frac{1}{6} n(n+1)(2n+1)$$

N23b

Summation

$$\sum_{k=1}^n k^2 = \frac{1}{6} n(n+1)(2n+1)$$

Find the following

$$(1) \sum_{k=1}^4 (k^2 + 2)$$

$$(2) \sum_{k=1}^n (3k^2)$$

$$(3) \sum_{k=1}^n (6k^2)$$

Answers: in order $k, k, \frac{1}{2}n(n+1), n+1, \frac{1}{2}n(n+1)(2n+1), \frac{1}{6}n(n+1)(2n+1)$

N23b

Summation Formula II

$$\sum_{k=1}^n k^2 = \frac{1}{6}n(n+1)(2n+1)$$

Find the following sums.

$$\begin{aligned} (1) \quad \sum_{k=1}^4 (k^2 + 2k) &= \sum_{k=1}^4 k^2 + 2 \sum_{k=1}^4 k \\ &= \frac{1}{6} \cdot 4(4+1)(2 \cdot 4+1) + 2 \cdot \frac{1}{2} \cdot 4(4+1) \\ &= 50 \end{aligned}$$

$$\begin{aligned} (2) \quad \sum_{k=1}^n (3k^2 - k) &= 3 \sum_{k=1}^n k^2 - \sum_{k=1}^n k \\ &= 3 \cdot \frac{1}{6}n(n+1)(2n+1) - \frac{1}{2}n(n+1) \\ &= \frac{1}{2}n(n+1)[(2n+1) - 1] \\ &= n^2(n+1) \end{aligned}$$

$$\begin{aligned} (3) \quad \sum_{k=1}^n (6k^2 - 7k - 3) &= 6 \sum_{k=1}^n k^2 - 7 \sum_{k=1}^n k - \sum_{k=1}^n 3 \\ &= 6 \cdot \frac{1}{6}n(n+1)(2n+1) - 7 \cdot \frac{1}{2}n(n+1) - 3n \\ &= \frac{1}{2}n[2(n+1)(2n+1) - 7(n+1) - 6] \\ &= \frac{1}{2}n(4n^2 - n - 11) \end{aligned}$$

N24a

Various Sequences I

KUMON

N 24

Find $\sum_{k=1}^n k^3$.

[Sol] Use $(k+1)^4 - k^4 = 4k^3 + 6k^2 + 4k + 1$. $\leftarrow (k+1)^4 = k^4 + 4k^3 + 6k^2 + 4k + 1$

When $k=1$,

$$2^4 - 1^4 = 4 \cdot 1^3 + 6 \cdot 1^2 + 4 \cdot 1 + 1$$

When $k=2$,

$$3^4 - 2^4 = 4 \cdot 2^3 + 6 \cdot 2^2 + 4 \cdot 2 + 1$$

When $k=3$,

$$4^4 - 3^4 = 4 \cdot 3^3 + 6 \cdot 3^2 + 4 \cdot 3 + 1$$

$\vdots$

$\vdots$

When $k=n$,

$$(n+1)^4 - n^4 = 4 \cdot n^3 + 6 \cdot n^2 + 4 \cdot n + 1$$

Adding up the terms on each side from all these n equations,

$$\begin{aligned} (n+1)^4 - 1^4 &= 4(1^3 + 2^3 + \dots + n^3) + 6(1^2 + 2^2 + \dots + n^2) \\ &\quad + 4(1 + 2 + \dots + n) + (1 + 1 + \dots + 1) \\ &= 4 \sum_{k=1}^n k^3 + 6 \sum_{k=1}^n k^2 + 4 \sum_{k=1}^n k + n \end{aligned}$$

Therefore,

$$\begin{aligned} 4 \sum_{k=1}^n k^3 &= (n+1)^4 - 1 - 6 \sum_{k=1}^n k^2 - 4 \sum_{k=1}^n k - n \\ &= (n+1)^4 - 6 \cdot \frac{1}{6} n(n+1)(2n+1) - 4 \cdot \frac{1}{2} n(n+1) - (n+1) \\ &= (n+1)[(n+1)^3 - n(2n+1) - 2n - 1] \\ &= n^2(n+1)^2 \end{aligned}$$

$$\therefore \sum_{k=1}^n k^3 = \left[\frac{1}{2} n(n+1) \right]^2$$

Rearranging into

$$4 \sum_{k=1}^n k^3 =$$

N24b

Sum

$$\sum_{k=1}^n k^3$$

Find the fo

$$(1) \sum_{k=1}^6 (k$$

$$(2) \sum_{k=1}^n$$

$$(3) \sum_{k=1}^n$$

Answers: in order $k_2, k_2, \frac{1}{6} n(n+1)(2n+1), n+1, n+1, (1+u)u, \frac{1}{2} n(n+1), \frac{1}{2} n(n+1)$

N24b

Summation Formula III

$$\sum_{k=1}^n k^3 = \left[\frac{1}{2}n(n+1) \right]^2$$

Find the following sums.

$$\begin{aligned} (1) \quad \sum_{k=1}^6 (k^3 + k) &= \sum_{k=1}^6 k^3 + \sum_{k=1}^6 k \\ &= \left[\frac{1}{2} \cdot 6(6+1) \right]^2 + \frac{1}{2} \cdot 6(6+1) \\ &= 462 \end{aligned}$$

$$\begin{aligned} (2) \quad \sum_{k=1}^n (2k^3 - 3k) &= 2 \sum_{k=1}^n k^3 - 3 \sum_{k=1}^n k \\ &= 2 \left[\frac{1}{2}n(n+1) \right]^2 - 3 \cdot \frac{1}{2}n(n+1) \\ &= \frac{1}{2}n(n+1)[n(n+1) - 3] \\ &= \frac{1}{2}n(n+1)(n^2 + n - 3) \end{aligned}$$

$$\begin{aligned} (3) \quad \sum_{k=1}^n (4k^3 - 6k^2 + 3) &= 4 \sum_{k=1}^n k^3 - 6 \sum_{k=1}^n k^2 + \sum_{k=1}^n 3 \\ &= 4 \left[\frac{1}{2}n(n+1) \right]^2 - 6 \cdot \frac{1}{6}n(n+1)(2n+1) + 3n \\ &= n[n(n+1)^2 - (n+1)(2n+1) + 3] \\ &= n(n^3 - 2n + 2) \end{aligned}$$

into

n+1)

Various Sequences I

Since the sum S_n of the geometric sequence with 1st term a , common ratio r ($r \neq 1$) and number of terms n is

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1}, \quad \leftarrow \dots \text{N16}$$

when $r \neq 1$, the sum of the geometric sequence $a, ar, ar^2, \dots, ar^{n-1}$ becomes

$$a + ar + ar^2 + \dots + ar^{n-1} = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1}$$

Rewriting the sum of the sequence using Σ , the following formula is obtained.

Summation Formula IV

$$\sum_{k=1}^n ar^{k-1} = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1} \quad (r \neq 1)$$

Find the following sums.

Ex. $\sum_{k=1}^4 2 \cdot 3^{k-1} = \frac{2(3^4-1)}{3-1} = 80$

(1) $\sum_{k=1}^6 5 \cdot 2^{k-1} = \frac{5(2^6-1)}{2-1} = 315$

(2) $\sum_{k=1}^3 (-3)^{k-1} = \frac{1-(-3)^3}{1-(-3)} = 7$

(3) $\sum_{k=1}^n 3 \cdot 4^{k-1} = \frac{3(4^n-1)}{4-1} = 4^n - 1$

(4) $\sum_{k=1}^n 6(-5)^{k-1} = \frac{6[1-(-5)^n]}{1-(-5)} = 1 - (-5)^n$

(5) $\sum_{k=1}^n 4^k =$

(6) $\sum_{k=1}^n 6^k =$

(7) $\sum_{k=1}^n 3^k =$

(8) $\sum_{k=1}^n (-1)^k =$

(9) $\sum_{k=1}^{n-1} (-1)^k =$

N25b

$$(5) \sum_{k=1}^n 4^k = \sum_{k=1}^n 4 \cdot 4^{k-1} = \frac{4(4^n - 1)}{4 - 1} = \frac{4^{n+1} - 4}{3}$$

$$(6) \sum_{k=1}^n 6^k = \sum_{k=1}^n 6 \cdot 6^{k-1} = \frac{6(6^n - 1)}{6 - 1} = \frac{6^{n+1} - 6}{5}$$

$$(7) \sum_{k=1}^n 3(-2)^k = \sum_{k=1}^n [-6(-2)^{k-1}] = \frac{-6[1 - (-2)^n]}{1 - (-2)} = -2 - (-2)^{n+1}$$

$$\begin{aligned} (8) \sum_{k=1}^n (4 \cdot 3^k - 2^{k+1}) &= \sum_{k=1}^n 12 \cdot 3^{k-1} - \sum_{k=1}^n 4 \cdot 2^{k-1} \\ &= \frac{12(3^n - 1)}{3 - 1} - \frac{4(2^n - 1)}{2 - 1} \\ &= (2 \cdot 3^{n+1} - 6) - (2^{n+2} - 4) \\ &= 2 \cdot 3^{n+1} - 2^{n+2} - 2 \\ &= [2(3^{n+1} - 2^{n+1} - 1)] \end{aligned}$$

$$\begin{aligned} (9) \sum_{k=1}^{n-1} (5^k + 3^{k+1}) &= \sum_{k=1}^{n-1} 5 \cdot 5^{k-1} + \sum_{k=1}^{n-1} 9 \cdot 3^{k-1} \\ &= \frac{5(5^{n-1} - 1)}{5 - 1} + \frac{9(3^{n-1} - 1)}{3 - 1} \\ &= \left(\frac{5^n}{4} - \frac{5}{4} \right) + \left(\frac{3^{n+1}}{2} - \frac{9}{2} \right) \\ &= \frac{5^n}{4} + \frac{3^{n+1}}{2} - \frac{23}{4} \\ &= \left[\frac{1}{4}(5^n + 2 \cdot 3^{n+1} - 23) \right] \end{aligned}$$

N26a
Various Sequences I

KUMON

N 26

Find the following sums.

$$\begin{aligned}
 (1) \quad & \sum_{k=1}^n (2k+3^k) \\
 &= 2 \sum_{k=1}^n k + \sum_{k=1}^n 3 \cdot 3^{k-1} \\
 &= 2 \cdot \frac{1}{2} n(n+1) + \frac{3(3^n-1)}{3-1} \\
 &= n(n+1) + \frac{3^{n+1}-3}{2}
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad & \sum_{k=1}^n [3k^2 + (-5)^k] \\
 &= 3 \sum_{k=1}^n k^2 + \sum_{k=1}^n [-5(-5)^{k-1}] \\
 &= 3 \cdot \frac{1}{6} n(n+1)(2n+1) + \frac{-5[1-(-5)^n]}{1-(-5)} \\
 &= \frac{1}{2} n(n+1)(2n+1) - \frac{5+(-5)^{n+1}}{6}
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad & \sum_{k=1}^n (6k^2 + 2k + 4^{k+1}) \\
 &= 6 \sum_{k=1}^n k^2 + 2 \sum_{k=1}^n k + \sum_{k=1}^n 16 \cdot 4^{k-1} \\
 &= 6 \cdot \frac{1}{6} n(n+1)(2n+1) + 2 \cdot \frac{1}{2} n(n+1) + \frac{16(4^n-1)}{4-1} \\
 &= n(n+1)[(2n+1)+1] + \frac{4^{n+2}-16}{3} \\
 &= 2n(n+1)^2 + \frac{4^{n+2}-16}{3}
 \end{aligned}$$

N26b

$$\begin{aligned}
 (4) \quad & \sum_{k=1}^{2n} (2k^3 + 3k^2 + 6^{k-1}) \\
 &= 2 \sum_{k=1}^{2n} k^3 + 3 \sum_{k=1}^{2n} k^2 + \sum_{k=1}^{2n} 6^{k-1} \\
 &= 2 \left[\frac{1}{2} \cdot 2n(2n+1) \right]^2 + 3 \cdot \frac{1}{6} \cdot 2n(2n+1)(2 \cdot 2n+1) + \frac{6^{2n}-1}{6-1} \\
 &= n(2n+1)[2n(2n+1) + (4n+1)] + \frac{6^{2n}-1}{5} \\
 &= n(2n+1)(4n^2 + 6n + 1) + \frac{6^{2n}-1}{5}
 \end{aligned}$$

$$\begin{aligned}
 (5) \quad & \sum_{k=1}^{n+1} \left(4k^{\frac{3}{2}} + 3^k \right) \left(4k^{\frac{3}{2}} - 3^k \right) \\
 &= \sum_{k=1}^{n+1} (16k^3 - 9^k) \\
 &= 16 \sum_{k=1}^{n+1} k^3 - \sum_{k=1}^{n+1} 9 \cdot 9^{k-1} \\
 &= 16 \left\{ \frac{1}{2} (n+1)[(n+1)+1] \right\}^2 - \frac{9(9^{n+1}-1)}{9-1} \\
 &= 4(n+1)^2(n+2)^2 - \frac{9^{n+2}-9}{8}
 \end{aligned}$$

$$\begin{aligned}
 (6) \quad & \sum_{k=1}^{n-1} (-3k + 2^{k+1})(3k + 2^{k+1}) \\
 &= \sum_{k=1}^{n-1} (-9k^2 + 4^{k+1}) \\
 &= -9 \sum_{k=1}^{n-1} k^2 + \sum_{k=1}^{n-1} 16 \cdot 4^{k-1} \\
 &= -9 \cdot \frac{1}{6} (n-1)[(n-1)+1][2(n-1)+1] + \frac{16(4^{n-1}-1)}{4-1} \\
 &= -\frac{3}{2} n(n-1)(2n-1) + \frac{4^{n+1}-16}{3}
 \end{aligned}$$

N27a

Various Sequences I

KUMON

N 27

Find the following sums.

Ex. $1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + n(n+2)$

[Sol] Since this is the sum of the first n terms of the sequence whose k^{th} term is $k(k+2)$,

$$\begin{aligned}\sum_{k=1}^n k(k+2) &= \sum_{k=1}^n (k^2 + 2k) \\ &= \sum_{k=1}^n k^2 + 2 \sum_{k=1}^n k \\ &= \frac{1}{6}n(n+1)(2n+1) + 2 \cdot \frac{1}{2}n(n+1) \\ &= \frac{1}{6}n(n+1)[(2n+1)+6] \\ &= \frac{1}{6}n(n+1)(2n+7)\end{aligned}$$

(1) $2 \cdot 2 + 4 \cdot 5 + 6 \cdot 8 + \cdots + 2n(3n-1)$

[Sol] Since this is the sum of the first n terms of the sequence whose k^{th} term is $2k(3k-1)$,

$$\begin{aligned}\sum_{k=1}^n 2k(3k-1) &= \sum_{k=1}^n (6k^2 - 2k) \\ &= 6 \sum_{k=1}^n k^2 - 2 \sum_{k=1}^n k \\ &= 6 \cdot \frac{1}{6}n(n+1)(2n+1) - 2 \cdot \frac{1}{2}n(n+1) \\ &= n(n+1)[(2n+1)-1] \\ &= 2n^2(n+1)\end{aligned}$$

N27b

$$(2) \quad 1^2 \cdot 2 + 2^2 \cdot 3 + 3^2 \cdot 4 + \cdots + n^2(n+1)$$

[Sol] Since this is the sum of the first n terms of the sequence whose k^{th} term is $k^2(k+1)$,

$$\begin{aligned} \sum_{k=1}^n k^2(k+1) &= \sum_{k=1}^n (k^3 + k^2) \\ &= \sum_{k=1}^n k^3 + \sum_{k=1}^n k^2 \\ &= \left[\frac{1}{2}n(n+1) \right]^2 + \frac{1}{6}n(n+1)(2n+1) \\ &= \frac{1}{12}n(n+1)[3n(n+1) + 2(2n+1)] \\ &= \frac{1}{12}n(n+1)(n+2)(3n+1) \end{aligned}$$

$$(3) \quad 1^3 + 3^3 + 5^3 + \cdots + (2n-1)^3$$

[Sol] Since this is the sum of the first n terms of the sequence whose k^{th} term is $(2k-1)^3$,

$$\begin{aligned} \sum_{k=1}^n (2k-1)^3 &= \sum_{k=1}^n (8k^3 - 12k^2 + 6k - 1) \\ &= 8 \sum_{k=1}^n k^3 - 12 \sum_{k=1}^n k^2 + 6 \sum_{k=1}^n k - \sum_{k=1}^n 1 \\ &= 8 \left[\frac{1}{2}n(n+1) \right]^2 - 12 \cdot \frac{1}{6}n(n+1)(2n+1) + 6 \cdot \frac{1}{2}n(n+1) - n \\ &= n[2n(n+1)^2 - 2(n+1)(2n+1) + 3(n+1) - 1] \\ &= n^2(2n^2 - 1) \end{aligned}$$

Find the sum S_n of the first n terms of each given sequence.

Ex. $1 \cdot 2, 2 \cdot 3, 3 \cdot 4, 4 \cdot 5, \dots$

[Sol] The k^{th} term is $k(k+1) = k^2 + k$.

$$\begin{aligned}\therefore S_n &= \sum_{k=1}^n (k^2 + k) = \sum_{k=1}^n k^2 + \sum_{k=1}^n k \\ &= \frac{1}{6}n(n+1)(2n+1) + \frac{1}{2}n(n+1) \\ &= \frac{1}{6}n(n+1)[(2n+1)+3] \\ &= \frac{1}{3}n(n+1)(n+2)\end{aligned}$$

(1) $1 \cdot 4, 2 \cdot 7, 3 \cdot 10, 4 \cdot 13, \dots$

[Sol] The k^{th} term is $k(3k+1) = 3k^2 + k$.

$$\begin{aligned}\therefore S_n &= \sum_{k=1}^n (3k^2 + k) = 3 \sum_{k=1}^n k^2 + \sum_{k=1}^n k \\ &= 3 \cdot \frac{1}{6}n(n+1)(2n+1) + \frac{1}{2}n(n+1) \\ &= \frac{1}{2}n(n+1)[(2n+1)+1] \\ &= n(n+1)^2\end{aligned}$$

(2) $1 \cdot 2 \cdot 3, 2 \cdot 3 \cdot 4, 3 \cdot 4 \cdot 5, 4 \cdot 5 \cdot 6, \dots$

[Sol] The k^{th} term is $k(k+1)(k+2) = k^3 + 3k^2 + 2k$.

$$\begin{aligned}\therefore S_n &= \sum_{k=1}^n (k^3 + 3k^2 + 2k) \\ &= \sum_{k=1}^n k^3 + 3 \sum_{k=1}^n k^2 + 2 \sum_{k=1}^n k \\ &= \left[\frac{1}{2}n(n+1) \right]^2 + 3 \cdot \frac{1}{6}n(n+1)(2n+1) + 2 \cdot \frac{1}{2}n(n+1) \\ &= \frac{1}{4}n(n+1)[n(n+1) + 2(2n+1) + 4] \\ &= \frac{1}{4}n(n+1)(n+2)(n+3)\end{aligned}$$

N28b

(3) 1, 1+2, 1+2+3, 1+2+3+4, ...

[Sol] The k^{th} term is $1+2+3+\dots+k=\frac{1}{2}k(k+1)=\frac{1}{2}(k^2+k)$.

$$\begin{aligned}\therefore S_n &= \sum_{k=1}^n \frac{1}{2}(k^2+k) = \frac{1}{2} \left(\sum_{k=1}^n k^2 + \sum_{k=1}^n k \right) && 1+2+3+\dots+n = \sum_{k=1}^n k \\ &= \frac{1}{2} \left[\frac{1}{6}n(n+1)(2n+1) + \frac{1}{2}n(n+1) \right] \\ &= \frac{1}{12}n(n+1)[(2n+1)+3] \\ &= \frac{1}{6}n(n+1)(n+2)\end{aligned}$$

(4) $1^2, 1^2+2^2, 1^2+2^2+3^2, 1^2+2^2+3^2+4^2, \dots$

[Sol] The k^{th} term is

$$1^2+2^2+3^2+\dots+k^2=\frac{1}{6}k(k+1)(2k+1)=\frac{1}{6}(2k^3+3k^2+k).$$

$$\begin{aligned}\therefore S_n &= \sum_{k=1}^n \frac{1}{6}(2k^3+3k^2+k) && 1^2+2^2+3^2+\dots+n^2 = \sum_{k=1}^n k^2 \\ &= \frac{1}{6} \left(2 \sum_{k=1}^n k^3 + 3 \sum_{k=1}^n k^2 + \sum_{k=1}^n k \right) \\ &= \frac{1}{6} \left\{ 2 \left[\frac{1}{2}n(n+1) \right]^2 + 3 \cdot \frac{1}{6}n(n+1)(2n+1) + \frac{1}{2}n(n+1) \right\} \\ &= \frac{1}{12}n(n+1)[n(n+1)+(2n+1)+1] \\ &= \frac{1}{12}n(n+1)^2(n+2)\end{aligned}$$

Various Sequences I

1. Find the sum of the first 30 terms of the sequence $\{a_n = \alpha + \beta n + \gamma n^2\}$ whose 1st term is 3, 2nd term is 5 and whose sum of the first 3 terms is 17.

[Sol]
$$\begin{cases} a_1 = 3 & \dots \textcircled{1} \\ a_2 = 5 & \dots \textcircled{2} \\ a_1 + a_2 + a_3 = 17 & \dots \textcircled{3} \end{cases}$$

From $\textcircled{1} \sim \textcircled{3}$,

$$a_3 = 9$$

$$\therefore \begin{cases} a_1 = \alpha + \beta + \gamma = 3 & \dots \textcircled{4} \\ a_2 = \alpha + 2\beta + 4\gamma = 5 & \dots \textcircled{5} \\ a_3 = \alpha + 3\beta + 9\gamma = 9 & \dots \textcircled{6} \end{cases}$$

From $\textcircled{4} \sim \textcircled{6}$,

$$\alpha = 3, \beta = -1, \gamma = 1$$

$$\therefore a_n = 3 - n + n^2$$

Therefore, the sum is

$$\sum_{k=1}^{30} a_k = \sum_{k=1}^{30} (3 - k + k^2)$$

$$= \sum_{k=1}^{30} 3 - \sum_{k=1}^{30} k + \sum_{k=1}^{30} k^2$$

$$= 30 \cdot 3 - \frac{1}{2} \cdot 30(30+1) + \frac{1}{6} \cdot 30(30+1)(2 \cdot 30+1)$$

$$= 9080$$

N29b

2. Given $S_n = \sum_{k=1}^n k$ ($n=1, 2, \dots$), find the following sums.

(1) $T_n = \sum_{k=1}^n S_k$

[Sol] $S_n = \sum_{k=1}^n k = \frac{1}{2}n(n+1) = \frac{1}{2}(n^2+n)$

$$\therefore T_n = \sum_{k=1}^n \frac{1}{2}(k^2+k) \quad \leftarrow S_k = \frac{1}{2}(k^2+k)$$

$$= \frac{1}{2} \left(\sum_{k=1}^n k^2 + \sum_{k=1}^n k \right)$$

$$= \frac{1}{2} \left[\frac{1}{6}n(n+1)(2n+1) + \frac{1}{2}n(n+1) \right]$$

$$= \frac{1}{12}n(n+1)[(2n+1)+3]$$

$$= \frac{1}{6}n(n+1)(n+2)$$

(2) $U_n = \sum_{k=1}^n T_k$

[Sol] From (1), $T_n = \frac{1}{6}(n^3+3n^2+2n)$

$$\therefore U_n = \sum_{k=1}^n \frac{1}{6}(k^3+3k^2+2k) \quad \leftarrow T_k = \frac{1}{6}(k^3+3k^2+2k)$$

$$= \frac{1}{6} \left(\sum_{k=1}^n k^3 + 3 \sum_{k=1}^n k^2 + 2 \sum_{k=1}^n k \right)$$

$$= \frac{1}{6} \left\{ \left[\frac{1}{2}n(n+1) \right]^2 + 3 \cdot \frac{1}{6}n(n+1)(2n+1) + 2 \cdot \frac{1}{2}n(n+1) \right\}$$

$$= \frac{1}{24}n(n+1)[n(n+1)+2(2n+1)+4]$$

$$= \frac{1}{24}n(n+1)(n+2)(n+3)$$

Various Sequences I

1. Find the following sums.

$$\begin{aligned}
 (1) \quad \sum_{k=1}^n (6k^2 - 2k + 3) &= 6 \sum_{k=1}^n k^2 - 2 \sum_{k=1}^n k + \sum_{k=1}^n 3 \quad \rightarrow \text{N23} \\
 &= 6 \cdot \frac{1}{6} n(n+1)(2n+1) - 2 \cdot \frac{1}{2} n(n+1) + 3n \\
 &= n[(n+1)(2n+1) - (n+1) + 3] \\
 &= n(2n^2 + 2n + 3)
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad \sum_{k=1}^n (k^3 + 2k) &= \sum_{k=1}^n k^3 + 2 \sum_{k=1}^n k \quad \rightarrow \text{N24} \\
 &= \left[\frac{1}{2} n(n+1) \right]^2 + 2 \cdot \frac{1}{2} n(n+1) \\
 &= \frac{1}{4} n(n+1)[n(n+1) + 4] \\
 &= \frac{1}{4} n(n+1)(n^2 + n + 4)
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad \sum_{k=1}^n (2^k + 4 \cdot 5^{k-1}) &= \sum_{k=1}^n 2 \cdot 2^{k-1} + \sum_{k=1}^n 4 \cdot 5^{k-1} \quad \rightarrow \text{N25} \\
 &= \frac{2(2^n - 1)}{2 - 1} + \frac{4(5^n - 1)}{5 - 1} \\
 &= (2^{n+1} - 2) + (5^n - 1) \\
 &= 2^{n+1} + 5^n - 3
 \end{aligned}$$

N30b

(4) $2^2 \cdot 2 + 4^2 \cdot 3 + 6^2 \cdot 4 + \dots + (2n)^2(n+1)$

→ N27

[Sol] Since this is the sum of the first n terms of the sequence whose k^{th} term is $(2k)^2(k+1)$,

$$\begin{aligned} \sum_{k=1}^n (2k)^2(k+1) &= \sum_{k=1}^n 4(k^3 + k^2) \\ &= 4 \left(\sum_{k=1}^n k^3 + \sum_{k=1}^n k^2 \right) \\ &= 4 \left\{ \left[\frac{1}{2}n(n+1) \right]^2 + \frac{1}{6}n(n+1)(2n+1) \right\} \\ &= \frac{1}{3}n(n+1)[3n(n+1) + 2(2n+1)] \\ &= \frac{1}{3}n(n+1)(n+2)(3n+1) \end{aligned}$$

2. Find the sum S_n of the first n terms of the following sequence.

→ N28

$1^2 \cdot 3, 2^2 \cdot 5, 3^2 \cdot 7, 4^2 \cdot 9, \dots$

[Sol] The k^{th} term is $k^2(2k+1) = 2k^3 + k^2$.

$$\begin{aligned} \therefore S_n &= \sum_{k=1}^n (2k^3 + k^2) = 2 \sum_{k=1}^n k^3 + \sum_{k=1}^n k^2 \\ &= 2 \left[\frac{1}{2}n(n+1) \right]^2 + \frac{1}{6}n(n+1)(2n+1) \\ &= \frac{1}{6}n(n+1)[3n(n+1) + (2n+1)] \\ &= \frac{1}{6}n(n+1)(3n^2 + 5n + 1) \end{aligned}$$

N31a

Various Sequences II

KUMON

N 31

- Find the sequence by writing out the difference between each two consecutive terms in each given sequence.

Ex 6, 10, 15, 21, 28, 36, ...

[Sol] 4, 5, 6, 7, 8, ...

- 1, 2, 4, 7, 11, 16, ...

[Sol] 1, 2, 3, 4, 5, ...

- 3, 6, 11, 18, 27, 38, ...

[Sol] 3, 5, 7, 9, 11, ...

- 1, 4, 11, 22, 37, 56, ...

[Sol] 3, 7, 11, 15, 19, ...

- 18, 17, 14, 9, 2, -7, ...

[Sol] -1, -3, -5, -7, -9, ...

- 2, 4, 6, 8, 10, 12, ...

[Sol] 2, 2, 2, 2, 2, ...

N31b

The term which
in the sequence

$$b_n = a_{n+1} - a_n$$

Generally, the

- Given that
differences

$$(n \geq 2)$$

[Sol]

$$3$$

$$6$$

$$10$$

$$a_n$$

Addin

$$a_n$$

$$\therefore$$

- Let $\{b_n\}$
the gene

[Sol]

$$a_2$$

$$a_3$$

$$a_4$$

$$a_5$$

$$a_6$$

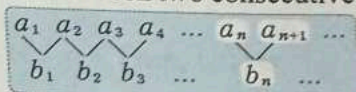
Add

$$a_7$$

N31b

The term which is derived by taking the difference between two consecutive terms in the sequence $\{a_n\}$ is

$$b_n = a_{n+1} - a_n \quad (n=1, 2, 3, \dots)$$



Generally, the sequence $\{b_n\}$ is called the **sequence of differences** of $\{a_n\}$.

2. Given that the sequence $\{a_n\}$ is 1, 3, 6, 10, 15, ..., its sequence of differences is 2, 3, 4, 5, Using this, find the general term of $\{a_n\}$. ($n \geq 2$)

[Sol]

$$\begin{array}{rcl} 3 - 1 & = & 2 \\ 6 - 3 & = & 3 \\ 10 - 6 & = & 4 \\ & \vdots & \\ a_n - a_{n-1} & = & n \end{array}$$

Adding up the terms on each side from all these $(n-1)$ equations,

$$a_n - \boxed{1} = 2 + 3 + 4 + \dots + n$$

$$\therefore a_n = \boxed{1} + (2 + 3 + 4 + \dots + n)$$

$$= \boxed{\frac{1}{2}n(n+1)}$$

3. Let $\{b_n\}$ be the sequence of differences of the sequence $\{a_n\}$. Using $\{b_n\}$, find the general term of $\{a_n\}$. ($n \geq 2$)

[Sol]

$$\begin{array}{rcl} a_2 - a_1 & = & b_1 \\ a_3 - a_2 & = & b_2 \\ a_4 - a_3 & = & b_3 \\ & \vdots & \\ a_n - a_{n-1} & = & b_{n-1} \end{array}$$

Adding up the terms on each side from all these $(n-1)$ equations,

$$a_n - \boxed{a_1} = b_1 + b_2 + b_3 + \dots + b_{n-1}$$

$$\therefore a_n = \boxed{a_1} + (b_1 + b_2 + b_3 + \dots + b_{n-1})$$

$$= \boxed{a_1} + \sum_{k=1}^{n-1} b_k$$

Sequence of Differences and General Term

Let $\{b_n\}$ be the sequence of differences of the sequence $\{a_n\}$.

$$\text{When } n \geq 2, a_n = a_1 + \sum_{k=1}^{n-1} b_k$$

Find the general term of each given sequence $\{a_n\}$.

Ex. 2, 6, 12, 20, 30, 42, ...

[Sol] Let $\{b_n\}$ be the sequence of differences of $\{a_n\}$. Then, $\{b_n\}$ is

$$4, 6, 8, 10, 12, \dots$$

$$\therefore b_n = 4 + (n-1) \cdot 2 = 2n + 2$$

When $n \geq 2$,

$$a_n = a_1 + \sum_{k=1}^{n-1} (2k + 2)$$

$$= 2 + 2 \cdot \frac{1}{2} (n-1)n + 2(n-1)$$

$$= n(n+1) \dots \textcircled{1}$$

Since $a_1 = 2$, $\textcircled{1}$ is also true when $n = 1$.

$$\therefore a_n = n(n+1)$$

$\{b_n\}$ is the arithmetic sequence with 1st term 4 and common difference 2.

$$\sum_{k=1}^n k = \frac{1}{2}n(n+1), \sum_{k=1}^n c = nc$$

Substituting $n = 1$ into $\textcircled{1}$, $a_1 = 2$ and it coincides with the 1st term of the given sequence.

(1) 2, 10, 24, 44, 70, 102, ...

[Sol] Let $\{b_n\}$ be the sequence of differences of $\{a_n\}$. Then, $\{b_n\}$ is

$$8, 14, 20, 26, 32, \dots$$

$$\therefore b_n = 8 + (n-1) \cdot 6 = 6n + 2$$

When $n \geq 2$,

$$a_n = a_1 + \sum_{k=1}^{n-1} (6k + 2)$$

$$= 2 + 6 \cdot \frac{1}{2} (n-1)n + 2(n-1)$$

$$= n(3n-1) \dots \textcircled{1}$$

Since $a_1 = 2$, $\textcircled{1}$ is also true when $n = 1$.

$$\therefore a_n = n(3n-1)$$

N32b

(2) 10, 10, 9, 7, 4, 0, ...

[Sol] Let $\{b_n\}$ be the sequence of differences of $\{a_n\}$. Then, $\{b_n\}$ is

$$0, -1, -2, -3, -4, \dots$$

$$\therefore b_n = 0 + (n-1) \cdot (-1) = -n+1$$

When $n \geq 2$,

$$a_n = a_1 + \sum_{k=1}^{n-1} (-k+1)$$

$$= 10 - \frac{1}{2}(n-1)n + (n-1)$$

$$= -\frac{1}{2}(n+3)(n-6) \dots \textcircled{1}$$

Since $a_1 = 10$, $\textcircled{1}$ is also true when $n = 1$.

$$\therefore a_n = -\frac{1}{2}(n+3)(n-6)$$

(3) $1, \frac{9}{2}, \frac{20}{2}, \frac{35}{2}, \frac{54}{2}, \frac{77}{2}, \dots$

[Sol] Let $\{b_n\}$ be the sequence of differences of $\{a_n\}$. Then, $\{b_n\}$ is

$$\frac{7}{2}, \frac{11}{2}, \frac{15}{2}, \frac{19}{2}, \frac{23}{2}, \dots$$

$$\therefore b_n = \frac{7}{2} + (n-1) \cdot 2 = 2n + \frac{3}{2}$$

When $n \geq 2$,

$$a_n = a_1 + \sum_{k=1}^{n-1} \left(2k + \frac{3}{2}\right)$$

$$= 1 + 2 \cdot \frac{1}{2}(n-1)n + \frac{3}{2}(n-1)$$

$$= \frac{1}{2}(n+1)(2n-1) \dots \textcircled{1}$$

Since $a_1 = 1$, $\textcircled{1}$ is also true when $n = 1$.

$$\therefore a_n = \frac{1}{2}(n+1)(2n-1)$$

Various Sequences II

Find the general term of each given sequence $\{a_n\}$.

(1) 1, 4, 10, 22, 46, 94, ...

[Sol] Let $\{b_n\}$ be the sequence of differences of $\{a_n\}$. Then, $\{b_n\}$ is

3, 6, 12, 24, 48, ...

$\therefore b_n = 3 \cdot 2^{n-1}$

$\{b_n\}$ is the geometric sequence with 1st term 3 and common ratio 2.

When $n \geq 2$,

$$a_n = a_1 + \sum_{k=1}^{n-1} 3 \cdot 2^{k-1}$$

$$= 1 + \frac{3(2^{n-1} - 1)}{2 - 1}$$

$$= 3 \cdot 2^{n-1} - 2 \quad \dots \textcircled{1}$$

$$\sum_{k=1}^n ar^{k-1} = \frac{a(r^n - 1)}{r - 1} \quad (r \neq 1)$$

Since $a_1 = 1$, $\textcircled{1}$ is also true when $n = 1$.

$\therefore a_n = 3 \cdot 2^{n-1} - 2$

(2) 1, 3, 7, 15, 31, 63, ...

[Sol] Let $\{b_n\}$ be the sequence of differences of $\{a_n\}$. Then, $\{b_n\}$ is

2, 4, 8, 16, 32, ...

$\therefore b_n = 2 \cdot 2^{n-1} = 2^n$

When $n \geq 2$,

$$a_n = a_1 + \sum_{k=1}^{n-1} 2^k$$

$$= 1 + \sum_{k=1}^{n-1} 2 \cdot 2^{k-1}$$

$$= 1 + \frac{2(2^{n-1} - 1)}{2 - 1}$$

$$= 2^n - 1 \quad \dots \textcircled{1}$$

Since $a_1 = 1$, $\textcircled{1}$ is also true when $n = 1$.

$\therefore a_n = 2^n - 1$

N33b

(3) 3, 4, 1, 10, -17, 64, ...

[Sol] Let $\{b_n\}$ be the sequence of differences of $\{a_n\}$. Then, $\{b_n\}$ is

1, -3, 9, -27, 81, ...

$$\therefore b_n = 1 \cdot (-3)^{n-1} = (-3)^{n-1}$$

When $n \geq 2$,

$$\begin{aligned} a_n &= a_1 + \sum_{k=1}^{n-1} (-3)^{k-1} \\ &= 3 + \frac{1 - (-3)^{n-1}}{1 - (-3)} \\ &= \frac{13 - (-3)^{n-1}}{4} \dots \textcircled{1} \end{aligned}$$

Since $a_1 = 3$, $\textcircled{1}$ is also true when $n = 1$.

$$\therefore a_n = \frac{13 - (-3)^{n-1}}{4}$$

(4) $\frac{1}{6}, \frac{1}{9}, \frac{1}{14}, \frac{1}{21}, \frac{1}{30}, \frac{1}{41}, \dots$

(Let $b_n = \frac{1}{a_n}$ and $\{c_n\}$ be the sequence of differences of $\{b_n\}$.)

[Sol] Let $b_n = \frac{1}{a_n}$. $\{b_n\}$ is 6, 9, 14, 21, 30, 41,

Let $\{c_n\}$ be the sequence of differences of $\{b_n\}$. Then, $\{c_n\}$ is

3, 5, 7, 9, 11, ...

$$\therefore c_n = 3 + (n-1) \cdot 2 = 2n + 1$$

When $n \geq 2$,

$$\begin{aligned} b_n &= b_1 + \sum_{k=1}^{n-1} (2k + 1) \\ &= 6 + 2 \cdot \frac{1}{2} (n-1)n + (n-1) \\ &= n^2 + 5 \dots \textcircled{1} \end{aligned}$$

Since $b_1 = 6$, $\textcircled{1}$ is also true when $n = 1$.

$$\therefore b_n = n^2 + 5$$

$$\therefore a_n = \frac{1}{b_n} = \frac{1}{n^2 + 5}$$

Various Sequences II

Given that S_n is the sum of the first n terms of the sequence $\{a_n\}$, fill in the following blanks and find the relationship between S_n and a_n .

[Sol] When $n \geq 2$, $S_n = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n \cdots \textcircled{1}$

$$S_{n-1} = a_1 + a_2 + a_3 + \cdots + a_{n-1} \cdots \textcircled{2}$$

From $\textcircled{1} - \textcircled{2}$, $S_n - \boxed{S_{n-1}} = a_n$

Also, $S_1 = \boxed{a_1}$

From the above, the following is true.

Sum of a Sequence and General Term

Let S_n be the sum of the first n terms of the sequence $\{a_n\}$.

The 1st term a_1 is $a_1 = S_1$.

When $n \geq 2$, $a_n = S_n - S_{n-1}$

Find the general term of each given sequence $\{a_n\}$ whose sum of the first n terms is S_n .

Ex. $S_n = 2n^2 - 3n$

[Sol] The 1st term a_1 is $a_1 = S_1 = 2 \cdot 1^2 - 3 \cdot 1 = -1$.

When $n \geq 2$,

$$a_n = S_n - \boxed{S_{n-1}}$$

$$= (2n^2 - 3n) - [2(\boxed{n-1})^2 - 3(\boxed{n-1})]$$

$$= \boxed{4n-5} \cdots \textcircled{1}$$

Since $a_1 = -1$, $\textcircled{1}$ is also true when $n=1$.

$$\therefore a_n = \boxed{4n-5}$$

N34b

(1) $S_n = n^2 + 4n$

[Sol] The 1st term a_1 is $a_1 = S_1 = 1^2 + 4 \cdot 1 = 5$.

When $n \geq 2$,

$$\begin{aligned} a_n &= S_n - S_{n-1} \\ &= (n^2 + 4n) - [(n-1)^2 + 4(n-1)] \\ &= 2n + 3 \quad \dots \textcircled{1} \end{aligned}$$

Since $a_1 = 5$, $\textcircled{1}$ is also true when $n = 1$.

$\therefore a_n = 2n + 3$

(2) $S_n = 2^n - 1$

[Sol] The 1st term a_1 is $a_1 = S_1 = 2^1 - 1 = 1$.

When $n \geq 2$,

$$\begin{aligned} a_n &= S_n - S_{n-1} \\ &= (2^n - 1) - (2^{n-1} - 1) \\ &= 2^{n-1} \quad \dots \textcircled{1} \end{aligned}$$

$$\begin{aligned} 2^n - 2^{n-1} &= 2 \cdot 2^{n-1} - 2^{n-1} \\ &= (2-1)2^{n-1} \end{aligned}$$

Since $a_1 = 1$, $\textcircled{1}$ is also true when $n = 1$.

$\therefore a_n = 2^{n-1}$

Various Sequences II

Find the sum S for each given question.

Ex. $S = 2 \cdot 1 + 4 \cdot 3 + 6 \cdot 3^2 + 8 \cdot 3^3 + \dots + 2n \cdot 3^{n-1} \dots \textcircled{1}$

[Sol] Multiplying both sides of $\textcircled{1}$ by 3,

$$3S = 2 \cdot 3 + 4 \cdot 3^2 + 6 \cdot 3^3 + \dots + 2(n-1) \cdot 3^{n-1} + 2n \cdot 3^n \dots \textcircled{2}$$

From $\textcircled{1} - \textcircled{2}$,

$$-2S = (2 + 2 \cdot 3 + 2 \cdot 3^2 + 2 \cdot 3^3 + \dots + 2 \cdot 3^{n-1}) - 2n \cdot 3^n$$

$$= \frac{2(3^n - 1)}{3 - 1} - 2n \cdot 3^n$$

$$= -(2n - 1) \cdot 3^n - 1$$

$$\therefore S = \frac{(2n - 1) \cdot 3^n + 1}{2}$$

Since $\textcircled{1}$ is

$$S = 2 \cdot 1 + 4 \cdot 3 + \dots + 2(n-1) \cdot 3^{n-1} + 2n \cdot 3^n$$

$$\begin{array}{r} S = 2 \cdot 1 + 4 \cdot 3 + 6 \cdot 3^2 + 8 \cdot 3^3 + \dots + 2n \cdot 3^{n-1} \\ -) 3S = 2 \cdot 3 + 4 \cdot 3^2 + 6 \cdot 3^3 + \dots + 2(n-1) \cdot 3^{n-1} + 2n \cdot 3^n \\ \hline S - 3S = 2 + 2 \cdot 3 + 2 \cdot 3^2 + 2 \cdot 3^3 + \dots + 2 \cdot 3^{n-1} - 2n \cdot 3^n \end{array}$$

(1) $S = 1 \cdot 1 + 2 \cdot 5 + 3 \cdot 5^2 + 4 \cdot 5^3 + \dots + n \cdot 5^{n-1} \dots \textcircled{1}$

[Sol] Multiplying both sides of $\textcircled{1}$ by 5,

$$5S = 1 \cdot 5 + 2 \cdot 5^2 + 3 \cdot 5^3 + \dots + (n-1) \cdot 5^{n-1} + n \cdot 5^n \dots \textcircled{2}$$

From $\textcircled{1} - \textcircled{2}$,

$$-4S = (1 + 5 + 5^2 + 5^3 + \dots + 5^{n-1}) - n \cdot 5^n$$

$$= \frac{5^n - 1}{5 - 1} - n \cdot 5^n$$

$$= -\frac{(4n - 1) \cdot 5^n + 1}{4}$$

$$\therefore S = \frac{(4n - 1) \cdot 5^n + 1}{16}$$

N35b

$$(2) \quad S = n \cdot 1 + (n-1) \cdot 3 + (n-2) \cdot 3^2 + (n-3) \cdot 3^3 + \dots + 2 \cdot 3^{n-2} + 1 \cdot 3^{n-1} \dots \textcircled{1}$$

[Sol] Multiplying both sides of $\textcircled{1}$ by 3,

$$3S = n \cdot 3 + (n-1) \cdot 3^2 + (n-2) \cdot 3^3 + \dots + 3 \cdot 3^{n-2} + 2 \cdot 3^{n-1} + 3^n \dots \textcircled{2}$$

From $\textcircled{1} - \textcircled{2}$,

$$-2S = n - (3 + 3^2 + 3^3 + \dots + 3^{n-2} + 3^{n-1} + 3^n)$$

$$= n - \frac{3(3^n - 1)}{3 - 1}$$

$$= \frac{2n - 3^{n+1} + 3}{2}$$

$$\therefore S = -\frac{2n - 3^{n+1} + 3}{4}$$

$$(3) \quad S = 4 \cdot 1 + 7 \cdot 2 + 10 \cdot 2^2 + 13 \cdot 2^3 + \dots + (3n+1) \cdot 2^{n-1} \dots \textcircled{1}$$

[Sol] Multiplying both sides of $\textcircled{1}$ by 2,

$$2S = 4 \cdot 2 + 7 \cdot 2^2 + 10 \cdot 2^3 + \dots + (3n-2) \cdot 2^{n-1} + (3n+1) \cdot 2^n \dots \textcircled{2}$$

From $\textcircled{1} - \textcircled{2}$,

$$-S = 4 + (3 \cdot 2 + 3 \cdot 2^2 + 3 \cdot 2^3 + \dots + 3 \cdot 2^{n-1}) - (3n+1) \cdot 2^n$$

$$= 4 + \frac{6(2^{n-1} - 1)}{2 - 1} - (3n+1) \cdot 2^n$$

$$= -(3n-2) \cdot 2^n - 2$$

$$\therefore S = (3n-2) \cdot 2^n + 2$$

Alternative Solution

Multiplying both sides of $\textcircled{1}$ by 2,

$$2S = 4 \cdot 2 + 7 \cdot 2^2 + 10 \cdot 2^3 + \dots + (3n-2) \cdot 2^{n-1} + (3n+1) \cdot 2^n \dots \textcircled{2}$$

From $\textcircled{1} - \textcircled{2}$,

$$-S = 1 + (3 + 3 \cdot 2 + 3 \cdot 2^2 + \dots + 3 \cdot 2^{n-1}) - (3n+1) \cdot 2^n$$

$$= 1 + \frac{3(2^n - 1)}{2 - 1} - (3n+1) \cdot 2^n$$

$$= -(3n-2) \cdot 2^n - 2$$

$$\therefore S = (3n-2) \cdot 2^n + 2$$

Various Sequences II

Find the sum S for each given question.

Ex. $S = \frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \dots + \frac{1}{\sqrt{n}+\sqrt{n+1}}$

[Sol] $\frac{1}{\sqrt{k}+\sqrt{k+1}} = \frac{\sqrt{k}-\sqrt{k+1}}{(\sqrt{k}+\sqrt{k+1})(\sqrt{k}-\sqrt{k+1})}$
 $= -\sqrt{k} + \sqrt{k+1}$

Therefore,

$$S = (-\sqrt{1} + \sqrt{2}) + (-\sqrt{2} + \sqrt{3}) + (-\sqrt{3} + \sqrt{4}) + \dots + (-\sqrt{n} + \sqrt{n+1})$$

$$= -1 + \sqrt{n+1}$$

$$\begin{array}{r} -\sqrt{1} + \sqrt{2} \\ -\sqrt{2} + \sqrt{3} \\ -\sqrt{3} + \sqrt{4} \\ \vdots \\ +) -\sqrt{n} + \sqrt{n+1} \\ \hline -1 + \sqrt{n+1} \end{array}$$

(1) $S = \frac{2}{\sqrt{1}+\sqrt{3}} + \frac{2}{\sqrt{3}+\sqrt{5}} + \frac{2}{\sqrt{5}+\sqrt{7}} + \dots + \frac{2}{\sqrt{2n-1}+\sqrt{2n+1}}$

[Sol] $\frac{2}{\sqrt{2k-1}+\sqrt{2k+1}} = \frac{2(\sqrt{2k-1}-\sqrt{2k+1})}{(\sqrt{2k-1}+\sqrt{2k+1})(\sqrt{2k-1}-\sqrt{2k+1})}$
 $= -\sqrt{2k-1} + \sqrt{2k+1}$

Therefore,

$$S = (-\sqrt{1} + \sqrt{3}) + (-\sqrt{3} + \sqrt{5}) + (-\sqrt{5} + \sqrt{7}) + \dots + (-\sqrt{2n-1} + \sqrt{2n+1})$$

$$= -1 + \sqrt{2n+1}$$

N36b

$$(2) S = \frac{1}{\sqrt{2}+\sqrt{5}} + \frac{1}{\sqrt{5}+\sqrt{8}} + \frac{1}{\sqrt{8}+\sqrt{11}} + \dots + \frac{1}{\sqrt{3n-1}+\sqrt{3n+2}}$$

$$\begin{aligned} [\text{Sol}] \frac{1}{\sqrt{3k-1}+\sqrt{3k+2}} &= \frac{\sqrt{3k-1}-\sqrt{3k+2}}{(\sqrt{3k-1}+\sqrt{3k+2})(\sqrt{3k-1}-\sqrt{3k+2})} \\ &= -\frac{1}{3}(\sqrt{3k-1}-\sqrt{3k+2}) \end{aligned}$$

Therefore,

$$\begin{aligned} S &= -\frac{1}{3}(\sqrt{2}-\sqrt{5}) - \frac{1}{3}(\sqrt{5}-\sqrt{8}) - \frac{1}{3}(\sqrt{8}-\sqrt{11}) - \dots - \frac{1}{3}(\sqrt{3n-1}-\sqrt{3n+2}) \\ &= -\frac{1}{3}(\sqrt{2}-\sqrt{3n+2}) \end{aligned}$$

$$(3) S = \frac{1}{\sqrt{1}+\sqrt{3}} + \frac{1}{\sqrt{2}+\sqrt{4}} + \frac{1}{\sqrt{3}+\sqrt{5}} + \dots + \frac{1}{\sqrt{n}+\sqrt{n+2}}$$

$$\begin{aligned} [\text{Sol}] \frac{1}{\sqrt{k}+\sqrt{k+2}} &= \frac{\sqrt{k}-\sqrt{k+2}}{(\sqrt{k}+\sqrt{k+2})(\sqrt{k}-\sqrt{k+2})} \\ &= -\frac{1}{2}(\sqrt{k}-\sqrt{k+2}) \end{aligned}$$

Therefore,

$$\begin{aligned} S &= -\frac{1}{2}(\sqrt{1}-\sqrt{3}) - \frac{1}{2}(\sqrt{2}-\sqrt{4}) - \frac{1}{2}(\sqrt{3}-\sqrt{5}) - \dots \\ &\quad \dots - \frac{1}{2}(\sqrt{n-2}-\sqrt{n}) - \frac{1}{2}(\sqrt{n-1}-\sqrt{n+1}) - \frac{1}{2}(\sqrt{n}-\sqrt{n+2}) \\ &= -\frac{1}{2}(1+\sqrt{2}-\sqrt{n+1}-\sqrt{n+2}) \end{aligned}$$

$$\begin{aligned} &-\frac{1}{2}(\sqrt{1}-\sqrt{3}) \\ &-\frac{1}{2}(\sqrt{2}-\sqrt{4}) \\ &-\frac{1}{2}(\sqrt{3}-\sqrt{5}) \\ &\vdots \\ &-\frac{1}{2}(\sqrt{n-2}-\sqrt{n}) \\ &-\frac{1}{2}(\sqrt{n-1}-\sqrt{n+1}) \\ &+\frac{1}{2}(\sqrt{n}-\sqrt{n+2}) \\ &-\frac{1}{2}(1+\sqrt{2}-\sqrt{n+1}-\sqrt{n+2}) \end{aligned}$$

Various Sequences II

Find the sum S for each given question.

Ex. $S = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)}$

[Sol] Let $\frac{1}{k(k+1)} = \frac{a}{k} - \frac{b}{k+1}$.

$$1 = a(k+1) - bk$$

$$= (a-b)k + a$$

$$\therefore \begin{cases} a-b=0 \\ a=1 \end{cases}$$

$$\therefore a=1, b=1$$

Multiplying both sides by $k(k+1)$

Coefficient Comparison Method (J | 8 |)

Therefore,

$$\begin{aligned} S &= \left(\frac{1}{1} - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1} \right) \\ &= 1 - \frac{1}{n+1} = \frac{n}{n+1} \end{aligned}$$

(1) $S = \frac{2}{1 \cdot 3} + \frac{2}{3 \cdot 5} + \frac{2}{5 \cdot 7} + \cdots + \frac{2}{(2n-1)(2n+1)}$

[Sol] Let $\frac{2}{(2k-1)(2k+1)} = \frac{a}{2k-1} - \frac{b}{2k+1}$.

$$2 = a(2k+1) - b(2k-1)$$

$$= 2(a-b)k + (a+b)$$

$$\therefore \begin{cases} 2(a-b)=0 \\ a+b=2 \end{cases}$$

$$\therefore a=1, b=1$$

Therefore,

$$\begin{aligned} S &= \left(\frac{1}{1} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{5} \right) + \left(\frac{1}{5} - \frac{1}{7} \right) + \cdots + \left(\frac{1}{2n-1} - \frac{1}{2n+1} \right) \\ &= 1 - \frac{1}{2n+1} = \frac{2n}{2n+1} \end{aligned}$$

N37b

$$(2) S = \frac{1}{2 \cdot 4} + \frac{1}{4 \cdot 6} + \frac{1}{6 \cdot 8} + \dots + \frac{1}{2n(2n+2)}$$

[Sol] Let $\frac{1}{2k(2k+2)} = \frac{a}{2k} - \frac{b}{2k+2}$.

$$1 = a(2k+2) - b \cdot 2k$$

$$= 2(a-b)k + 2a$$

$$\therefore \begin{cases} 2(a-b) = 0 \\ 2a = 1 \end{cases}$$

$$\therefore a = \frac{1}{2}, b = \frac{1}{2}$$

Therefore,

$$\begin{aligned} S &= \frac{1}{2} \left(\frac{1}{2} - \frac{1}{4} \right) + \frac{1}{2} \left(\frac{1}{4} - \frac{1}{6} \right) + \frac{1}{2} \left(\frac{1}{6} - \frac{1}{8} \right) + \dots + \frac{1}{2} \left(\frac{1}{2n} - \frac{1}{2n+2} \right) \\ &= \frac{1}{2} \left(\frac{1}{2} - \frac{1}{2n+2} \right) = \frac{n}{4(n+1)} \end{aligned}$$

$$(3) S = \frac{1}{1 \cdot 3} + \frac{1}{2 \cdot 4} + \frac{1}{3 \cdot 5} + \dots + \frac{1}{n(n+2)}$$

[Sol] Let $\frac{1}{k(k+2)} = \frac{a}{k} - \frac{b}{k+2}$.

$$1 = a(k+2) - bk$$

$$= (a-b)k + 2a$$

$$\therefore \begin{cases} a-b = 0 \\ 2a = 1 \end{cases}$$

$$\therefore a = \frac{1}{2}, b = \frac{1}{2}$$

Therefore,

$$\begin{aligned} S &= \frac{1}{2} \left(\frac{1}{1} - \frac{1}{3} \right) + \frac{1}{2} \left(\frac{1}{2} - \frac{1}{4} \right) + \frac{1}{2} \left(\frac{1}{3} - \frac{1}{5} \right) + \dots \\ &\quad \dots + \frac{1}{2} \left(\frac{1}{n-2} - \frac{1}{n} \right) + \frac{1}{2} \left(\frac{1}{n-1} - \frac{1}{n+1} \right) + \frac{1}{2} \left(\frac{1}{n} - \frac{1}{n+2} \right) \\ &= \frac{1}{2} \left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} \right) = \frac{n(3n+5)}{4(n+1)(n+2)} \end{aligned}$$

$$\begin{aligned} &\frac{1}{2} \left(\frac{1}{1} - \frac{1}{3} \right) \\ &\frac{1}{2} \left(\frac{1}{2} - \frac{1}{4} \right) \\ &\frac{1}{2} \left(\frac{1}{3} - \frac{1}{5} \right) \\ &\vdots \\ &\frac{1}{2} \left(\frac{1}{n-2} - \frac{1}{n} \right) \\ &\frac{1}{2} \left(\frac{1}{n-1} - \frac{1}{n+1} \right) \\ &+ \frac{1}{2} \left(\frac{1}{n} - \frac{1}{n+2} \right) \\ &= \frac{1}{2} \left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} \right) \end{aligned}$$

below, assume that the n^{th} group contains ...

1	3, 5	7, 9, 11	13, 15, 17, 19	21, ...
1 st group	2 nd group	3 rd group	4 th group	

[Sol] The number of terms from the 1st to the 9th group is:

1 | 3, 5 | 7, 9, 11 | ... | 9th group
1 term 2 terms 3 terms 9 terms

Since the last term in the 9th group is the 45th term, the first term in the 10th group is $45 + 1 = 46^{(th)}$.

[Sol] The n^{th} group contains n terms.

$$1+2+3+\dots+(n-1)=\sum_{k=1}^{n-1} k = \frac{1}{2}(n-1)n$$

Therefore, the first odd number in the n^{th} group is the $\left[\frac{1}{2}(n-1)n + 1 \right]^{\text{th}}$ term.

Since $x=1$ when $n=1$, ① is also true when $n=1$.

Since the sequence of positive odd numbers is the arithmetic sequence with 1st term 1 and common difference 2, the k^{th} term is $1 + (k-1) \cdot 2 = 2k-1$.

[Sol] Since S is the sum of the arithmetic sequence with 1st term $n^2 - n + 1$, common difference 2 and number of terms n ,

$$S = \frac{1}{2}n[2(n^2 - n + 1) + (n-1) \cdot 2] = n^3 \quad \leftarrow \dots S_n = \frac{1}{2}n[2a + (n-1)d]$$

A set of numbers partitioned under a certain rule is called a **group**.

Answers: in order 46, $\frac{1}{2}(n-1)n$, $\frac{1}{2}(n-1)n$, n^2-n+1 , n^2-n+1 , n^3

N38b

Partitioning the sequence of natural numbers into groups as shown below, assume that the n^{th} group contains $2n$ terms.

1, 2 | 3, 4, 5, 6 | 7, 8, 9, 10, 11, 12 | 13, ...
 1st group 2nd group 3rd group

- (1) The m^{th} term is the first natural number in the 8th group. Find m .

[Sol] The number of terms from the 1st to the 7th group is:

$$2 + 4 + 6 + \dots + 14 = \sum_{k=1}^7 2k = 2 \cdot \frac{1}{2} \cdot 7(7+1) = 56$$

$$\therefore m = 57$$

1, 2 | 3, 4, 5, 6 | ... | 7th group |
 2 terms 4 terms 14 terms

- (2) Find the first natural number x in the n^{th} group.

[Sol] The n^{th} group contains $2n$ terms.

When $n \geq 2$, the number of terms from the 1st to the $(n-1)^{\text{th}}$ group is:

$$2 + 4 + 6 + \dots + 2(n-1) = \sum_{k=1}^{n-1} 2k = 2 \cdot \frac{1}{2} (n-1)n = (n-1)n$$

Therefore, the first natural number in the n^{th} group is the $[(n-1)n + 1]^{\text{th}}$ term.

$$\therefore x = (n-1)n + 1 = n^2 - n + 1 \quad \dots \textcircled{1}$$

The k^{th} term of the sequence of natural numbers is k .

Since $x = 1$ when $n = 1$, $\textcircled{1}$ is also true when $n = 1$.

$$\therefore x = n^2 - n + 1$$

- (3) Find the sum S of all natural numbers in the n^{th} group.

[Sol] Since S is the sum of the arithmetic sequence with 1st term $n^2 - n + 1$, common difference 1 and number of terms $2n$,

$$S = \frac{1}{2} \cdot 2n [2(n^2 - n + 1) + (2n - 1) \cdot 1] = n(2n^2 + 1)$$

Various Sequences II

1. Find the general term of the following sequence.

$$6, 24, 60, 120, 210, 336, 504, \dots$$

[Sol] Let $\{a_n\}$ be the given sequence, $\{b_n\}$ be the sequence of differences of $\{a_n\}$, and $\{c_n\}$ be the sequence of the differences of $\{b_n\}$.

$$\{a_n\}: 6, 24, 60, 120, 210, 336, 504, \dots$$

$$\{b_n\}: 18, 36, 60, 90, 126, 168, \dots$$

$$\{c_n\}: 18, 24, 30, 36, 42, \dots$$

$$\therefore c_n = 18 + (n-1) \cdot 6 = 6n + 12$$

When $n \geq 2$,

$$\begin{aligned} b_n &= b_1 + \sum_{k=1}^{n-1} (6k + 12) \\ &= 18 + 6 \cdot \frac{1}{2}(n-1)n + 12(n-1) \\ &= 3n^2 + 9n + 6 \quad \dots \textcircled{1} \end{aligned}$$

Since $b_1 = 18$, $\textcircled{1}$ is also true when $n = 1$.

$$\therefore b_n = 3n^2 + 9n + 6$$

When $n \geq 2$,

$$\begin{aligned} a_n &= a_1 + \sum_{k=1}^{n-1} (3k^2 + 9k + 6) \\ &= 6 + 3 \cdot \frac{1}{6}(n-1)n(2n-1) + 9 \cdot \frac{1}{2}(n-1)n + 6(n-1) \\ &= n(n+1)(n+2) \quad \dots \textcircled{2} \end{aligned}$$

Since $a_1 = 6$, $\textcircled{2}$ is also true when $n = 1$.

$$\therefore a_n = n(n+1)(n+2)$$

2. Given the sequence $1, \frac{1}{2}, \frac{2}{2}, \frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4}, \dots$, the m^{th} term is $\frac{5}{11}$. Find m . Also, find the 48^{th} term.

[Sol] Partitioning the sequence into groups by the common denominator,

$$1 \left| \frac{1}{2}, \frac{2}{2} \right| \frac{1}{3}, \frac{2}{3}, \frac{3}{3} \left| \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4} \right| \dots$$

Therefore, $\frac{5}{11}$ is the 5^{th} term in the 11^{th} group. $\leftarrow \dots$

Since the n^{th} group contains n terms,

the number of terms from the 1^{st} to

the 10^{th} group is:

$$1 + 2 + 3 + \dots + 10 = \sum_{k=1}^{10} k = \frac{1}{2} \cdot 10(10+1) = 55$$

$$\therefore m = 60$$

The 11^{th} group contains the fractions with denominator 11.

$$\dots \left| \frac{1}{11}, \frac{2}{11}, \frac{3}{11}, \frac{4}{11}, \frac{5}{11}, \dots, \frac{11}{11} \right| \dots$$

$1^{\text{st}} \qquad \qquad \qquad 5^{\text{th}}$

Since $\frac{5}{11}$ is the 5^{th} term in the 11^{th} group,
 $55 + 5 = 60$

Also, let the 48^{th} term be in the n^{th} group.

$$\frac{1}{2}(n-1)n < 48 \leq \frac{1}{2}n(n+1)$$

$$(n-1)n < 96 \leq n(n+1)$$

The natural number n that satisfies this $\leftarrow \dots$

is $n = 10$.

Finding the natural number n that satisfies $(n-1)n < 96 \leq n(n+1)$ by substituting some values into n

The number of terms from the 1^{st} to the 9^{th} group is:

$$1 + 2 + 3 + \dots + 9 = \sum_{k=1}^9 k = \frac{1}{2} \cdot 9(9+1) = 45$$

Therefore, the 48^{th} term is $\frac{3}{10}$. $\leftarrow \dots$

Since $48 - 45 = 3$, the 48^{th} term is the 3^{rd} term in the 10^{th} group.

Various Sequences II

1. Find the general term of the following sequence $\{a_n\}$.

➡ N32

7, 19, 37, 61, 91, 127, ...

[Sol] Let $\{b_n\}$ be the sequence of differences of $\{a_n\}$. Then, $\{b_n\}$ is

12, 18, 24, 30, 36, ...

$$\therefore b_n = 12 + (n-1) \cdot 6 = 6n + 6$$

When $n \geq 2$,

$$\begin{aligned} a_n &= a_1 + \sum_{k=1}^{n-1} (6k+6) \\ &= 7 + 6 \cdot \frac{1}{2} (n-1)n + 6(n-1) \\ &= 3n^2 + 3n + 1 \quad \dots \textcircled{1} \end{aligned}$$

Since $a_1 = 7$, $\textcircled{1}$ is also true when $n=1$.

$$\therefore a_n = 3n^2 + 3n + 1$$

2. Find the general term of the sequence $\{a_n\}$ whose sum of the first n terms is S_n .

➡ N34

$$S_n = 5n - n^2$$

[Sol] The 1st term a_1 is $a_1 = S_1 = 5 \cdot 1 - 1^2 = 4$.

When $n \geq 2$,

$$\begin{aligned} a_n &= S_n - S_{n-1} \\ &= (5n - n^2) - [5(n-1) - (n-1)^2] \\ &= -2n + 6 \quad \dots \textcircled{1} \end{aligned}$$

Since $a_1 = 4$, $\textcircled{1}$ is also true when $n=1$.

$$\therefore a_n = -2n + 6$$

N40b

3. Find the sum S for each given question.

$$(1) S = 1 \cdot 1 + 2 \cdot 3 + 3 \cdot 3^2 + 4 \cdot 3^3 + \dots + n \cdot 3^{n-1} \dots \textcircled{1}$$

[Sol] Multiplying both sides of $\textcircled{1}$ by 3,

➡ N35

$$3S = 1 \cdot 3 + 2 \cdot 3^2 + 3 \cdot 3^3 + \dots + (n-1) \cdot 3^{n-1} + n \cdot 3^n \dots \textcircled{2}$$

From $\textcircled{1} - \textcircled{2}$,

$$-2S = (1 + 3 + 3^2 + 3^3 + \dots + 3^{n-1}) - n \cdot 3^n$$

$$= \frac{3^n - 1}{3 - 1} - n \cdot 3^n$$

$$= -\frac{(2n-1) \cdot 3^n + 1}{2}$$

$$\therefore S = \frac{(2n-1) \cdot 3^n + 1}{4}$$

$$(2) S = \frac{3}{1 \cdot 4} + \frac{3}{4 \cdot 7} + \frac{3}{7 \cdot 10} + \dots + \frac{3}{(3n-2)(3n+1)}$$

➡ N37

[Sol] Let $\frac{3}{(3k-2)(3k+1)} = \frac{a}{3k-2} - \frac{b}{3k+1}$.

$$3 = a(3k+1) - b(3k-2)$$

$$= 3(a-b)k + (a+2b)$$

$$\therefore \begin{cases} 3(a-b) = 0 \\ a+2b = 3 \end{cases}$$

$$\therefore a = 1, b = 1$$

Therefore,

$$S = \left(\frac{1}{1} - \frac{1}{4} \right) + \left(\frac{1}{4} - \frac{1}{7} \right) + \left(\frac{1}{7} - \frac{1}{10} \right) + \dots + \left(\frac{1}{3n-2} - \frac{1}{3n+1} \right)$$

$$= 1 - \frac{1}{3n+1} = \frac{3n}{3n+1}$$

Given that the sequence $\{a_n\}$ satisfies the following two conditions:

(i) $a_1 = 2$

(ii) $a_{n+1} = 3a_n - 1 \quad (n = 1, 2, 3, \dots),$

fill in the following blanks.

[Sol] $a_2 = 3a_1 - 1 = 3 \cdot \boxed{2} - 1 = \boxed{5}$

$a_3 = 3a_2 - 1 = 3 \cdot \boxed{5} - 1 = \boxed{14}$

$a_4 = 3a_3 - 1 = 3 \cdot \boxed{14} - 1 = \boxed{41}$

Answers: in order 2, 5, 14, 41

From the above, each term is successively determined from a_1 to $a_2, a_3, a_4, \dots$ one after another, then one sequence $\{a_n\}$ will be defined. A condition such as (ii), where the expression defines the relationship between two or more consecutive terms in a sequence, is called a **recurrence relation**.

Find the 4th term of each given sequence $\{a_n\}$ defined by the following conditions.

Ex. $a_1 = 4, a_{n+1} = 2a_n - 3$

[Sol] $a_2 = 2a_1 - 3 = 5$

$a_3 = 2a_2 - 3 = 7$

$\therefore a_4 = 2a_3 - 3 = 11$

(1) $a_1 = 1, a_{n+1} = 3a_n + 2$

[Sol] $a_2 = 3a_1 + 2 = 5$

$a_3 = 3a_2 + 2 = 17$

$\therefore a_4 = 3a_3 + 2 = 53$

Unless otherwise specified, the recurrence relation is defined by $n = 1, 2, 3, \dots$

N41b

$$(2) \quad a_1 = -2, \quad a_{n+1} = a_n - 5$$

$$[\text{Sol}] \quad a_2 = a_1 - 5 = -7$$

$$a_3 = a_2 - 5 = -12$$

$$\therefore a_4 = a_3 - 5 = -17$$

$$(3) \quad a_1 = 2, \quad a_{n+1} = 3a_n$$

$$[\text{Sol}] \quad a_2 = 3a_1 = 6$$

$$a_3 = 3a_2 = 18$$

$$\therefore a_4 = 3a_3 = 54$$

$$(4) \quad a_1 = 1, \quad a_{n+1} = 2 - \frac{1}{a_n}$$

$$[\text{Sol}] \quad a_2 = 2 - \frac{1}{a_1} = 1$$

$$a_3 = 2 - \frac{1}{a_2} = 1$$

$$\therefore a_4 = 2 - \frac{1}{a_3} = 1$$

$$(5) \quad a_1 = 5, \quad a_{n+1} = a_n + n$$

$$[\text{Sol}] \quad a_2 = a_1 + 1 = 6$$

$$a_3 = a_2 + 2 = 8$$

$$\therefore a_4 = a_3 + 3 = 11$$

$$(6) \quad a_1 = 1, \quad a_2 = 2, \quad a_{n+2} = a_{n+1} + a_n$$

$$[\text{Sol}] \quad a_3 = a_2 + a_1 = 3$$

$$\therefore a_4 = a_3 + a_2 = 5$$

N42a

Recurrence Relations

Find the general term of each given sequence $\{a_n\}$ defined by the following conditions.

Ex. $a_1=1, a_{n+1}=a_n+5$

[Sol] $a_1=1$

$$a_2=a_1+5=6$$

$$a_3=a_2+5=11$$

$\vdots$

Therefore, $\{a_n\}$ is the arithmetic sequence with 1st term 1 and common difference 5.

$$\therefore a_n = 1 + (n-1) \cdot 5$$

$$= 5n - 4$$

The general term of an arithmetic sequence is $a_n = a + (n-1)d$.

$$a_1=3, a_{n+1}=2a_n$$

[Sol] $a_1=3$

$$a_2=2a_1=6$$

$$a_3=2a_2=12$$

$\vdots$

Therefore, $\{a_n\}$ is the geometric sequence with 1st term 3 and common ratio 2.

$$\therefore a_n = 3 \cdot 2^{n-1}$$

The general term of a geometric sequence is $a_n = ar^{n-1}$.

(1) $a_1=4, a_{n+1}=a_n+8$

[Sol] $a_1=4$

$$a_2=a_1+8=12$$

$$a_3=a_2+8=20$$

$\vdots$

Therefore, $\{a_n\}$ is the arithmetic sequence with 1st term 4 and common difference 8.

$$\therefore a_n = 4 + (n-1) \cdot 8$$

$$= 8n - 4$$

(2) $a_1=a, a_{n+1}=a_n+d$

[Sol] $a_1=a$

$$a_2=a_1+d=a+d$$

$$a_3=a_2+d=a+2d$$

$\vdots$

Therefore, $\{a_n\}$ is the arithmetic sequence with 1st term a and common difference d .

$$\therefore a_n = a + (n-1)d$$

(3) $a_1=5, a_{n+1}=4a_n$

[Sol] $a_1=5$

$$a_2=4a_1=20$$

$$a_3=4a_2=80$$

$\vdots$

Therefore, $\{a_n\}$ is the geometric sequence with 1st term 5 and common ratio 4.

$$\therefore a_n = 5 \cdot 4^{n-1}$$

(4) $a_1=a, a_{n+1}=ra_n$

[Sol] $a_1=a$

$$a_2=ra_1=ar$$

$$a_3=ra_2=ar^2$$

$\vdots$

Therefore, $\{a_n\}$ is the geometric sequence with 1st term a and common ratio r .

$$\therefore a_n = ar^{n-1}$$

N42b

Based on the results on side a, the following statements are true.
 The sequence $\{a_n\}$ defined by $a_1 = a$, $a_{n+1} = a_n + d$ is an arithmetic sequence with 1st term a and common difference d .
 The sequence $\{a_n\}$ defined by $a_1 = a$, $a_{n+1} = ra_n$ is a geometric sequence with 1st term a and common ratio r .

Ex $a_1 = 2$, $a_{n+1} = a_n + 3$

[Sol] $a_n = 2 + (n-1) \cdot 3$
 $= 3n - 1$

$\{a_n\}$ is the arithmetic sequence with 1st term 2 and common difference 3.

$a_1 = 3$, $a_{n+1} = 7a_n$

[Sol] $a_n = 3 \cdot 7^{n-1}$

$\{a_n\}$ is the geometric sequence with 1st term 3 and common ratio 7.

(5) $a_1 = -7$, $a_{n+1} = a_n + 5$

[Sol] $a_n = -7 + (n-1) \cdot 5$
 $= 5n - 12$

(8) $a_1 = 2$, $a_{n+1} = 3a_n$

[Sol] $a_n = 2 \cdot 3^{n-1}$

(6) $a_1 = 3$, $a_{n+1} = a_n - 7$

[Sol] $a_n = 3 + (n-1) \cdot (-7)$
 $= -7n + 10$

(9) $a_1 = -5$, $a_{n+1} = -2a_n$

[Sol] $a_n = -5(-2)^{n-1}$

(7) $a_1 = -4$, $a_{n+1} = a_n - 9$

[Sol] $a_n = -4 + (n-1) \cdot (-9)$
 $= -9n + 5$

(10) $a_1 = -7$, $a_{n+1} = 7a_n$

[Sol] $a_n = -7 \cdot 7^{n-1}$
 $= -7^n$

Find the general term of each given sequence $\{a_n\}$ defined by the following conditions.

Ex. $a_1=1, a_{n+1}=a_n+3n$

[Sol] $a_{n+1}-a_n=3n$

Since the general term of the sequence of differences of $\{a_n\}$ is $3n$,

when $n \geq 2$,

$$\begin{aligned} a_n &= a_1 + \sum_{k=1}^{n-1} 3k = 1 + 3 \cdot \frac{1}{2} (n-1)n \\ &= \frac{1}{2} (3n^2 - 3n + 2) \quad \dots \textcircled{1} \end{aligned}$$

Since $a_1=1$, $\textcircled{1}$ is also true when $n=1$.

$$\therefore a_n = \frac{1}{2} (3n^2 - 3n + 2)$$

Let $\{b_n\}$ be the sequence of differences of $\{a_n\}$.

$\{a_n\}: a_1, a_2, a_3, \dots, a_n, a_{n+1}, \dots$
 $\{b_n\}: 3, 6, \dots, 3n, \dots$
 Therefore, the general term of $\{b_n\}$ is $3n$.

When $n \geq 2$,

$$a_n = a_1 + \sum_{k=1}^{n-1} b_k \quad (\text{N32})$$

(1) $a_1=3, a_{n+1}=a_n+6n$

[Sol] $a_{n+1}-a_n=6n$

Since the general term of the sequence of differences of $\{a_n\}$ is $6n$,
 when $n \geq 2$,

$$\begin{aligned} a_n &= a_1 + \sum_{k=1}^{n-1} 6k = 3 + 6 \cdot \frac{1}{2} (n-1)n \\ &= 3(n^2 - n + 1) \quad \dots \textcircled{1} \end{aligned}$$

Since $a_1=3$, $\textcircled{1}$ is also true when $n=1$.

$$\therefore a_n = 3(n^2 - n + 1)$$

(2) $a_1=2, a_{n+1}=a_n+n^2$

[Sol] $a_{n+1}-a_n=n^2$

Since the general term of the sequence of differences of $\{a_n\}$ is n^2 ,
 when $n \geq 2$,

$$\begin{aligned} a_n &= a_1 + \sum_{k=1}^{n-1} k^2 = 2 + \frac{1}{6} (n-1)n(2n-1) \\ &= \frac{1}{6} (2n^3 - 3n^2 + n + 12) \quad \dots \textcircled{1} \end{aligned}$$

Since $a_1=2$, $\textcircled{1}$ is also true when $n=1$.

$$\therefore a_n = \frac{1}{6} (2n^3 - 3n^2 + n + 12)$$

$$\sum_{k=1}^n k^2 = \frac{1}{6} n(n+1)(2n+1)$$

N43b

(3) $a_1=4, a_{n+1}=a_n+4n^3$

[Sol] $a_{n+1}-a_n=4n^3$

Since the general term of the sequence of differences of $\{a_n\}$ is $4n^3$,
when $n \geq 2$,

$$a_n = a_1 + \sum_{k=1}^{n-1} 4k^3 = 4 + 4 \left[\frac{1}{2}(n-1)n \right]^2 \quad \leftarrow \sum_{k=1}^n k^3 = \left[\frac{1}{2}n(n+1) \right]^2$$

$$= n^4 - 2n^3 + n^2 + 4 \quad \dots \textcircled{1}$$

Since $a_1=4$, $\textcircled{1}$ is also true when $n=1$.

$\therefore a_n = n^4 - 2n^3 + n^2 + 4$

(4) $a_1=3, a_{n+1}=a_n+2^{n-1}$

[Sol] $a_{n+1}-a_n=2^{n-1}$

Since the general term of the sequence of differences of $\{a_n\}$ is 2^{n-1} ,
when $n \geq 2$,

$$a_n = a_1 + \sum_{k=1}^{n-1} 2^{k-1} = 3 + \frac{2^{n-1}-1}{2-1} \quad \leftarrow \sum_{k=1}^n ar^{k-1} = \frac{a(r^n-1)}{r-1} \quad (r \neq 1)$$

$$= 2^{n-1} + 2 \quad \dots \textcircled{1}$$

Since $a_1=3$, $\textcircled{1}$ is also true when $n=1$.

$\therefore a_n = 2^{n-1} + 2$

(5) $a_1=4, a_{n+1}=a_n+5^n$

[Sol] $a_{n+1}-a_n=5^n$

Since the general term of the sequence of differences of $\{a_n\}$ is 5^n ,
when $n \geq 2$,

$$a_n = a_1 + \sum_{k=1}^{n-1} 5^k = 4 + \sum_{k=1}^{n-1} 5 \cdot 5^{k-1}$$

$$= 4 + \frac{5(5^{n-1}-1)}{5-1}$$

$$= \frac{5^n + 11}{4} \quad \dots \textcircled{1}$$

Since $a_1=4$, $\textcircled{1}$ is also true when $n=1$.

$\therefore a_n = \frac{5^n + 11}{4}$

Recurrence Relations

Assume that the recurrence relation $a_{n+1} = pa_n + q \dots ①$ is true when p and q are constants ($p \neq 0, p \neq 1$). By using x which satisfies $x = px + q$ where a_{n+1} and a_n are both replaced with x , prove that the recurrence relation ① can be rearranged into $a_{n+1} - x = p(a_n - x)$.

[Sol] Since $x = px + q, x = \frac{q}{1-p} \dots ②$

Subtracting the RHS of ② from the both sides of ①,

$$\text{LHS} = a_{n+1} - \frac{q}{1-p} \dots ③$$

$$\begin{aligned} \text{RHS} &= pa_n + q - \frac{q}{1-p} = pa_n + \frac{q(1-p) - q}{1-p} \\ &= pa_n - \frac{pq}{1-p} = p \left(a_n - \frac{q}{1-p} \right) \dots ④ \end{aligned}$$

From ③ and ④, $a_{n+1} - x = p(a_n - x)$

Find the general term of each given sequence $\{a_n\}$ defined by the following conditions.

Ex. $a_1 = 3, a_{n+1} = 3a_n - 2$

[Sol] Since $a_{n+1} = 3a_n - 2, a_{n+1} - 1 = 3(a_n - 1)$

Let $b_n = a_n - 1$.

$$\begin{cases} b_{n+1} = 3b_n \dots ① \end{cases}$$

$$\begin{cases} b_1 = a_1 - 1 = 2 \dots ② \end{cases}$$

From ① and ②, $b_n = 2 \cdot 3^{n-1}$

$$\therefore a_n = b_n + 1 = 2 \cdot 3^{n-1} + 1$$

Replacing both a_{n+1} and a_n with x ,
 $x = 3x - 2$
 $\therefore x = 1$

$\{b_n\}$ is the geometric sequence with 1st term $b_1 = 2$ and common ratio 3.

N44b

(1) $a_1=4, a_{n+1}=2a_n+1$

[Sol] Since $a_{n+1}=2a_n+1$, $a_{n+1}+1=2(a_n+1)$
Let $b_n=a_n+1$.

$$\begin{cases} b_{n+1}=2b_n & \dots \textcircled{1} \\ b_1=a_1+1=5 & \dots \textcircled{2} \end{cases}$$

From ① and ②, $b_n=5 \cdot 2^{n-1}$

$$\therefore a_n=b_n-1=5 \cdot 2^{n-1}-1$$

(2) $a_1=1, a_{n+1}=2a_n+3$

[Sol] Since $a_{n+1}=2a_n+3$, $a_{n+1}+3=2(a_n+3)$

Let $b_n=a_n+3$.

$$\begin{cases} b_{n+1}=2b_n & \dots \textcircled{1} \\ b_1=a_1+3=4 & \dots \textcircled{2} \end{cases}$$

From ① and ②, $b_n=4 \cdot 2^{n-1}=2^{n+1}$

$$\therefore a_n=b_n-3=2^{n+1}-3$$

(3) $a_1=1, 3a_{n+1}-2a_n=3$

[Sol] Since $3a_{n+1}-2a_n=3$, $a_{n+1}=\frac{2}{3}a_n+1$; therefore,

$$a_{n+1}-3=\frac{2}{3}(a_n-3)$$

Let $b_n=a_n-3$.

$$\begin{cases} b_{n+1}=\frac{2}{3}b_n & \dots \textcircled{1} \\ b_1=a_1-3=-2 & \dots \textcircled{2} \end{cases}$$

From ① and ②, $b_n=-2\left(\frac{2}{3}\right)^{n-1}$

$$\therefore a_n=b_n+3=-2\left(\frac{2}{3}\right)^{n-1}+3$$

← Replacing both a_{n+1} and a_n with x ,
 $x=2x+1$
 $\therefore x=-1$

← Replacing both a_{n+1} and a_n with x ,
 $x=2x+3$
 $\therefore x=-3$

← Replacing both a_{n+1} and a_n with x ,
 $x=\frac{2}{3}x+1$
 $\therefore x=3$

Note Summary

The recurrence relation $a_{n+1}=pa_n+q$ can be rearranged into $a_{n+1}-x=p(a_n-x)$ by using x which satisfies $x=px+q$. If $b_n=a_n-x$, then the sequence $\{b_n\}$ is a geometric sequence. Therefore, the general term of the sequence $\{a_n\}$ can be determined.

Recurrence Relations

Find the general term of each given sequence $\{a_n\}$ defined by the following conditions.

Ex.

$$a_1 = 1, a_{n+1} = \frac{a_n}{4a_n + 2}$$

[Sol] Since $a_{n+1} = \frac{a_n}{4a_n + 2}$, $\frac{1}{a_{n+1}} = \frac{4a_n + 2}{a_n} = \frac{2}{a_n} + 4$

Let $b_n = \frac{1}{a_n}$. Since $b_{n+1} = 2b_n + 4$,

$$\begin{cases} b_{n+1} + 4 = 2(b_n + 4) \dots \textcircled{1} \\ b_1 + 4 = \frac{1}{a_1} + 4 = 5 \dots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}$, $b_n + 4 = 5 \cdot 2^{n-1}$

$$b_n = 5 \cdot 2^{n-1} - 4$$

$$\therefore a_n = \frac{1}{b_n} = \frac{1}{5 \cdot 2^{n-1} - 4}$$

Substituting $\frac{1}{a_n} = b_n$

and $\frac{1}{a_{n+1}} = b_{n+1}$ into

$$\frac{1}{a_{n+1}} = \frac{2}{a_n} + 4$$

Replacing both b_{n+1} and b_n with x , $x = 2x + 4$

$$\therefore x = -4$$

$\{b_n + 4\}$ is the geometric sequence with 1st term $b_1 + 4 = 5$ and common ratio 2.

(1) $a_1 = 2, a_{n+1} = \frac{a_n}{2a_n + 3}$

[Sol] Since $a_{n+1} = \frac{a_n}{2a_n + 3}$, $\frac{1}{a_{n+1}} = \frac{2a_n + 3}{a_n} = \frac{3}{a_n} + 2$

Let $b_n = \frac{1}{a_n}$. Since $b_{n+1} = 3b_n + 2$,

$$\begin{cases} b_{n+1} + 1 = 3(b_n + 1) \dots \textcircled{1} \\ b_1 + 1 = \frac{1}{a_1} + 1 = \frac{3}{2} \dots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}$, $b_n + 1 = \frac{3}{2} \cdot 3^{n-1} = \frac{3^n}{2}$

$$b_n = \frac{3^n}{2} - 1$$

$$\therefore a_n = \frac{1}{b_n} = \frac{1}{\frac{3^n}{2} - 1} = \frac{2}{3^n - 2}$$

N45b

(2) $a_1=1, a_{n+1}=\frac{2a_n}{a_n+10}$

[Sol] Since $a_{n+1}=\frac{2a_n}{a_n+10}, \frac{1}{a_{n+1}}=\frac{a_n+10}{2a_n}=\frac{5}{a_n}+\frac{1}{2}$

Let $b_n=\frac{1}{a_n}$. Since $b_{n+1}=5b_n+\frac{1}{2}$,

$$\begin{cases} b_{n+1}+\frac{1}{8}=5\left(b_n+\frac{1}{8}\right) \dots \textcircled{1} \end{cases}$$

$$\begin{cases} b_1+\frac{1}{8}=\frac{1}{a_1}+\frac{1}{8}=\frac{9}{8} \dots \textcircled{2} \end{cases}$$

From ① and ②, $b_n+\frac{1}{8}=\frac{9}{8}\cdot 5^{n-1}$

$$b_n=\frac{9}{8}\cdot 5^{n-1}-\frac{1}{8}$$

$$\therefore a_n=\frac{1}{b_n}=\frac{1}{\frac{9}{8}\cdot 5^{n-1}-\frac{1}{8}}=\frac{8}{9\cdot 5^{n-1}-1}$$

(3) $a_1=3, a_{n+1}=\frac{2a_n}{3a_n+5}$

[Sol] Since $a_{n+1}=\frac{2a_n}{3a_n+5}, \frac{1}{a_{n+1}}=\frac{3a_n+5}{2a_n}=\frac{5}{2a_n}+\frac{3}{2}$

Let $b_n=\frac{1}{a_n}$. Since $b_{n+1}=\frac{5}{2}b_n+\frac{3}{2}$,

$$\begin{cases} b_{n+1}+1=\frac{5}{2}(b_n+1) \dots \textcircled{1} \end{cases}$$

$$\begin{cases} b_1+1=\frac{1}{a_1}+1=\frac{4}{3} \dots \textcircled{2} \end{cases}$$

From ① and ②, $b_n+1=\frac{4}{3}\left(\frac{5}{2}\right)^{n-1}$

$$b_n=\frac{4}{3}\left(\frac{5}{2}\right)^{n-1}-1$$

$$\therefore a_n=\frac{1}{b_n}=\frac{1}{\frac{4}{3}\left(\frac{5}{2}\right)^{n-1}-1}=\frac{3\cdot 2^{n-1}}{4\cdot 5^{n-1}-3\cdot 2^{n-1}}$$

Recurrence Relations

Find the general term of each given sequence $\{a_n\}$ defined by the following conditions.

Ex. $a_1 = 1, a_{n+1} = 2a_n - 3^n$

[Sol] Dividing both sides of $a_{n+1} = 2a_n - 3^n$ by 3^{n+1} ,

$$\frac{a_{n+1}}{3^{n+1}} = \frac{2}{3} \cdot \frac{a_n}{3^n} - \frac{1}{3}$$

Let $b_n = \frac{a_n}{3^n}$. Since $b_{n+1} = \frac{2}{3}b_n - \frac{1}{3}$,

$$\begin{cases} b_{n+1} + 1 = \frac{2}{3}(b_n + 1) \quad \dots ① \end{cases}$$

$$\begin{cases} b_1 + 1 = \frac{a_1}{3^1} + 1 = \frac{4}{3} \quad \dots ② \end{cases}$$

From ① and ②, $b_n + 1 = \frac{4}{3} \left(\frac{2}{3} \right)^{n-1} = 2 \left(\frac{2}{3} \right)^n$

$$b_n = 2 \left(\frac{2}{3} \right)^n - 1$$

$$\therefore a_n = 3^n b_n = 3^n \left[2 \left(\frac{2}{3} \right)^n - 1 \right] = 2^{n+1} - 3^n$$

Substituting $\frac{a_n}{3^n} = b_n$

and $\frac{a_{n+1}}{3^{n+1}} = b_{n+1}$ into

$$\frac{a_{n+1}}{3^{n+1}} = \frac{2}{3} \cdot \frac{a_n}{3^n} - \frac{1}{3}$$

Replacing both b_{n+1} and

$$b_n \text{ with } x, x = \frac{2}{3}x - \frac{1}{3}$$

$$\therefore x = -1$$

$\{b_n + 1\}$ is the geometric sequence with 1st term

$$b_1 + 1 = \frac{4}{3} \text{ and}$$

common ratio $\frac{2}{3}$.

(1) $a_1 = 1, a_{n+1} = 3a_n + 2^n$

[Sol] Dividing both sides of $a_{n+1} = 3a_n + 2^n$ by 2^{n+1} ,

$$\frac{a_{n+1}}{2^{n+1}} = \frac{3}{2} \cdot \frac{a_n}{2^n} + \frac{1}{2}$$

Let $b_n = \frac{a_n}{2^n}$. Since $b_{n+1} = \frac{3}{2}b_n + \frac{1}{2}$,

$$\begin{cases} b_{n+1} + 1 = \frac{3}{2}(b_n + 1) \quad \dots ① \end{cases}$$

$$\begin{cases} b_1 + 1 = \frac{a_1}{2^1} + 1 = \frac{3}{2} \quad \dots ② \end{cases}$$

From ① and ②, $b_n + 1 = \frac{3}{2} \left(\frac{3}{2} \right)^{n-1} = \left(\frac{3}{2} \right)^n$

$$b_n = \left(\frac{3}{2} \right)^n - 1$$

$$\therefore a_n = 2^n b_n = 2^n \left[\left(\frac{3}{2} \right)^n - 1 \right] = 3^n - 2^n$$

N46b

(2) $a_1 = \frac{1}{8}, a_{n+1} = 3^{n+1} - 5a_n$

[Sol] Dividing both sides of $a_{n+1} = 3^{n+1} - 5a_n$ by 3^{n+1} ,

$$\frac{a_{n+1}}{3^{n+1}} = 1 - \frac{5}{3} \cdot \frac{a_n}{3^n}$$

Let $b_n = \frac{a_n}{3^n}$. Since $b_{n+1} = 1 - \frac{5}{3}b_n$,

$$\begin{cases} b_{n+1} - \frac{3}{8} = -\frac{5}{3} \left(b_n - \frac{3}{8} \right) \dots \textcircled{1} \end{cases}$$

$$\begin{cases} b_1 - \frac{3}{8} = \frac{a_1}{3^1} - \frac{3}{8} = -\frac{1}{3} \dots \textcircled{2} \end{cases}$$

From ① and ②, $b_n - \frac{3}{8} = -\frac{1}{3} \left(-\frac{5}{3} \right)^{n-1}$

$$b_n = -\frac{1}{3} \left(-\frac{5}{3} \right)^{n-1} + \frac{3}{8}$$

$$\therefore a_n = 3^n b_n = 3^n \left[-\frac{1}{3} \left(-\frac{5}{3} \right)^{n-1} + \frac{3}{8} \right] = -(-5)^{n-1} + \frac{3^{n+1}}{8}$$

(3) $a_1 = 1, a_{n+1} = a_n + \frac{1}{2^n}$

[Sol] Multiplying both sides of $a_{n+1} = a_n + \frac{1}{2^n}$ by 2^{n+1} ,

$$2^{n+1} a_{n+1} = 2 \cdot 2^n a_n + 2$$

Let $b_n = 2^n a_n$. Since $b_{n+1} = 2b_n + 2$,

$$\begin{cases} b_{n+1} + 2 = 2(b_n + 2) \dots \textcircled{1} \end{cases}$$

$$\begin{cases} b_1 + 2 = 2^1 a_1 + 2 = 4 \dots \textcircled{2} \end{cases}$$

From ① and ②, $b_n + 2 = 4 \cdot 2^{n-1} = 2^{n+1}$

$$b_n = 2^{n+1} - 2$$

$$\therefore a_n = \frac{b_n}{2^n} = \frac{1}{2^n} (2^{n+1} - 2) = 2 - \frac{1}{2^{n-1}}$$

[Sol] Since two solutions of $x^2 + px + q = 0$ are α and β , from root-coefficient relationships,

From root-coefficient relationships, let two solutions of $ax^2 + bx + c = 0$ ($a \neq 0$) be α and β .

$$\alpha + \beta = -\frac{b}{a}, \quad \alpha\beta = \frac{c}{a} \quad (J131)$$

$$\therefore a_{n+2} - \alpha a_{n+1} = \beta(a_{n+1} - \alpha a_n) \dots \textcircled{4}$$

Find the general term of each given sequence $\{a_n\}$ defined by the following conditions.

Replacing

$$\begin{aligned} & a_{n+2}, a_{n+1}, a_n \text{ with} \\ & x^2, x, 1 \text{ respectively,} \\ & x^2 - 5x + 6 = 0 \\ & (x-2)(x-3) = 0 \\ & \therefore x = 2, 3 \end{aligned}$$

From ③ and ④, $a_n = 2 \cdot 3^{n-1} - 2^n$

③—④

N47b

(1) $a_1=1, a_2=5, a_{n+2}-7a_{n+1}+12a_n=0$

[Sol] Rearranging $a_{n+2}-7a_{n+1}+12a_n=0$,

$$\begin{cases} a_{n+2}-3a_{n+1}=4(a_{n+1}-3a_n) \dots \textcircled{1} \\ a_{n+2}-4a_{n+1}=3(a_{n+1}-4a_n) \dots \textcircled{2} \end{cases}$$

Replacing a_{n+2}, a_{n+1}, a_n with $x^2, x, 1$ respectively,
 $x^2-7x+12=0$
 $(x-3)(x-4)=0$
 $\therefore x=3, 4$

From $\textcircled{1}$, the sequence $\{a_{n+1}-3a_n\}$ is the geometric sequence with 1st term $a_2-3a_1=2$ and common ratio 4.

$$\therefore a_{n+1}-3a_n=2 \cdot 4^{n-1} \dots \textcircled{3}$$

From $\textcircled{2}$, the sequence $\{a_{n+1}-4a_n\}$ is the geometric sequence with 1st term $a_2-4a_1=1$ and common ratio 3.

$$\therefore a_{n+1}-4a_n=3^{n-1} \dots \textcircled{4}$$

From $\textcircled{3}$ and $\textcircled{4}$, $a_n=2 \cdot 4^{n-1}-3^{n-1} \quad [=2^{2n-1}-3^{n-1}]$

(2) $a_1=0, a_2=1, 5a_{n+1}=3a_{n+2}+2a_n$

[Sol] Since $5a_{n+1}=3a_{n+2}+2a_n$, $3a_{n+2}-5a_{n+1}+2a_n=0 \dots \textcircled{1}$

Rearranging $\textcircled{1}$,

$$\begin{cases} a_{n+2}-\frac{2}{3}a_{n+1}=a_{n+1}-\frac{2}{3}a_n \dots \textcircled{2} \\ a_{n+2}-a_{n+1}=\frac{2}{3}(a_{n+1}-a_n) \dots \textcircled{3} \end{cases}$$

Replacing a_{n+2}, a_{n+1}, a_n with $x^2, x, 1$ respectively,
 $3x^2-5x+2=0$
 $(3x-2)(x-1)=0$
 $\therefore x=\frac{2}{3}, 1$

From $\textcircled{2}$, the sequence $\{a_{n+1}-\frac{2}{3}a_n\}$ is the geometric sequence with 1st term

$$a_2-\frac{2}{3}a_1=1 \text{ and common ratio } 1.$$

$$\therefore a_{n+1}-\frac{2}{3}a_n=1 \dots \textcircled{4}$$

From $\textcircled{3}$, the sequence $\{a_{n+1}-a_n\}$ is the geometric sequence with 1st term

$$a_2-a_1=1 \text{ and common ratio } \frac{2}{3}.$$

$$\therefore a_{n+1}-a_n=\left(\frac{2}{3}\right)^{n-1} \dots \textcircled{5}$$

From $\textcircled{4}$ and $\textcircled{5}$, $a_n=3\left[1-\left(\frac{2}{3}\right)^{n-1}\right]$

Note Summary

The recurrence relation $a_{n+2}+pa_{n+1}+qa_n=0$ can be rearranged into $a_{n+2}-\alpha a_{n+1}=\beta(a_{n+1}-\alpha a_n)$ by using two solutions α and β of quadratic equation $x^2+px+q=0$. The sequence $\{a_{n+1}-\alpha a_n\}$ is a geometric sequence. Therefore, the general term of the sequence $\{a_n\}$ can be determined.

Find the general terms of the following sequences $\{a_n\}$ and $\{b_n\}$ defined by the following conditions.

Ex.

$$\begin{cases} a_1=2, b_1=1 \\ a_{n+1}=3a_n+b_n \cdots \textcircled{1} \\ b_{n+1}=a_n+3b_n \cdots \textcircled{2} \end{cases}$$

[Sol] From $\textcircled{1}$, $b_n = -3a_n + a_{n+1} \cdots \textcircled{3}$

From $\textcircled{3}$, $b_{n+1} = -3 \boxed{a_{n+1}} + \boxed{a_{n+2}} \cdots \textcircled{4}$

In $\textcircled{3}$, replacing n with $n+1$

Substituting $\textcircled{3}$ and $\textcircled{4}$ into $\textcircled{2}$,

$$\boxed{-3a_{n+1} + a_{n+2}} = a_n + 3(\boxed{-3a_n + a_{n+1}})$$

$$a_{n+2} - 6a_{n+1} + 3a_n = 0 \cdots \textcircled{5}$$

Rearranging $\textcircled{5}$,

$$\begin{cases} a_{n+2} - 2a_{n+1} = \boxed{4} (a_{n+1} - 2a_n) \cdots \textcircled{6} \\ a_{n+2} - 4a_{n+1} = \boxed{2} (a_{n+1} - 4a_n) \cdots \textcircled{7} \end{cases}$$

Replacing

a_{n+2}, a_{n+1}, a_n with $x^2, x, 1$ respectively,
 $x^2 - 6x + 8 = 0$
 $(x-2)(x-4) = 0$
 $\therefore x = 2, 4$

From $\textcircled{6}$, the sequence $\{a_{n+1} - 2a_n\}$ is the geometric sequence with 1st term $a_2 - 2a_1 = (3a_1 + b_1) - 2a_1 = 3$ and common ratio $\boxed{4}$.

$$\therefore a_{n+1} - 2a_n = 3 \cdot \boxed{4}^{n-1} \cdots \textcircled{8}$$

From $\textcircled{1}$,
 $a_2 = 3a_1 + b_1$

From $\textcircled{7}$, the sequence $\{a_{n+1} - 4a_n\}$ is the geometric sequence with 1st term $a_2 - 4a_1 = (3a_1 + b_1) - 4a_1 = -1$ and common ratio $\boxed{2}$.

$$\therefore a_{n+1} - 4a_n = -\boxed{2}^{n-1} \cdots \textcircled{9}$$

From $\textcircled{8}$ and $\textcircled{9}$, $a_n = \frac{1}{2}(3 \cdot \boxed{4}^{n-1} + \boxed{2}^{n-1}) \cdots \textcircled{10}$

From $\textcircled{8} - \textcircled{9}$,
 $2a_n = 3 \cdot 4^{n-1} + 2^{n-1}$

From $\textcircled{3}$ and $\textcircled{10}$, $b_n = -3 \cdot \frac{1}{2}(3 \cdot \boxed{4}^{n-1} + \boxed{2}^{n-1}) + \frac{1}{2}(3 \cdot \boxed{4}^n + \boxed{2}^n)$
 $= \frac{1}{2}(\boxed{3 \cdot 4^{n-1} - 2^{n-1}})$

Substituting $a_n = \frac{1}{2}(3 \cdot 4^{n-1} + 2^{n-1})$,
 $a_{n+1} = \frac{1}{2}(3 \cdot 4^n + 2^n)$ into
 $b_n = -3a_n + a_{n+1}$

Answers: in order $a_{n+1}, a_{n+2}, -3a_{n+1} + a_{n+2}, -3a_n + a_{n+1}, 4, 2, 4, 4, 2, 2, 4, 2, 4, 2, 4, 2, 3 \cdot 4^{n-1} - 2^{n-1}$

N48b

$$(1) \begin{cases} a_1 = 1, b_1 = 3 \\ a_{n+1} = 3a_n + b_n \dots ① \\ b_{n+1} = 2a_n + 4b_n \dots ② \end{cases}$$

[Sol] From ①, $b_n = -3a_n + a_{n+1} \dots ③$

From ③, $b_{n+1} = -3a_{n+1} + a_{n+2} \dots ④$

Substituting ③ and ④ into ②,

$$-3a_{n+1} + a_{n+2} = 2a_n + 4(-3a_n + a_{n+1})$$

$$a_{n+2} - 7a_{n+1} + 10a_n = 0 \dots ⑤$$

Rearranging ⑤,

$$\begin{cases} a_{n+2} - 2a_{n+1} = 5(a_{n+1} - 2a_n) \dots ⑥ \\ a_{n+2} - 5a_{n+1} = 2(a_{n+1} - 5a_n) \dots ⑦ \end{cases}$$

From ⑥, the sequence $\{a_{n+1} - 2a_n\}$ is the geometric sequence with 1st term

$$a_2 - 2a_1 = (3a_1 + b_1) - 2a_1 = 4 \text{ and common ratio } 5.$$

$$\therefore a_{n+1} - 2a_n = 4 \cdot 5^{n-1} \dots ⑧$$

From ⑦, the sequence $\{a_{n+1} - 5a_n\}$ is the geometric sequence with 1st term

$$a_2 - 5a_1 = (3a_1 + b_1) - 5a_1 = 1 \text{ and common ratio } 2.$$

$$\therefore a_{n+1} - 5a_n = 2^{n-1} \dots ⑨$$

$$\text{From ⑧ and ⑨, } a_n = \frac{1}{3}(4 \cdot 5^{n-1} - 2^{n-1}) \dots ⑩$$

$$\begin{aligned} \text{From ③ and ⑩, } b_n &= -3 \cdot \frac{1}{3}(4 \cdot 5^{n-1} - 2^{n-1}) + \frac{1}{3}(4 \cdot 5^n - 2^n) \\ &= \frac{1}{3}(8 \cdot 5^{n-1} + 2^{n-1}) \end{aligned}$$

1. The sequence $\{a_n\}$ is defined by the following conditions.

$$a_1 = \frac{1}{2}, a_2 = \frac{1}{3}, a_{n+2} = \frac{a_n a_{n+1}}{2a_n - a_{n+1} + 2a_n a_{n+1}}$$

- (1) Let $b_n = \frac{1}{a_n}$. Express b_{n+2} in terms of b_{n+1} and b_n .

[Sol] Since $a_{n+2} = \frac{a_n a_{n+1}}{2a_n - a_{n+1} + 2a_n a_{n+1}}$,

$$\frac{1}{a_{n+2}} = \frac{2a_n - a_{n+1} + 2a_n a_{n+1}}{a_n a_{n+1}} = \frac{2}{a_{n+1}} - \frac{1}{a_n} + 2$$

$$\text{Let } b_n = \frac{1}{a_n}, b_{n+2} = 2b_{n+1} - b_n + 2$$

- (2) Let $b_{n+1} - b_n = c_n$. Express c_n in terms of n .

[Sol] Let $b_{n+1} - b_n = c_n$.

$$\begin{aligned} c_{n+1} &= b_{n+2} - b_{n+1} \\ &= (2b_{n+1} - b_n + 2) - b_{n+1} \\ &= b_{n+1} - b_n + 2 \\ &= c_n + 2 \end{aligned} \quad \dots \textcircled{1}$$

$$c_1 = b_2 - b_1 = \frac{1}{a_2} - \frac{1}{a_1} = 1 \quad \dots \textcircled{2}$$

From ① and ②, $c_n = 1 + (n-1) \cdot 2 = 2n-1$

From (1),

$$b_{n+2} = 2b_{n+1} - b_n + 2$$

$\{c_n\}$ is the arithmetic sequence with 1st term $c_1 = 1$ and common difference 2.

- (3) Express a_n in terms of n .

[Sol] From (2), $b_{n+1} - b_n = 2n-1$

Since the general term of the sequence of differences of $\{b_n\}$ is $2n-1$, when $n \geq 2$,

$$\begin{aligned} b_n &= b_1 + \sum_{k=1}^{n-1} (2k-1) = 2 + 2 \cdot \frac{1}{2} (n-1)n - (n-1) \\ &= n^2 - 2n + 3 \quad \dots \textcircled{3} \end{aligned}$$

Since $b_1 = 2$, ③ is also true when $n = 1$.

$$\therefore b_n = n^2 - 2n + 3$$

$$\therefore a_n = \frac{1}{b_n} = \frac{1}{n^2 - 2n + 3}$$

N49b

2. Given the sequence $\{a_n\}$, let S_n be the sum of the first n terms. Then,
 $S_1=1, S_2=-13, S_{n+2}=-13S_{n+1}-36S_n$
 are true. Express S_n and a_n in terms of n .

[Sol] Since $S_{n+2}=-13S_{n+1}-36S_n$, $S_{n+2}+13S_{n+1}+36S_n=0 \dots \textcircled{1}$

Rearranging $\textcircled{1}$,

$$\begin{cases} S_{n+2}+9S_{n+1}=-4(S_{n+1}+9S_n) \dots \textcircled{2} \\ S_{n+2}+4S_{n+1}=-9(S_{n+1}+4S_n) \dots \textcircled{3} \end{cases}$$

Replacing S_{n+2}, S_{n+1}, S_n
 with $x^2, x, 1$ respectively,
 $x^2+13x+36=0$
 $(x+9)(x+4)=0$
 $\therefore x=-9, -4$

From $\textcircled{2}$, the sequence $\{S_{n+1}+9S_n\}$ is the geometric sequence with 1st
 term $S_2+9S_1=-4$ and common ratio -4 .

$$\therefore S_{n+1}+9S_n=-4(-4)^{n-1}=(-4)^n \dots \textcircled{4}$$

From $\textcircled{3}$, the sequence $\{S_{n+1}+4S_n\}$ is the geometric sequence with 1st
 term $S_2+4S_1=-9$ and common ratio -9 .

$$\therefore S_{n+1}+4S_n=-9(-9)^{n-1}=(-9)^n \dots \textcircled{5}$$

$$\text{From } \textcircled{4} \text{ and } \textcircled{5}, S_n=\frac{1}{5}[(-4)^n-(-9)^n]$$

The 1st term a_1 is $a_1=S_1=1$.

When $n \geq 2$,

$$a_n=S_n-S_{n-1}$$

$$=\frac{1}{5}[(-4)^n-(-9)^n]-\frac{1}{5}[(-4)^{n-1}-(-9)^{n-1}]$$

$$=\frac{1}{5}[(-4-1)(-4)^{n-1}+(9+1)(-9)^{n-1}]$$

$$=-(-4)^{n-1}+2(-9)^{n-1} \dots \textcircled{6}$$

Since $a_1=1$, $\textcircled{6}$ is also true when $n=1$.

$$\therefore a_n=-(-4)^{n-1}+2(-9)^{n-1}$$

Sum of a Sequence and
 General Term (N34)

N50a

Recurrence Relations

KUMON

N 50

Find the general term of each given sequence $\{a_n\}$ defined by the following conditions.

(1) $a_1=6, a_{n+1}=a_n+4n$ → N43

[Sol] $a_{n+1}-a_n=4n$

Since the general term of the sequence of differences of $\{a_n\}$ is $4n$,
when $n \geq 2$,

$$\begin{aligned} a_n &= a_1 + \sum_{k=1}^{n-1} 4k = 6 + 4 \cdot \frac{1}{2}(n-1)n \\ &= 2(n^2 - n + 3) \quad \dots \textcircled{1} \end{aligned}$$

Since $a_1=6$, $\textcircled{1}$ is also true when $n=1$.

$$\therefore a_n = 2(n^2 - n + 3)$$

(2) $a_1=2, a_{n+1}=2a_n+5$ → N44

[Sol] Since $a_{n+1}=2a_n+5, a_{n+1}+5=2(a_n+5)$

Let $b_n=a_n+5$.

$$\begin{cases} b_{n+1}=2b_n & \dots \textcircled{1} \\ b_1=a_1+5=7 & \dots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}, b_n=7 \cdot 2^{n-1}$

$$\therefore a_n = b_n - 5 = 7 \cdot 2^{n-1} - 5$$

N50b

$$(3) \quad a_1 = \frac{1}{3}, \quad a_{n+1} = \frac{a_n}{4a_n - 1}$$

➡ N45

[Sol] Since $a_{n+1} = \frac{a_n}{4a_n - 1}$, $\frac{1}{a_{n+1}} = \frac{4a_n - 1}{a_n} = -\frac{1}{a_n} + 4$

Let $b_n = \frac{1}{a_n}$. Since $b_{n+1} = -b_n + 4$,

$$\begin{cases} b_{n+1} - 2 = -(b_n - 2) \quad \dots \textcircled{1} \\ b_1 - 2 = \frac{1}{a_1} - 2 = 1 \quad \dots \textcircled{2} \end{cases}$$

From ① and ②, $b_n - 2 = (-1)^{n-1}$

$$b_n = (-1)^{n-1} + 2$$

$$\therefore a_n = \frac{1}{b_n} = \frac{1}{(-1)^{n-1} + 2}$$

$$(4) \quad a_1 = 2, \quad a_2 = 1, \quad a_{n+2} = a_{n+1} + 6a_n$$

➡ N47

[Sol] Since $a_{n+2} = a_{n+1} + 6a_n$, $a_{n+2} - a_{n+1} - 6a_n = 0 \quad \dots \textcircled{1}$

Rearranging ①,

$$\begin{cases} a_{n+2} + 2a_{n+1} = 3(a_{n+1} + 2a_n) \quad \dots \textcircled{2} \\ a_{n+2} - 3a_{n+1} = -2(a_{n+1} - 3a_n) \quad \dots \textcircled{3} \end{cases}$$

From ②, the sequence $\{a_{n+1} + 2a_n\}$ is the geometric sequence with 1st term $a_2 + 2a_1 = 5$ and common ratio 3.

$$\therefore a_{n+1} + 2a_n = 5 \cdot 3^{n-1} \quad \dots \textcircled{4}$$

From ③, the sequence $\{a_{n+1} - 3a_n\}$ is the geometric sequence with 1st term $a_2 - 3a_1 = -5$ and common ratio -2.

$$\therefore a_{n+1} - 3a_n = -5(-2)^{n-1} \quad \dots \textcircled{5}$$

From ④ and ⑤, $a_n = 3^{n-1} + (-2)^{n-1}$

Ex. Given that n is a natural number, prove the following equation.
 $2+4+6+\dots+2n=n(n+1) \dots \textcircled{1}$

[Sol] (i) When $n=1$,

$$\text{LHS} = \boxed{2}, \text{ RHS} = 1 \cdot (1+1) = \boxed{2}$$

Therefore, $\textcircled{1}$ is true when $n=1$.

(ii) If $\textcircled{1}$ is true when $n=k$,

$$2+4+6+\dots+2k = \boxed{k(k+1)} \dots \textcircled{2}$$

When $n=k+1$, rearranging the LHS of $\textcircled{1}$ by using $\textcircled{2}$,

$$2+4+6+\dots+2k+2(k+1) \\ = \boxed{k(k+1)} + 2(k+1) \quad \leftarrow \text{From } \textcircled{2}$$

$$= (k+1)(\boxed{k+2})$$

$$= (k+1)[(\boxed{k+1}) + 1] \quad \leftarrow \text{The RHS of } \textcircled{1} \text{ when } n=k+1$$

Therefore, $\textcircled{1}$ is also true when $n=k+1$.

From (i) and (ii), $\textcircled{1}$ is true for all natural numbers n .

The method of proof shown above is called **mathematical induction**.

Mathematical Induction

To prove that a proposition P is true for all natural numbers n by mathematical induction, the following two statements have to be proved.

(i) P is true when $n=1$.

(ii) If P is true when $n=k$, then P is also true when $n=k+1$.

A statement that can be either true or false is called a **proposition**.

1. Given that n is a natural number, prove the following equation by mathematical induction.

$$1+3+5+\dots+(2n-1)=n^2 \dots \textcircled{1}$$

[Sol] (i) When $n=1$,

$$\text{LHS}=1, \text{RHS}=1^2=1$$

Therefore, $\textcircled{1}$ is true when $n=1$.

(ii) If $\textcircled{1}$ is true when $n=k$,

$$1+3+5+\dots+(2k-1)=k^2 \dots \textcircled{2}$$

When $n=k+1$, rearranging the LHS of $\textcircled{1}$ by using $\textcircled{2}$,

$$1+3+5+\dots+(2k-1)+[2(k+1)-1]$$

$$=k^2+2k+1$$

$$=(k+1)^2$$

Therefore, $\textcircled{1}$ is also true when $n=k+1$.

From (i) and (ii), $\textcircled{1}$ is true for all natural numbers n .

2. Given that n is a natural number, prove the following equation by mathematical induction.

$$4+16+28+\dots+(12n-8)=2n(3n-1) \dots \textcircled{1}$$

[Sol] (i) When $n=1$,

$$\text{LHS}=4, \text{RHS}=2 \cdot 1 \cdot (3 \cdot 1 - 1) = 4$$

Therefore, $\textcircled{1}$ is true when $n=1$.

(ii) If $\textcircled{1}$ is true when $n=k$,

$$4+16+28+\dots+(12k-8)=2k(3k-1) \dots \textcircled{2}$$

When $n=k+1$, rearranging the LHS of $\textcircled{1}$ by using $\textcircled{2}$,

$$4+16+28+\dots+(12k-8)+[12(k+1)-8]$$

$$=2k(3k-1)+12k+4$$

$$=6k^2+10k+4$$

$$=2(k+1)(3k+2)$$

$$=2(k+1)[3(k+1)-1]$$

Therefore, $\textcircled{1}$ is also true when $n=k+1$.

From (i) and (ii), $\textcircled{1}$ is true for all natural numbers n .

Mathematical Induction

1. Given that n is a natural number, prove the following equation by mathematical induction.

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \frac{1}{3}n(n+1)(n+2) \quad \cdots \textcircled{1}$$

[Sol] (i) When $n=1$,

$$\text{LHS} = 1 \cdot 2 = 2, \text{ RHS} = \frac{1}{3} \cdot 1 \cdot (1+1)(1+2) = 2$$

Therefore, $\textcircled{1}$ is true when $n=1$.

(ii) If $\textcircled{1}$ is true when $n=k$,

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + k(k+1) = \frac{1}{3}k(k+1)(k+2) \quad \cdots \textcircled{2}$$

When $n=k+1$, rearranging the LHS of $\textcircled{1}$ by using $\textcircled{2}$,

$$\begin{aligned} & 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + k(k+1) + (k+1)[(k+1)+1] \\ &= \frac{1}{3}k(k+1)(k+2) + (k+1)(k+2) \end{aligned}$$

$$= \frac{1}{3}(k+1)(k+2)(k+3)$$

$$= \frac{1}{3}(k+1)[(k+1)+1][(k+1)+2]$$

Therefore, $\textcircled{1}$ is also true when $n=k+1$.

From (i) and (ii), $\textcircled{1}$ is true for all natural numbers n .

N52b

2. Given that n is a natural number, prove the following equation by mathematical induction.

$$10 + 10^2 + 10^3 + \dots + 10^n = \frac{10}{9}(10^n - 1) \dots \textcircled{1}$$

[Sol] (i) When $n=1$,

$$\text{LHS} = 10, \text{ RHS} = \frac{10}{9}(10^1 - 1) = 10$$

Therefore, $\textcircled{1}$ is true when $n=1$.

(ii) If $\textcircled{1}$ is true when $n=k$,

$$10 + 10^2 + 10^3 + \dots + 10^k = \frac{10}{9}(10^k - 1) \dots \textcircled{2}$$

When $n=k+1$, rearranging the LHS of $\textcircled{1}$ by using $\textcircled{2}$,

$$10 + 10^2 + 10^3 + \dots + 10^k + 10^{k+1}$$

$$= \frac{10}{9}(10^k - 1) + 10^{k+1}$$

$$= \frac{10}{9}(10^k - 1 + 9 \cdot 10^k)$$

$$= \frac{10}{9}(10^{k+1} - 1)$$

$$10^k + 9 \cdot 10^k = (1+9) \cdot 10^k = 10^{k+1}$$

Therefore, $\textcircled{1}$ is also true when $n=k+1$.

From (i) and (ii), $\textcircled{1}$ is true for all natural numbers n .

From Kumon Hub

Mathematical Induction

1. Given that n is a natural number, prove the following equation by mathematical induction.

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1} \quad \dots \textcircled{1}$$

[Sol] (i) When $n=1$,

$$\text{LHS} = \frac{1}{1 \cdot 2} = \frac{1}{2}, \quad \text{RHS} = \frac{1}{1+1} = \frac{1}{2}$$

Therefore, $\textcircled{1}$ is true when $n=1$.

(ii) If $\textcircled{1}$ is true when $n=k$,

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{k(k+1)} = \frac{k}{k+1} \quad \dots \textcircled{2}$$

When $n=k+1$, rearranging the LHS of $\textcircled{1}$ by using $\textcircled{2}$,

$$\begin{aligned} & \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{k(k+1)} + \frac{1}{(k+1)[(k+1)+1]} \\ &= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \\ &= \frac{k(k+2)+1}{(k+1)(k+2)} \\ &= \frac{(k+1)^2}{(k+1)(k+2)} \\ &= \frac{k+1}{k+2} \\ &= \frac{k+1}{(k+1)+1} \end{aligned}$$

Therefore, $\textcircled{1}$ is also true when $n=k+1$.

From (i) and (ii), $\textcircled{1}$ is true for all natural numbers n .

N53b

2. Given that n is a natural number, prove the following equation by mathematical induction.

$$1 + \frac{3}{2^2} + \frac{4}{2^3} + \dots + \frac{n+1}{2^n} = 3 - \frac{n+3}{2^n} \dots \textcircled{1}$$

[Sol] (i) When $n=1$,

$$\text{LHS} = 1, \text{ RHS} = 3 - \frac{1+3}{2^1} = 1$$

Therefore, $\textcircled{1}$ is true when $n=1$.

(ii) If $\textcircled{1}$ is true when $n=k$,

$$1 + \frac{3}{2^2} + \frac{4}{2^3} + \dots + \frac{k+1}{2^k} = 3 - \frac{k+3}{2^k} \dots \textcircled{2}$$

When $n=k+1$, rearranging the LHS of $\textcircled{1}$ by using $\textcircled{2}$,

$$1 + \frac{3}{2^2} + \frac{4}{2^3} + \dots + \frac{k+1}{2^k} + \frac{(k+1)+1}{2^{k+1}}$$

$$= 3 - \frac{k+3}{2^k} + \frac{k+2}{2^{k+1}}$$

$$= 3 - \frac{2(k+3) - (k+2)}{2^{k+1}}$$

$$= 3 - \frac{k+4}{2^{k+1}}$$

$$= 3 - \frac{(k+1)+3}{2^{k+1}}$$

Therefore, $\textcircled{1}$ is also true when $n=k+1$.

From (i) and (ii), $\textcircled{1}$ is true for all natural numbers n .

from Kumon Hub

Ex. Given that n is a natural number, prove the following statement by mathematical induction.

$n^3 + 2n$ is a multiple of 3.

[Sol] Let the statement " $n^3 + 2n$ is a multiple of 3" be ①.

(i) When $n=1$,

$$n^3 + 2n = \boxed{1}^3 + 2 \cdot \boxed{1} = \boxed{3}$$

Therefore, ① is true when $n=1$.

(ii) If ① is true when $n=k$,

using an integer m ,

$$k^3 + 2k = \boxed{3} m \cdots \textcircled{2} \quad \leftarrow k^3 + 2k \text{ is a multiple of 3.}$$

When $n=k+1$, rearranging by using ②,

$$\begin{aligned} & (k+1)^3 + 2(k+1) \\ &= (k^3 + 3k^2 + 3k + 1) + (2k + 2) \\ &= (k^3 + 2k) + \boxed{3}(k^2 + k + 1) \\ &= \boxed{3} m + \boxed{3}(k^2 + k + 1) \\ &= \boxed{3}(m + k^2 + k + 1) \end{aligned}$$

From ②

Since $m + k^2 + k + 1$ is an integer,

$(k+1)^3 + 2(k+1)$ is a multiple of 3.

Therefore, ① is also true when $n=k+1$.

From (i) and (ii), ① is true for all natural numbers n .

N54b

1. Given that n is a natural number, prove the following statement by mathematical induction.

$2n^3 - 3n^2 + n + 6$ is a multiple of 6.

[Sol] Let the statement " $2n^3 - 3n^2 + n + 6$ is a multiple of 6" be ①.

(i) When $n=1$,

$$2n^3 - 3n^2 + n + 6 = 2 \cdot 1^3 - 3 \cdot 1^2 + 1 + 6 = 6$$

Therefore, ① is true when $n=1$.

(ii) If ① is true when $n=k$,

using an integer m ,

$$2k^3 - 3k^2 + k + 6 = 6m \dots \textcircled{2}$$

When $n=k+1$, rearranging by using ②,

$$\begin{aligned} & 2(k+1)^3 - 3(k+1)^2 + (k+1) + 6 \\ &= 2k^3 + 6k^2 + 6k + 2 - 3k^2 - 6k - 3 + k + 1 + 6 \\ &= (2k^3 - 3k^2 + k + 6) + 6k^2 \\ &= 6m + 6k^2 \\ &= 6(m + k^2) \end{aligned}$$

Since $m + k^2$ is an integer,

$2(k+1)^3 - 3(k+1)^2 + (k+1) + 6$ is a multiple of 6.

Therefore, ① is also true when $n=k+1$.

From (i) and (ii), ① is true for all natural numbers n .

from Kumon Hub

Mathematical Induction

1. Given that n is a natural number, prove the following statement by mathematical induction.

$3^{2n} - 1$ is a multiple of 8.

[Sol] Let the statement " $3^{2n} - 1$ is a multiple of 8" be ①.

(i) When $n=1$,

$$3^{2n} - 1 = 3^2 - 1 = 8$$

Therefore, ① is true when $n=1$.

(ii) If ① is true when $n=k$,

using an integer m ,

$$3^{2k} - 1 = 8m \quad \dots \text{②}$$

When $n=k+1$, rearranging by using ②,

$$3^{2(k+1)} - 1$$

$$= 9 \cdot 3^{2k} - 1$$

$$= 9(8m+1) - 1$$

$$= 9 \cdot 8m + 8$$

$$= 8(9m+1)$$

Since $9m+1$ is an integer,

$3^{2(k+1)} - 1$ is a multiple of 8.

Therefore, ① is also true when $n=k+1$.

From (i) and (ii), ① is true for all natural numbers n .

$$3^{2(k+1)} = 3^{2k+2} = 3^2 \cdot 3^{2k}$$

$$\text{From ②,} \\ 3^{2k} = 8m+1$$

N55b

2. Given that n is a natural number, prove the following statement by mathematical induction.

$11^{n+1} + 12^{2n-1}$ is a multiple of 19.

[Sol] Let the statement " $11^{n+1} + 12^{2n-1}$ is a multiple of 19" be ①.

(i) When $n=1$,

$$11^{n+1} + 12^{2n-1} = 11^2 + 12^1 = 133 = 19 \cdot 7$$

Therefore, ① is true when $n=1$.

(ii) If ① is true when $n=k$,

using an integer m ,

$$11^{k+1} + 12^{2k-1} = 19m \quad \dots \textcircled{2}$$

When $n=k+1$, rearranging by using ②,

$$11^{(k+1)+1} + 12^{2(k+1)-1}$$

$$= 11 \cdot 11^{k+1} + 12^{2k+1}$$

$$= 11(19m - 12^{2k-1}) + 12^2 \cdot 12^{2k-1}$$

$$= 11 \cdot 19m + 133 \cdot 12^{2k-1}$$

$$= 19(11m + 7 \cdot 12^{2k-1})$$

Since $11m + 7 \cdot 12^{2k-1}$ is an integer,

$11^{(k+1)+1} + 12^{2(k+1)-1}$ is a multiple of 19.

Therefore, ① is also true when $n=k+1$.

From (i) and (ii), ① is true for all natural numbers n .

From ②,

$$11^{k+1} = 19m - 12^{2k-1}$$

From Kumon Hub

Mathematical Induction

Ex. Given that n is a natural number greater than or equal to 3, prove the following inequality by mathematical induction.

$$3^n > 8n \cdots \textcircled{1}$$

[Sol] (i) When $n=3$,

$$\text{LHS} = 3^3 = 27$$

$$\text{RHS} = 8 \cdot 3 = 24$$

Therefore, $\text{LHS} > \text{RHS}$

Thus, $\textcircled{1}$ is true when $n=3$.

(ii) For $k \geq 3$, if $\textcircled{1}$ is true when $n=k$,

$$3^k > 8k \cdots \textcircled{2}$$

When $n=k+1$, considering the difference of both sides of $\textcircled{1}$, then from $\textcircled{2}$,

$$3^{k+1} - 8(k+1) = 3 \cdot 3^k - 8(k+1)$$

$$> 3 \cdot 8k - 8(k+1)$$

$$= 8(2k-1) > 0$$

$$\therefore 3^{k+1} > 8(k+1)$$

Therefore, $\textcircled{1}$ is also true when $n=k+1$.

From (i) and (ii), $\textcircled{1}$ is true for all natural numbers n greater than or equal to 3.

Proof of
 $\text{LHS} - \text{RHS} > 0$
(J194)

From $\textcircled{2}$

Since $k \geq 3$,
 $2k-1 > 0$

Answers in order 3, 24, $k+1$, $k+1$, $8k$, $k+1$, $2k-1$, $k+1$

As shown in **Ex.**, if the statement is proved when $n=3$, then from (i) and (ii), the statement is true for all natural numbers n greater than or equal to 3.

N56b

1. Given that n is a natural number greater than or equal to 3, prove the following inequality by mathematical induction.
 $2^n > 2n + 1 \dots \textcircled{1}$

[Sol] (i) When $n=3$,

$$\text{LHS} = 2^3 = 8$$

$$\text{RHS} = 2 \cdot 3 + 1 = 7$$

Therefore, $\text{LHS} > \text{RHS}$

Thus, $\textcircled{1}$ is true when $n=3$.

(ii) For $k \geq 3$, if $\textcircled{1}$ is true when $n=k$,

$$2^k > 2k + 1 \dots \textcircled{2}$$

When $n=k+1$, considering the difference of both sides of $\textcircled{1}$, then from $\textcircled{2}$,

$$\begin{aligned} 2^{k+1} - [2(k+1) + 1] &= 2 \cdot 2^k - (2k + 3) \\ &> 2(2k + 1) - (2k + 3) \\ &= 2k - 1 > 0 \end{aligned}$$

$$\therefore 2^{k+1} > 2(k+1) + 1$$

Therefore, $\textcircled{1}$ is also true when $n=k+1$.

From (i) and (ii), $\textcircled{1}$ is true for all natural numbers n greater than or equal to 3.

Mathematical Induction

Find the general term of each given sequence $\{a_n\}$ defined by the following conditions.

Ex. $a_1 = 2, a_{n+1} = 2 - \frac{1}{a_n} \dots \textcircled{1}$

$$\begin{aligned} a_1 &= 2 \\ a_2 &= 2 - \frac{1}{a_1} \\ a_3 &= 2 - \frac{1}{a_2} \\ &\vdots \end{aligned}$$

[Sol] From the conditions,

$$a_1 = 2, a_2 = \frac{3}{2}, a_3 = \frac{4}{3}, a_4 = \frac{5}{4}, \dots \leftarrow$$

Therefore, the general term of $\{a_n\}$ is assumed to be

$$a_n = \frac{n+1}{n} \dots \textcircled{2}$$

Prove that the assumption $\textcircled{2}$ is true by mathematical induction.

(i) When $n=1$, in $\textcircled{2}$,

$$a_1 = \frac{1+1}{1} = 2$$

Therefore, $\textcircled{2}$ is true when $n=1$. $\leftarrow$

It coincides with condition $a_1 = 2$.

(ii) If $\textcircled{2}$ is true when $n=k$,

$$a_k = \frac{k+1}{k}$$

When $n=k+1$, substituting $n=k$ into $\textcircled{1}$,

$$\begin{aligned} a_{k+1} &= 2 - \frac{1}{a_k} \\ &= 2 - \frac{k}{k+1} \\ &= \frac{k+2}{k+1} = \frac{(k+1)+1}{k+1} \end{aligned}$$

$$a_k = \frac{k+1}{k}$$

Therefore, $\textcircled{2}$ is also true when $n=k+1$.

From (i) and (ii), $\textcircled{2}$ is true for all natural numbers n .

$$\therefore a_n = \frac{n+1}{n}$$

N57b

$$(1) \quad a_1 = 2, \quad a_{n+1} = \frac{a_n}{1+a_n} \quad \dots \textcircled{1}$$

[Sol] From the conditions,

$$a_1 = 2, \quad a_2 = \frac{2}{3}, \quad a_3 = \frac{2}{5}, \quad a_4 = \frac{2}{7}, \quad \dots$$

Therefore, the general term of $\{a_n\}$ is assumed to be

$$a_n = \frac{2}{2n-1} \quad \dots \textcircled{2}$$

Prove that the assumption $\textcircled{2}$ is true by mathematical induction.

(i) When $n=1$, in $\textcircled{2}$,

$$a_1 = \frac{2}{2 \cdot 1 - 1} = 2$$

Therefore, $\textcircled{2}$ is true when $n=1$.

(ii) If $\textcircled{2}$ is true when $n=k$,

$$a_k = \frac{2}{2k-1}$$

When $n=k+1$, substituting $n=k$ into $\textcircled{1}$,

$$\begin{aligned} a_{k+1} &= \frac{a_k}{1+a_k} \\ &= \frac{\frac{2}{2k-1}}{1+\frac{2}{2k-1}} \\ &= \frac{2}{2k+1} = \frac{2}{2(k+1)-1} \end{aligned}$$

Therefore, $\textcircled{2}$ is also true when $n=k+1$.

From (i) and (ii), $\textcircled{2}$ is true for all natural numbers n .

$$\therefore a_n = \frac{2}{2n-1}$$

[Sol] The table below shows the values of 2^n and $3n + 1$ when $n = 1, 2, 3, 4, 5$.

n	1	2	3	4	5
2^n	2	4	8	16	32
$3n+1$	4	7	10	13	16

when $n \geq 4$, the following inequality is assumed to be true:

$$2^n > 3n+1 \quad \cdots \textcircled{1}$$

(i) ① is true when $n=4$.

$$2^k > 3k+1 \dots ②$$
$$2^{k+1} - [3(\boxed{k+1}) + 1] = 2 \cdot 2^k - (\boxed{3k+4})$$

$$\begin{aligned} &> 2(\boxed{3k+1}) - (\boxed{3k+4}) \\ &= \boxed{3k-2} > 0 \end{aligned}$$

$$\therefore 2^{k+1} > 3(\boxed{k+1}) + 1$$

Therefore, ① is also true when $n=k+1$.

From (i) and (ii), ① is true for all natural numbers n when $n \geq 4$.

Thus, when $n=1, 2, 3$, $2^n < 3n+1$ and

when $n \geq 4$, $2^n > 3n + 1$

58b

1. Given that n is a natural number, compare the values of 3^n and $5n+1$.

[Sol] The table below shows the values of 3^n and $5n+1$ when $n=1, 2, 3, 4, 5$.

n	1	2	3	4	5
3^n	3	9	27	81	243
$5n+1$	6	11	16	21	26

From the table,

when $n \geq 3$, the following inequality is assumed to be true:

$$3^n > 5n+1 \quad \dots \textcircled{1}$$

Prove that the assumption $\textcircled{1}$ is true by mathematical induction.

(i) $\textcircled{1}$ is true when $n=3$.

(ii) For $k \geq 3$, if $\textcircled{1}$ is true when $n=k$,

$$3^k > 5k+1 \quad \dots \textcircled{2}$$

When $n=k+1$, considering the difference of both sides of $\textcircled{1}$, then from $\textcircled{2}$,

$$\begin{aligned} 3^{k+1} - [5(k+1)+1] &= 3 \cdot 3^k - (5k+6) \\ &> 3(5k+1) - (5k+6) \\ &= 10k-3 > 0 \end{aligned}$$

$$\therefore 3^{k+1} > 5(k+1)+1$$

Therefore, $\textcircled{1}$ is also true when $n=k+1$.

From (i) and (ii), $\textcircled{1}$ is true for all natural numbers n when $n \geq 3$.

Thus, when $n=1, 2$, $3^n < 5n+1$ and

when $n \geq 3$, $3^n > 5n+1$

From Kumon Hub

1. Given that n is a natural number, prove the following equation by mathematical induction.

$$1 \cdot n + 2 \cdot (n-1) + 3 \cdot (n-2) + \dots + (n-1) \cdot 2 + n \cdot 1 = \frac{1}{6}n(n+1)(n+2)$$

[Sol] Let $1 \cdot n + 2 \cdot (n-1) + 3 \cdot (n-2) + \dots + (n-1) \cdot 2 + n \cdot 1 = \frac{1}{6}n(n+1)(n+2)$... ①

(i) When $n=1$,

$$\text{LHS} = 1 \cdot 1 = 1, \text{ RHS} = \frac{1}{6} \cdot 1 \cdot (1+1) \cdot (1+2) = 1$$

Therefore, ① is true when $n=1$.

(ii) If ① is true when $n=k$,

$$1 \cdot k + 2 \cdot (k-1) + 3 \cdot (k-2) + \dots + (k-1) \cdot 2 + k \cdot 1 = \frac{1}{6}k(k+1)(k+2) \dots ②$$

When $n=k+1$, rearranging the LHS of ① by using ②,

$$1 \cdot (k+1) + 2 \cdot [(k+1)-1] + 3 \cdot [(k+1)-2] + \dots + [(k+1)-1] \cdot 2 + (k+1) \cdot 1$$

$$= [1 \cdot k + 2 \cdot (k-1) + 3 \cdot (k-2) + \dots + k \cdot 1] + 1 + 2 + 3 + \dots + k + (k+1)$$

$$= \frac{1}{6}k(k+1)(k+2) + \frac{1}{2}(k+1)(k+2)$$

$$= \frac{1}{6}(k+1)(k+2)(k+3)$$

$$= \frac{1}{6}(k+1)[(k+1)+1][(k+1)+2]$$

$$\sum_{k=1}^n k = \frac{1}{2}n(n+1)$$

Therefore, ① is also true when $n=k+1$.

From (i) and (ii), ① is true for all natural numbers n .

※	$1 \cdot (k+1) + 2 \cdot k$	$+ 3 \cdot (k-1) + \dots + k \cdot 2 + (k+1) \cdot 1$
-)	$1 \cdot k$	$+ 2 \cdot (k-1) + 3 \cdot (k-2) + \dots + k \cdot 1$
	1	$+ 2 \quad + 3 \quad + \dots + k \quad + (k+1)$

2. Given that n is a natural number greater than or equal to 2, prove the following inequality by mathematical induction.

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} > \frac{2n}{n+1} \quad \dots \textcircled{1}$$

[Sol] (i) When $n=2$,

$$\text{LHS} = \frac{1}{1} + \frac{1}{2} = \frac{3}{2}$$

$$\text{RHS} = \frac{2 \cdot 2}{2+1} = \frac{4}{3}$$

Therefore, $\text{LHS} > \text{RHS}$

Thus, $\textcircled{1}$ is true when $n=2$.

(ii) For $k \geq 2$, if $\textcircled{1}$ is true when $n=k$,

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{k} > \frac{2k}{k+1} \quad \dots \textcircled{2}$$

When $n=k+1$, considering the difference of both sides of $\textcircled{1}$, then from $\textcircled{2}$,

$$\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{k} + \frac{1}{k+1} \right) - \frac{2(k+1)}{(k+1)+1}$$

From $\textcircled{2}$

$$> \left(\frac{2k}{k+1} + \frac{1}{k+1} \right) - \frac{2(k+1)}{k+2}$$

$$= \frac{2k+1}{k+1} - \frac{2(k+1)}{k+2}$$

$$= \frac{(2k+1)(k+2) - 2(k+1)^2}{(k+1)(k+2)}$$

$$= \frac{k}{(k+1)(k+2)} > 0 \quad \leftarrow \text{Since } k \geq 2$$

$$\therefore \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{k} + \frac{1}{k+1} > \frac{2(k+1)}{(k+1)+1}$$

Therefore, $\textcircled{1}$ is also true when $n=k+1$.

From (i) and (ii), $\textcircled{1}$ is true for all natural numbers n greater than or equal to 2.

N60a

Mathematical Induction

1. Given that n is a natural number, prove the following equation by mathematical induction. ➔ N51

$$1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{1}{6}n(n+1)(2n+1) \dots \textcircled{1}$$

[Sol] (i) When $n=1$,

$$\text{LHS} = 1^2 = 1, \text{ RHS} = \frac{1}{6} \cdot 1 \cdot (1+1) \cdot (2 \cdot 1 + 1) = 1$$

Therefore, $\textcircled{1}$ is true when $n=1$.

(ii) If $\textcircled{1}$ is true when $n=k$,

$$1^2 + 2^2 + 3^2 + \dots + k^2 = \frac{1}{6}k(k+1)(2k+1) \dots \textcircled{2}$$

When $n=k+1$, rearranging the LHS of $\textcircled{1}$ by using $\textcircled{2}$,

$$\begin{aligned} & 1^2 + 2^2 + 3^2 + \dots + k^2 + (k+1)^2 \\ &= \frac{1}{6}k(k+1)(2k+1) + (k+1)^2 \\ &= \frac{1}{6}(k+1)[k(2k+1) + 6(k+1)] \\ &= \frac{1}{6}(k+1)(k+2)(2k+3) \\ &= \frac{1}{6}(k+1)[(k+1)+1][2(k+1)+1] \end{aligned}$$

Therefore, $\textcircled{1}$ is also true when $n=k+1$.

From (i) and (ii), $\textcircled{1}$ is true for all natural numbers n .

N60b

2. Given that n is a natural number greater than or equal to 2, prove the following inequality by mathematical induction.
 $3^n > 4n \dots \textcircled{1}$

➡ N56

[Sol] (i) When $n=2$,

$$\text{LHS} = 3^2 = 9$$

$$\text{RHS} = 4 \cdot 2 = 8$$

Therefore, $\text{LHS} > \text{RHS}$

Thus, $\textcircled{1}$ is true when $n=2$.

(ii) For $k \geq 2$, if $\textcircled{1}$ is true when $n=k$,

$$3^k > 4k \dots \textcircled{2}$$

When $n=k+1$, considering the difference of both sides of $\textcircled{1}$, then from $\textcircled{2}$,

$$3^{k+1} - 4(k+1) = 3 \cdot 3^k - 4(k+1)$$

$$> 3 \cdot 4k - 4(k+1)$$

$$= 4(2k-1) > 0$$

$$\therefore 3^{k+1} > 4(k+1)$$

Therefore, $\textcircled{1}$ is also true when $n=k+1$.

From (i) and (ii), $\textcircled{1}$ is true for all natural numbers n greater than or equal to 2.

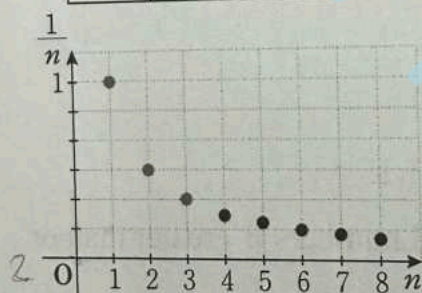
from Kumon Hub

A sequence with infinite terms $a_1, a_2, a_3, \dots, a_n, \dots$ is called an *infinite sequence*, and it is expressed as $\{a_n\}$.

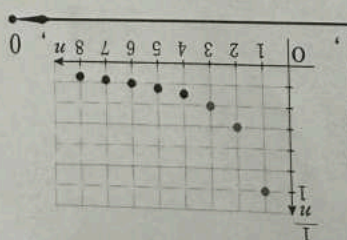
Given the sequence $1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots$, fill in the following table and plot the points representing the values of $\frac{1}{n}$ when $n=4, 5, 6, 7, 8$ on the coordinate plane below. Then, consider how the value of $\frac{1}{n}$ changes as n increases infinitely.

[Sol] When $n=1, 2, 3, 4, 5, 10, 100$, the values of $\frac{1}{n}$ are as shown in the table below.

n	1	2	3	4	5	...	10	...	100	...
$\frac{1}{n}$	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	...	$\frac{1}{10}$	...	$\frac{1}{100}$	...



Therefore, the value of $\frac{1}{n}$ decreases and approaches **0** as n increases infinitely.



n	1	2	3	4	5	...	10	...	100	...
$\frac{1}{n}$	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	...	$\frac{1}{10}$	...	$\frac{1}{100}$	...

Given the sequence $\{a_n\}$, if a_n approaches a constant value α as n approaches infinity, then $\{a_n\}$ is said to **converge** to α , where it is expressed as follows:

$$\lim_{n \rightarrow \infty} a_n = \alpha \quad \text{or} \quad a_n \rightarrow \alpha \text{ as } n \rightarrow \infty$$

The value α is called the **limit value** of $\{a_n\}$. In other words, the **limit** of $\{a_n\}$ is α . If the value of all terms of the sequence is constant c , then the limit value is also considered as c and expressed as follows: $\lim_{n \rightarrow \infty} c = c$

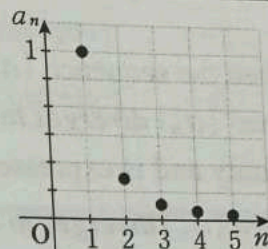
The symbol ∞ is read as "*infinity*" and represents an unbounded quantity that is greater than every real number.

N61b

For each given sequence $\{a_n\}$, plot the points representing the values of a_n when $n=1, 2, 3, 4, 5$ on the coordinate plane, and then find the limit value.

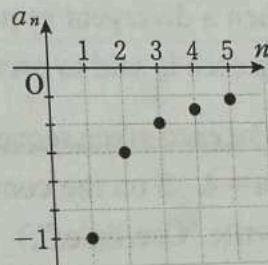
Ex. $1, \frac{1}{4}, \frac{1}{9}, \dots, \frac{1}{n^2}, \dots$

[Sol] $\lim_{n \rightarrow \infty} \frac{1}{n^2} = 0$



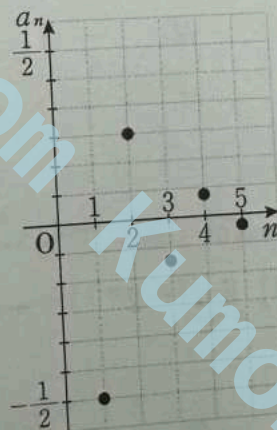
(1) $-1, -\frac{1}{2}, -\frac{1}{3}, \dots, -\frac{1}{n}, \dots$

[Sol] $\lim_{n \rightarrow \infty} \left(-\frac{1}{n}\right) = 0$



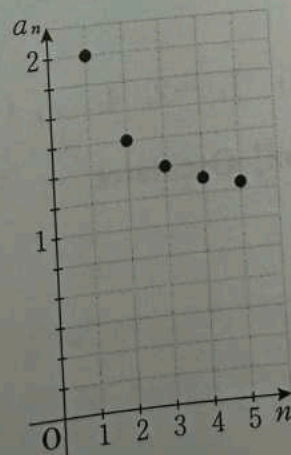
(2) $-\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \dots, \left(-\frac{1}{2}\right)^n, \dots$

[Sol] $\lim_{n \rightarrow \infty} \left(-\frac{1}{2}\right)^n = 0$



(3) $2, \frac{3}{2}, \frac{4}{3}, \dots, 1 + \frac{1}{n}, \dots$

[Sol] $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right) = 1$



N62a

Infinite Sequences

KUMON

N 62

When the sequence $\{a_n\}$ does not converge, $\{a_n\}$ is said to **diverge**.

When $\{a_n\}$ **diverges to positive infinity**, it is said that the limit of $\{a_n\}$ is positive infinity and is expressed as follows: $\lim_{n \rightarrow \infty} a_n = \infty$ or $a_n \rightarrow \infty$ as $n \rightarrow \infty$

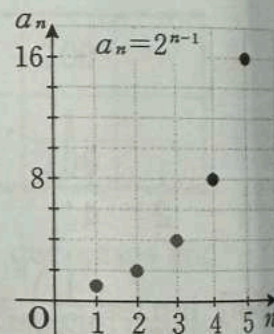
When $\{a_n\}$ **diverges to negative infinity**, it is said that the limit of $\{a_n\}$ is negative infinity and is expressed as follows: $\lim_{n \rightarrow \infty} a_n = -\infty$ or $a_n \rightarrow -\infty$ as $n \rightarrow \infty$

When a divergent sequence does not diverge to positive or negative infinity, the sequence is said to **oscillate**.

- For each given sequence $\{a_n\}$, plot the points representing the values of a_n when $n = 4, 5$ on the coordinate plane, and then find the limit. (When it oscillates, write "Oscillates".)

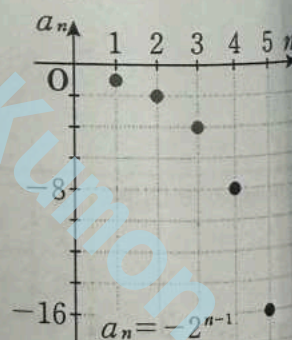
(1) $1, 2, 4, \dots, 2^{n-1}, \dots$

[Sol] $\lim_{n \rightarrow \infty} 2^{n-1} = \infty$



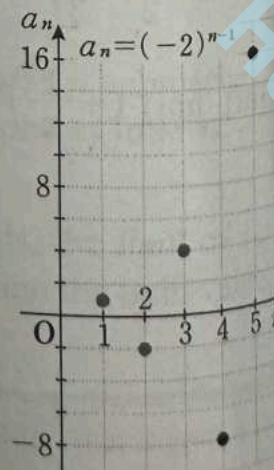
(2) $-1, -2, -4, \dots, -2^{n-1}, \dots$

[Sol] $\lim_{n \rightarrow \infty} (-2^{n-1}) = -\infty$



(3) $1, -2, 4, \dots, (-2)^{n-1}, \dots$

[Sol] **Oscillates**



Limit of a Sequence

Converges	$\lim_{n \rightarrow \infty} a_n = \alpha$	(converges to a constant value α) ...①
Diverges	$\lim_{n \rightarrow \infty} a_n = \infty$	(diverges to positive infinity) ...②
	$\lim_{n \rightarrow \infty} a_n = -\infty$	(diverges to negative infinity) ...③
	Oscillates	(no limit) ...④

2. Classify each given limit by indicating the appropriate number ① to ④.
If the limit is classified as ①, state the limit value.

Ex. $\lim_{n \rightarrow \infty} \frac{2}{n}$ (①, limit value 0) $\lim_{n \rightarrow \infty} (-3)^n$ (④)

(1) $\lim_{n \rightarrow \infty} \frac{1}{n^2 + 1}$ (①, limit value 0) (6) $\lim_{n \rightarrow \infty} (-5)^n$ (④)

(2) $\lim_{n \rightarrow \infty} \left(2 + \frac{1}{n}\right)$ (①, limit value 2) (7) $\lim_{n \rightarrow \infty} -4^n$ (③)

(3) $\lim_{n \rightarrow \infty} (2n + 5)$ (②) (8) $\lim_{n \rightarrow \infty} \frac{(-1)^n}{n}$ (①, limit value 0)

(4) $\lim_{n \rightarrow \infty} (-2n + 5)$ (③) (9) $\lim_{n \rightarrow \infty} \left(n - \frac{1}{n}\right)$ (②)

(5) $\lim_{n \rightarrow \infty} \cos \frac{\pi}{n}$ (①, limit value 1) (10) $\lim_{n \rightarrow \infty} \sin \frac{n\pi}{2}$ (④)

Substituting $n=10, 100, \dots$

$$\cos \frac{\pi}{10}, \cos \frac{\pi}{100}, \dots$$

Therefore, $\lim_{n \rightarrow \infty} \cos \frac{\pi}{n}$ approaches $\cos 0$.

Substituting $n=1, 2, 3, 4, 5, \dots$

$$\sin \frac{\pi}{2}, \sin \pi, \sin \frac{3\pi}{2}, \sin 2\pi, \sin \frac{5\pi}{2}, \dots$$

$$\therefore 1, 0, -1, 0, 1, \dots$$

Properties of Limits of Sequences

When the sequences $\{a_n\}$ and $\{b_n\}$ converge, where $\lim_{n \rightarrow \infty} a_n = \alpha$ and $\lim_{n \rightarrow \infty} b_n = \beta$,

$$\lim_{n \rightarrow \infty} k a_n = k\alpha$$

(k is a constant)

$$\lim_{n \rightarrow \infty} (a_n + b_n) = \alpha + \beta,$$

$$\lim_{n \rightarrow \infty} (a_n - b_n) = \alpha - \beta$$

$$\lim_{n \rightarrow \infty} a_n b_n = \alpha\beta$$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{\alpha}{\beta}$$

($\beta \neq 0$)

Find the following limits.

Ex.

$$\lim_{n \rightarrow \infty} \frac{2n+1}{3n^2+n} = \lim_{n \rightarrow \infty} \frac{\frac{2}{n} + \frac{1}{n^2}}{3 + \frac{1}{n}} = 0 \quad \leftarrow$$

Dividing the numerator and the denominator by the term with the highest power in the denominator (n^2 in this case)

$$(1) \quad \lim_{n \rightarrow \infty} \frac{2n+3}{4n^2+5n} = \lim_{n \rightarrow \infty} \frac{\frac{2}{n} + \frac{3}{n^2}}{4 + \frac{5}{n}} = 0$$

$$(2) \quad \lim_{n \rightarrow \infty} \frac{8-5n}{4+n} = \lim_{n \rightarrow \infty} \frac{\frac{8}{n} - 5}{\frac{4}{n} + 1} = -5$$

$$(3) \quad \lim_{n \rightarrow \infty} \frac{n^2+5n+4}{3-2n^2} = \lim_{n \rightarrow \infty} \frac{1 + \frac{5}{n} + \frac{4}{n^2}}{\frac{3}{n^2} - 2} = -\frac{1}{2}$$

N63b

$$(4) \lim_{n \rightarrow \infty} \frac{2n^2 - 3n}{n+1} = \lim_{n \rightarrow \infty} \frac{2n-3}{1+\frac{1}{n}} = \infty$$

$$(5) \lim_{n \rightarrow \infty} \frac{-2n^3 - 5n}{n^2 + 2n + 1} = \lim_{n \rightarrow \infty} \frac{-2n - \frac{5}{n}}{1 + \frac{2}{n} + \frac{1}{n^2}} = -\infty$$

$$(6) \lim_{n \rightarrow \infty} \frac{-2n^2(n+2)}{2n^3-5} = \lim_{n \rightarrow \infty} \frac{-2n^3-4n^2}{2n^3-5} = \lim_{n \rightarrow \infty} \frac{-2-\frac{4}{n}}{2-\frac{5}{n^3}} = -1$$

$$(7) \lim_{n \rightarrow \infty} \frac{5+n+n^5}{n(4-3n^2)} = \lim_{n \rightarrow \infty} \frac{5+n+n^5}{4n-3n^3} = \lim_{n \rightarrow \infty} \frac{\frac{5}{n^3} + \frac{1}{n^2} + n^2}{\frac{4}{n^2} - 3} = -\infty$$

$$\lim_{n \rightarrow \infty} \left(\frac{5}{n^3} + \frac{1}{n^2} + n^2 \right) = \infty$$

$$\lim_{n \rightarrow \infty} \left(\frac{4}{n^2} - 3 \right) = -3$$

$$(8) \lim_{n \rightarrow \infty} \frac{\sqrt{n^2+1}}{2n} = \lim_{n \rightarrow \infty} \frac{\sqrt{1+\frac{1}{n^2}}}{2} = \frac{1}{2}$$

$$\frac{\sqrt{n^2+1}}{n} = \sqrt{\frac{n^2+1}{n^2}} = \sqrt{\frac{n^2}{n^2} + \frac{1}{n^2}}$$

$$(9) \lim_{n \rightarrow \infty} \frac{\sqrt{4n^2+3n}}{6n} = \lim_{n \rightarrow \infty} \frac{\sqrt{4+\frac{3}{n}}}{6} = \frac{1}{3}$$

$$(10) \lim_{n \rightarrow \infty} \frac{2n^2+4n+3}{\sqrt{3n^2+1}} = \lim_{n \rightarrow \infty} \frac{2n+4+\frac{3}{n}}{\sqrt{3+\frac{1}{n^2}}} = \infty$$

In **Ex.**, since $\lim_{n \rightarrow \infty} (3n^2+n) = \infty$ and $\lim_{n \rightarrow \infty} (2n+1) = \infty$, the limit becomes an indeterminate form $\frac{\infty}{\infty}$. Therefore, in such cases, the expression has to be rearranged.

N64a

Infinite Sequences

KUMON

N 64

Find the following limits.

Ex.

$$\begin{aligned}\lim_{n \rightarrow \infty} \log_2 \frac{16n+1}{2n} &= \lim_{n \rightarrow \infty} \log_2 \frac{16 + \frac{1}{n}}{2} \\ &= \log_2 8 \\ &= 3\end{aligned}$$

$$\log_a a^m = m$$

$$\begin{aligned}(1) \quad \lim_{n \rightarrow \infty} \log_3 \frac{9n+1}{3n} &= \lim_{n \rightarrow \infty} \log_3 \frac{9 + \frac{1}{n}}{3} \\ &= \log_3 3 \\ &= 1\end{aligned}$$

$$\log_a a = 1$$

$$\begin{aligned}(2) \quad \lim_{n \rightarrow \infty} [\log_3 (2n^2 - 1) - \log_3 (6n^2 + 2)] &= \lim_{n \rightarrow \infty} \log_3 \frac{2n^2 - 1}{6n^2 + 2} \quad \leftarrow \log_a M - \log_a N = \log_a \frac{M}{N} \\ &= \lim_{n \rightarrow \infty} \log_3 \frac{2 - \frac{1}{n^2}}{6 + \frac{2}{n^2}} \\ &= \log_3 \frac{1}{3} \\ &= -1\end{aligned}$$

$$\begin{aligned}(3) \quad \lim_{n \rightarrow \infty} [\log_2 \sqrt{4n^4 + 3n} - \log_2 (2n^2 + n)] &= \lim_{n \rightarrow \infty} \log_2 \frac{\sqrt{4n^4 + 3n}}{2n^2 + n} \\ &= \lim_{n \rightarrow \infty} \log_2 \frac{\sqrt{4 + \frac{3}{n^3}}}{2 + \frac{1}{n}} \\ &= \log_2 1 \\ &= 0\end{aligned}$$

$$\log_a 1 = 0$$

$$\log_a a = 1, \quad \log_a 1 = 0, \quad \log_a \frac{M}{N} = \log_a M - \log_a N$$

N64b

Ex.

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{1+2+3+\cdots+n}{3n^2} &= \lim_{n \rightarrow \infty} \frac{\frac{1}{2}n(n+1)}{3n^2} \quad \leftarrow \dots \quad 1+2+3+\cdots+n = \sum_{k=1}^n k \\ &= \lim_{n \rightarrow \infty} \frac{n+1}{6n} \\ &= \lim_{n \rightarrow \infty} \frac{1+\frac{1}{n}}{6} \\ &= \frac{1}{6}\end{aligned}$$

$$\begin{aligned}(4) \quad \lim_{n \rightarrow \infty} \frac{1^2+2^2+3^2+\cdots+n^2}{n^3} &= \lim_{n \rightarrow \infty} \frac{\frac{1}{6}n(n+1)(2n+1)}{n^3} \quad \leftarrow \dots \quad 1^2+2^2+3^2+\cdots+n^2 = \sum_{k=1}^n k^2 \\ &= \lim_{n \rightarrow \infty} \frac{2n^2+3n+1}{6n^2} \\ &= \lim_{n \rightarrow \infty} \frac{2+\frac{3}{n}+\frac{1}{n^2}}{6} \\ &= \frac{1}{3}\end{aligned}$$

$$\begin{aligned}(5) \quad \lim_{n \rightarrow \infty} \frac{2^3+4^3+6^3+\cdots+(2n)^3}{n^4} &= \lim_{n \rightarrow \infty} \frac{8\left[\frac{1}{2}n(n+1)\right]^2}{n^4} \quad \leftarrow \dots \quad 2^3+4^3+6^3+\cdots+(2n)^3 \\ &= \lim_{n \rightarrow \infty} \frac{2n^2+4n+2}{n^2} \\ &= \lim_{n \rightarrow \infty} \left(2+\frac{4}{n}+\frac{2}{n^2}\right) \\ &= 2\end{aligned}$$

$$\sum_{k=1}^n k^2 = \frac{1}{6}n(n+1)(2n+1), \quad \sum_{k=1}^n k^3 = \left[\frac{1}{2}n(n+1)\right]^2$$

N65a

Infinite Sequences

KUMON

Find the following limits.

Ex. $\lim_{n \rightarrow \infty} (n^2 - 3n) = \lim_{n \rightarrow \infty} n^2 \left(1 - \frac{3}{n}\right) = \infty$ ← Taking out the term with the highest power

(1) $\lim_{n \rightarrow \infty} (n^2 - 10n) = \lim_{n \rightarrow \infty} n^2 \left(1 - \frac{10}{n}\right) = \infty$

(2) $\lim_{n \rightarrow \infty} (2n^3 - 3n^2 + 4) = \lim_{n \rightarrow \infty} n^3 \left(2 - \frac{3}{n} + \frac{4}{n^3}\right) = \infty$

(3) $\lim_{n \rightarrow \infty} (4n^2 - 2n^3) = \lim_{n \rightarrow \infty} n^3 \left(\frac{4}{n} - 2\right) = -\infty$ ← $\lim_{n \rightarrow \infty} n^3 = \infty$
 $\lim_{n \rightarrow \infty} \left(\frac{4}{n} - 2\right) = -2$

(4) $\lim_{n \rightarrow \infty} (n - \sqrt{n}) = \lim_{n \rightarrow \infty} n \left(1 - \frac{1}{\sqrt{n}}\right) = \infty$

(5) $\lim_{n \rightarrow \infty} (\sqrt{n^3} - n^2) = \lim_{n \rightarrow \infty} n^2 \left(\frac{1}{\sqrt{n}} - 1\right) = -\infty$

In **Ex.**, since $\lim_{n \rightarrow \infty} n^2 = \infty$ and $\lim_{n \rightarrow \infty} 3n = \infty$, the limit becomes an indeterminate form $\infty - \infty$. Therefore, in such cases, the expression has to be rearranged.

N65b

Ex.

$$\lim_{n \rightarrow \infty} (\sqrt{n^2 + 3n} - n) = \lim_{n \rightarrow \infty} \frac{(\sqrt{n^2 + 3n} - n)(\sqrt{n^2 + 3n} + n)}{\sqrt{n^2 + 3n} + n}$$

$$= \lim_{n \rightarrow \infty} \frac{3n}{\sqrt{n^2 + 3n} + n}$$

$$= \lim_{n \rightarrow \infty} \frac{3}{\sqrt{1 + \frac{3}{n}} + 1}$$

$$= \frac{3}{2}$$

Considering

$\sqrt{n^2 + 3n} - n = \frac{\sqrt{n^2 + 3n} - n}{1}$,
then multiplying the numerator and
the denominator by $\sqrt{n^2 + 3n} + n$

$$(6) \lim_{n \rightarrow \infty} (\sqrt{n^2 + n + 1} - n) = \lim_{n \rightarrow \infty} \frac{(\sqrt{n^2 + n + 1} - n)(\sqrt{n^2 + n + 1} + n)}{\sqrt{n^2 + n + 1} + n}$$

$$= \lim_{n \rightarrow \infty} \frac{n + 1}{\sqrt{n^2 + n + 1} + n}$$

$$= \lim_{n \rightarrow \infty} \frac{1 + \frac{1}{n}}{\sqrt{1 + \frac{1}{n} + \frac{1}{n^2}} + 1}$$

$$= \frac{1}{2}$$

$$(7) \lim_{n \rightarrow \infty} (\sqrt{2n} + \sqrt{n})(\sqrt{n+1} - \sqrt{n})$$

$$= \lim_{n \rightarrow \infty} \frac{(\sqrt{2n} + \sqrt{n})(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}}$$

$$= \lim_{n \rightarrow \infty} \frac{\sqrt{2n} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}}$$

$$= \lim_{n \rightarrow \infty} \frac{\sqrt{2} + 1}{\sqrt{1 + \frac{1}{n}} + 1}$$

$$= \frac{\sqrt{2} + 1}{2}$$

Dividing the numerator and
the denominator by $\sqrt{n}$

Find the following limits.

Ex.
$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n^2+5n}-n} = \lim_{n \rightarrow \infty} \frac{\sqrt{n^2+5n}+n}{(\sqrt{n^2+5n}-n)(\sqrt{n^2+5n}+n)}$$

$$= \lim_{n \rightarrow \infty} \frac{\sqrt{n^2+5n}+n}{5n}$$

$$= \lim_{n \rightarrow \infty} \frac{\sqrt{1+\frac{5}{n}}+1}{5}$$

$$= \frac{2}{5}$$

Multiplying the numerator and the denominator by $\sqrt{n^2+5n}+n$

(1)
$$\lim_{n \rightarrow \infty} \frac{3}{\sqrt{4n^2+2n}-2n} = \lim_{n \rightarrow \infty} \frac{3(\sqrt{4n^2+2n}+2n)}{(\sqrt{4n^2+2n}-2n)(\sqrt{4n^2+2n}+2n)}$$

$$= \lim_{n \rightarrow \infty} \frac{3(\sqrt{4n^2+2n}+2n)}{2n}$$

$$= \lim_{n \rightarrow \infty} \frac{3\left(\sqrt{4+\frac{2}{n}}+2\right)}{2}$$

$$= 6$$

(2)
$$\lim_{n \rightarrow \infty} \frac{5}{\sqrt{n^2+2}-\sqrt{n}} = \lim_{n \rightarrow \infty} \frac{5(\sqrt{n^2+2}+\sqrt{n})}{(\sqrt{n^2+2}-\sqrt{n})(\sqrt{n^2+2}+\sqrt{n})}$$

$$= \lim_{n \rightarrow \infty} \frac{5(\sqrt{n^2+2}+\sqrt{n})}{n^2-n+2}$$

$$= \lim_{n \rightarrow \infty} \frac{5\left(\sqrt{\frac{1}{n^2}+\frac{2}{n^4}}+\sqrt{\frac{1}{n^3}}\right)}{1-\frac{1}{n}+\frac{2}{n^2}}$$

$$= 0$$

N66b

$$(3) \lim_{n \rightarrow \infty} \frac{\sqrt{n+3} - \sqrt{n}}{\sqrt{n+1} - \sqrt{n}}$$

$$= \lim_{n \rightarrow \infty} \frac{(\sqrt{n+3} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}$$

Multiplying the numerator and the denominator by $\sqrt{n+1} + \sqrt{n}$

$$= \lim_{n \rightarrow \infty} (\sqrt{n+3} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})$$

$$= \lim_{n \rightarrow \infty} \frac{(\sqrt{n+3} - \sqrt{n})(\sqrt{n+3} + \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+3} + \sqrt{n}}$$

Considering

$$\sqrt{n+3} - \sqrt{n} = \frac{\sqrt{n+3} - \sqrt{n}}{1}$$

then multiplying the numerator and the denominator by $\sqrt{n+3} + \sqrt{n}$

$$= \lim_{n \rightarrow \infty} \frac{3(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+3} + \sqrt{n}}$$

Alternative Solution

$$\lim_{n \rightarrow \infty} \frac{\sqrt{n+3} - \sqrt{n}}{\sqrt{n+1} - \sqrt{n}}$$

$$= \lim_{n \rightarrow \infty} \frac{(\sqrt{n+3} - \sqrt{n})(\sqrt{n+3} + \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})(\sqrt{n+3} + \sqrt{n})}$$

$$= \lim_{n \rightarrow \infty} \frac{3\left(\sqrt{1 + \frac{1}{n}} + 1\right)}{\sqrt{1 + \frac{3}{n}} + 1}$$

$$= \lim_{n \rightarrow \infty} \frac{3(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+3} + \sqrt{n}} = \lim_{n \rightarrow \infty} \frac{3\left(\sqrt{1 + \frac{1}{n}} + 1\right)}{\sqrt{1 + \frac{3}{n}} + 1}$$

$$= 3$$

$$= 3$$

$$(4) \lim_{n \rightarrow \infty} \frac{2(\sqrt{n+5} - \sqrt{n+3})}{\sqrt{n+2} - \sqrt{n}}$$

$$= \lim_{n \rightarrow \infty} \frac{2(\sqrt{n+5} - \sqrt{n+3})(\sqrt{n+2} + \sqrt{n})}{(\sqrt{n+2} - \sqrt{n})(\sqrt{n+2} + \sqrt{n})}$$

$$= \lim_{n \rightarrow \infty} (\sqrt{n+5} - \sqrt{n+3})(\sqrt{n+2} + \sqrt{n})$$

$$= \lim_{n \rightarrow \infty} \frac{(\sqrt{n+5} - \sqrt{n+3})(\sqrt{n+5} + \sqrt{n+3})(\sqrt{n+2} + \sqrt{n})}{\sqrt{n+5} + \sqrt{n+3}}$$

$$= \lim_{n \rightarrow \infty} \frac{2(\sqrt{n+2} + \sqrt{n})}{\sqrt{n+5} + \sqrt{n+3}}$$

Alternative Solution

$$\lim_{n \rightarrow \infty} \frac{2(\sqrt{n+5} - \sqrt{n+3})}{\sqrt{n+2} - \sqrt{n}}$$

$$= \lim_{n \rightarrow \infty} \frac{2(\sqrt{n+5} - \sqrt{n+3})(\sqrt{n+5} + \sqrt{n+3})(\sqrt{n+2} + \sqrt{n})}{(\sqrt{n+2} - \sqrt{n})(\sqrt{n+2} + \sqrt{n})(\sqrt{n+5} + \sqrt{n+3})}$$

$$= \lim_{n \rightarrow \infty} \frac{2\left(\sqrt{1 + \frac{2}{n}} + 1\right)}{\sqrt{1 + \frac{5}{n}} + \sqrt{1 + \frac{3}{n}}}$$

$$= \lim_{n \rightarrow \infty} \frac{2(\sqrt{n+2} + \sqrt{n})}{\sqrt{n+5} + \sqrt{n+3}} = \lim_{n \rightarrow \infty} \frac{2\left(\sqrt{1 + \frac{2}{n}} + 1\right)}{\sqrt{1 + \frac{5}{n}} + \sqrt{1 + \frac{3}{n}}}$$

$$= 2$$

$$= 2$$

Limits of Sequences and Their Relationships

1. For all n , when $a_n \leq b_n$,
 if $\lim_{n \rightarrow \infty} a_n = \alpha$ and $\lim_{n \rightarrow \infty} b_n = \beta$, then $\alpha \leq \beta$
 if $\lim_{n \rightarrow \infty} a_n = \infty$, then $\lim_{n \rightarrow \infty} b_n = \infty$
2. For all n , when $a_n \leq c_n \leq b_n$,
 if $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = \alpha$, then $\lim_{n \rightarrow \infty} c_n = \alpha$

Find the following limits.

Ex.

$$\lim_{n \rightarrow \infty} \frac{(-1)^n}{n}$$

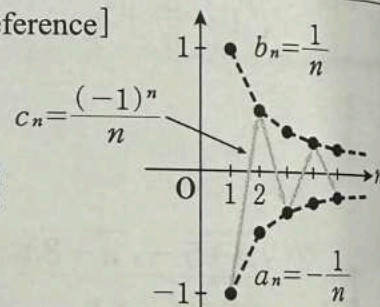
[Reference]

[Sol] Since $-1 \leq (-1)^n \leq 1$,

$$-\frac{1}{n} \leq \frac{(-1)^n}{n} \leq \frac{1}{n} \quad (n > 0)$$

Then, since $\lim_{n \rightarrow \infty} \left(-\frac{1}{n}\right) = 0$ and $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$,

$$\lim_{n \rightarrow \infty} \frac{(-1)^n}{n} = 0$$



$$(1) \lim_{n \rightarrow \infty} \frac{(-1)^n}{\sqrt{n}}$$

[Sol] Since $-1 \leq (-1)^n \leq 1$,

$$-\frac{1}{\sqrt{n}} \leq \frac{(-1)^n}{\sqrt{n}} \leq \frac{1}{\sqrt{n}}$$

Then, since $\lim_{n \rightarrow \infty} \left(-\frac{1}{\sqrt{n}}\right) = 0$ and $\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0$,

$$\lim_{n \rightarrow \infty} \frac{(-1)^n}{\sqrt{n}} = 0$$

As for Limits of Sequences and Their Relationships, statement 1 is also true when $a_n < b_n$, and statement 2 is also true when $a_n \leq c_n < b_n$, $a_n < c_n \leq b_n$, $a_n < c_n < b_n$.

N67b

(2) $\lim_{n \rightarrow \infty} \frac{1}{n} \sin \frac{n\pi}{2}$

[Sol] Since $-1 \leq \sin \frac{n\pi}{2} \leq 1$,

$$-\frac{1}{n} \leq \frac{1}{n} \sin \frac{n\pi}{2} \leq \frac{1}{n}$$

Then, since $\lim_{n \rightarrow \infty} \left(-\frac{1}{n}\right) = 0$ and $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sin \frac{n\pi}{2} = 0$$

← $-1 \leq \sin \theta \leq 1$ (M131)

(3) $\lim_{n \rightarrow \infty} \frac{n+1}{n^2} \cos n\theta$ (θ is a constant)

[Sol] Since $-1 \leq \cos n\theta \leq 1$,

$$-\frac{n+1}{n^2} \leq \frac{n+1}{n^2} \cos n\theta \leq \frac{n+1}{n^2}$$

Then, since $\lim_{n \rightarrow \infty} \left(-\frac{n+1}{n^2}\right) = \lim_{n \rightarrow \infty} \left(-\frac{1}{n} - \frac{1}{n^2}\right) = 0$ and

$$\lim_{n \rightarrow \infty} \frac{n+1}{n^2} = \lim_{n \rightarrow \infty} \left(\frac{1}{n} + \frac{1}{n^2}\right) = 0,$$

$$\lim_{n \rightarrow \infty} \frac{n+1}{n^2} \cos n\theta = 0$$

(4) $\lim_{n \rightarrow \infty} \frac{2^n}{n}$ (Using inequality $2^n > \frac{n^2}{2}$)

[Sol] Since $2^n > \frac{n^2}{2}$,

$$\frac{2^n}{n} > \frac{n}{2}$$

Then, since $\lim_{n \rightarrow \infty} \frac{n}{2} = \infty$,

$$\lim_{n \rightarrow \infty} \frac{2^n}{n} = \infty$$

Find the following limits.

Ex. $\lim_{n \rightarrow \infty} \frac{1}{2n-1} (n + \sin n\theta)$

[Sol] $\lim_{n \rightarrow \infty} \frac{1}{2n-1} (n + \sin n\theta)$

$$= \lim_{n \rightarrow \infty} \left(\frac{n}{2n-1} + \frac{1}{2n-1} \sin n\theta \right)$$

Then, $\lim_{n \rightarrow \infty} \frac{n}{2n-1} = \lim_{n \rightarrow \infty} \frac{1}{2 - \frac{1}{n}} = \boxed{\frac{1}{2}} \dots \textcircled{1}$

$\lim_{n \rightarrow \infty} \frac{n}{2n-1}$
converges.

Also, since $-1 \leq \sin n\theta \leq 1$,

$$\boxed{-\frac{1}{2n-1}} \leq \frac{1}{2n-1} \sin n\theta \leq \boxed{\frac{1}{2n-1}}$$

$2n-1 > 0$

Then, since $\lim_{n \rightarrow \infty} \left(\boxed{-\frac{1}{2n-1}} \right) = \boxed{0}$ and $\lim_{n \rightarrow \infty} \boxed{\frac{1}{2n-1}} = \boxed{0}$,

$$\lim_{n \rightarrow \infty} \frac{1}{2n-1} \sin n\theta = \boxed{0} \dots \textcircled{2}$$

$\lim_{n \rightarrow \infty} \frac{1}{2n-1} \sin n\theta$
converges.

From $\textcircled{1}$ and $\textcircled{2}$,

$$\lim_{n \rightarrow \infty} \frac{1}{2n-1} (n + \sin n\theta) = \boxed{\frac{1}{2}}$$

From $\textcircled{1}$ and $\textcircled{2}$,

$$\lim_{n \rightarrow \infty} \left(\frac{n}{2n-1} + \frac{1}{2n-1} \sin n\theta \right) = \lim_{n \rightarrow \infty} \frac{n}{2n-1} + \lim_{n \rightarrow \infty} \frac{1}{2n-1} \sin n\theta$$

Answers: in order $\frac{1}{2}, \frac{1}{2}, \frac{1}{2n-1}, \frac{1}{2n-1}, \frac{1}{2n-1}, 0, \frac{1}{2n-1}, 0, 0, \frac{1}{2}$

N68b

$$(1) \lim_{n \rightarrow \infty} \frac{1}{5n-2} \left(n + \cos \frac{n\pi}{6} \right)$$

$$[\text{Sol}] \lim_{n \rightarrow \infty} \frac{1}{5n-2} \left(n + \cos \frac{n\pi}{6} \right)$$

$$= \lim_{n \rightarrow \infty} \left(\frac{n}{5n-2} + \frac{1}{5n-2} \cos \frac{n\pi}{6} \right)$$

$$\text{Then, } \lim_{n \rightarrow \infty} \frac{n}{5n-2} = \lim_{n \rightarrow \infty} \frac{1}{5 - \frac{2}{n}} = \frac{1}{5} \dots \textcircled{1}$$

$$\text{Also, since } -1 \leq \cos \frac{n\pi}{6} \leq 1,$$

$$-\frac{1}{5n-2} \leq \frac{1}{5n-2} \cos \frac{n\pi}{6} \leq \frac{1}{5n-2}$$

$$\text{Then, since } \lim_{n \rightarrow \infty} \left(-\frac{1}{5n-2} \right) = 0 \text{ and } \lim_{n \rightarrow \infty} \frac{1}{5n-2} = 0,$$

$$\lim_{n \rightarrow \infty} \frac{1}{5n-2} \cos \frac{n\pi}{6} = 0 \dots \textcircled{2}$$

From $\textcircled{1}$ and $\textcircled{2}$,

$$\lim_{n \rightarrow \infty} \frac{1}{5n-2} \left(n + \cos \frac{n\pi}{6} \right) = \frac{1}{5}$$

from Kumon Hub

1. Given the sequence $\{a_n\} : a_1=5, a_2=12, a_3=21, a_4=32, \dots$, when the sequence of differences $\{b_n\}$ of this sequence becomes an arithmetic sequence, solve the following questions.

(1) Find a_n .

[Sol] $\{b_n\}$ is 7, 9, 11,

$$\therefore b_n = 7 + (n-1) \cdot 2 = 2n + 5$$

When $n \geq 2$,

$$\begin{aligned} a_n &= a_1 + \sum_{k=1}^{n-1} (2k+5) \\ &= 5 + 2 \cdot \frac{1}{2} (n-1)n + 5(n-1) \\ &= n^2 + 4n \quad \cdots \textcircled{1} \end{aligned}$$

Since $a_1=5$, $\textcircled{1}$ is also true when $n=1$.

$$\therefore a_n = n^2 + 4n$$

(2) Find $\lim_{n \rightarrow \infty} (\sqrt{a_n} - n)$ and $\lim_{n \rightarrow \infty} \left(\sqrt{\frac{a_n}{n}} - \sqrt{n} \right)$.

[Sol] From (1),

$$\begin{aligned} \lim_{n \rightarrow \infty} (\sqrt{a_n} - n) &= \lim_{n \rightarrow \infty} (\sqrt{n^2 + 4n} - n) = \lim_{n \rightarrow \infty} \frac{(\sqrt{n^2 + 4n} - n)(\sqrt{n^2 + 4n} + n)}{\sqrt{n^2 + 4n} + n} \\ &= \lim_{n \rightarrow \infty} \frac{4n}{\sqrt{n^2 + 4n} + n} \\ &= \lim_{n \rightarrow \infty} \frac{4}{\sqrt{1 + \frac{4}{n}} + 1} \\ &= 2 \end{aligned}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left(\sqrt{\frac{a_n}{n}} - \sqrt{n} \right) &= \lim_{n \rightarrow \infty} \left(\sqrt{\frac{n^2 + 4n}{n}} - \sqrt{n} \right) \\ &= \lim_{n \rightarrow \infty} (\sqrt{n+4} - \sqrt{n}) \\ &= \lim_{n \rightarrow \infty} \frac{(\sqrt{n+4} - \sqrt{n})(\sqrt{n+4} + \sqrt{n})}{\sqrt{n+4} + \sqrt{n}} \\ &= \lim_{n \rightarrow \infty} \frac{4}{\sqrt{n+4} + \sqrt{n}} \\ &= 0 \end{aligned}$$

$$\{a_n\} : 5 \quad 12 \quad 21 \quad 32 \dots$$

$$\{b_n\} : 7 \quad 9 \quad 11 \dots$$

$\{b_n\}$ is the arithmetic sequence with 1st term 7 and common difference 2.

When $n \geq 2$,

$$a_n = a_1 + \sum_{k=1}^{n-1} b_k$$

(N32)

N69b

2. Given that n is a positive integer and $x \geq 0$, solve the following questions.

(1) Using inequality $(1+x)^n \geq 1+nx+\frac{1}{2}n(n-1)x^2$, prove that

$$1+\sqrt{\frac{2}{n}} > n^{\frac{1}{n}} \text{ is true.}$$

[Sol] In $(1+x)^n \geq 1+nx+\frac{1}{2}n(n-1)x^2$, let $x = \sqrt{\frac{2}{n}}$.

$$\left(1+\sqrt{\frac{2}{n}}\right)^n \geq 1+n\sqrt{\frac{2}{n}}+\frac{1}{2}n(n-1)\cdot\frac{2}{n}$$

$$=1+\sqrt{2n}+n-1$$

$$=n+\sqrt{2n} > n \quad \leftarrow \sqrt{2n} > 0$$

$$\therefore \left(1+\sqrt{\frac{2}{n}}\right)^n > n$$

$$1+\sqrt{\frac{2}{n}} > 0, \quad n > 0; \text{ therefore,}$$

$$1+\sqrt{\frac{2}{n}} > n^{\frac{1}{n}}$$

When $a > 0$, $b > 0$ and $p > 0$,
 $a^p < b^p \Leftrightarrow a < b$
 (K188)

(2) Find the value of $\lim_{n \rightarrow \infty} n^{\frac{1}{n}}$.

[Sol] Since $n \geq 1$, $n^{\frac{1}{n}} \geq 1^{\frac{1}{n}} = 1 \dots \textcircled{1}$

For all positive integers n ,
 $1^{\frac{1}{n}} = 1$

From $\textcircled{1}$ and (1),

$$1 \leq n^{\frac{1}{n}} < 1+\sqrt{\frac{2}{n}}$$

Then, since $\lim_{n \rightarrow \infty} 1 = 1$ and $\lim_{n \rightarrow \infty} \left(1+\sqrt{\frac{2}{n}}\right) = 1$, $\leftarrow \lim_{n \rightarrow \infty} c = c$ (c is a constant)

$$\lim_{n \rightarrow \infty} n^{\frac{1}{n}} = 1$$

Find the following limits.

$$(1) \lim_{n \rightarrow \infty} \frac{5-2n^2}{3n^2+4} = \lim_{n \rightarrow \infty} \frac{\frac{5}{n^2}-2}{3+\frac{4}{n^2}} = -\frac{2}{3}$$

➡ N63

$$(2) \lim_{n \rightarrow \infty} \frac{2n^2+3n+1}{12n^3-18n} = \lim_{n \rightarrow \infty} \frac{\frac{2}{n} + \frac{3}{n^2} + \frac{1}{n^3}}{12 - \frac{18}{n^2}} = 0$$

➡ N63

$$(3) \lim_{n \rightarrow \infty} \frac{n^2+6}{\sqrt{5n^2+4}} = \lim_{n \rightarrow \infty} \frac{n + \frac{6}{n}}{\sqrt{5 + \frac{4}{n^2}}} = \infty$$

➡ N63

$$(4) \lim_{n \rightarrow \infty} (n^3 - 6n^2) = \lim_{n \rightarrow \infty} n^3 \left(1 - \frac{6}{n}\right) = \infty$$

➡ N65

N70b

$$(5) \lim_{n \rightarrow \infty} (\sqrt{n^2 - 9n} - n) = \lim_{n \rightarrow \infty} \frac{(\sqrt{n^2 - 9n} - n)(\sqrt{n^2 - 9n} + n)}{\sqrt{n^2 - 9n} + n}$$

➡ N65

$$= \lim_{n \rightarrow \infty} \frac{-9n}{\sqrt{n^2 - 9n} + n}$$

$$= \lim_{n \rightarrow \infty} \frac{-9}{\sqrt{1 - \frac{9}{n}} + 1}$$

$$= -\frac{9}{2}$$

$$(6) \lim_{n \rightarrow \infty} \frac{4}{\sqrt{n^2 + 3n} - n} = \lim_{n \rightarrow \infty} \frac{4(\sqrt{n^2 + 3n} + n)}{(\sqrt{n^2 + 3n} - n)(\sqrt{n^2 + 3n} + n)}$$

➡ N66

$$= \lim_{n \rightarrow \infty} \frac{4(\sqrt{n^2 + 3n} + n)}{3n}$$

$$= \lim_{n \rightarrow \infty} \frac{4\left(\sqrt{1 + \frac{3}{n}} + 1\right)}{3}$$

$$= \frac{8}{3}$$

$$(7) \lim_{n \rightarrow \infty} \frac{1}{n} \cos \frac{n\pi}{3}$$

➡ N67

[Sol] Since $-1 \leq \cos \frac{n\pi}{3} \leq 1$,

$$-\frac{1}{n} \leq \frac{1}{n} \cos \frac{n\pi}{3} \leq \frac{1}{n}$$

Then, since $\lim_{n \rightarrow \infty} \left(-\frac{1}{n}\right) = 0$ and $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \cos \frac{n\pi}{3} = 0$$

N71a

Infinite Geometric Sequences

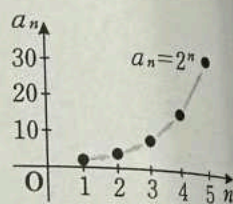
KUMON

A sequence $a, ar, ar^2, \dots, ar^{n-1}, \dots$ is called an *infinite geometric sequence* with 1st term a and common ratio r .

Find the limit for each given infinite geometric sequence. (When it oscillates, write "Oscillates".)

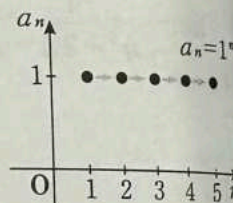
[1] $2, 4, 8, 16, \dots, 2^n, \dots$

[Sol] $\lim_{n \rightarrow \infty} 2^n = \boxed{\infty}$



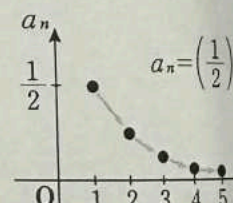
[2] $1, 1, 1, 1, \dots, 1^n, \dots$

[Sol] $\lim_{n \rightarrow \infty} 1^n = \boxed{1}$



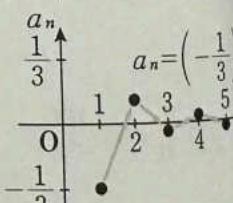
[3] $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots, \left(\frac{1}{2}\right)^n, \dots$

[Sol] $\lim_{n \rightarrow \infty} \left(\frac{1}{2}\right)^n = \boxed{0}$



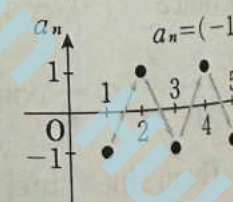
[4] $-\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \frac{1}{81}, \dots, \left(-\frac{1}{3}\right)^n, \dots$

[Sol] $\lim_{n \rightarrow \infty} \left(-\frac{1}{3}\right)^n = \boxed{0}$



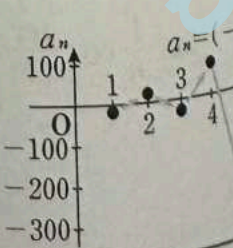
[5] $-1, 1, -1, 1, \dots, (-1)^n, \dots$

[Sol] $\boxed{\text{Oscillates}}$



[6] $-3, 9, -27, 81, \dots, (-3)^n, \dots$

[Sol] $\boxed{\text{Oscillates}}$



Answers: in order $\infty, 1, 0, 0$, Oscillates, Oscillates

Limit of an Infinite Geometric Sequence $\{r^n\}$

When $r > 1$, $\lim_{n \rightarrow \infty} r^n = \infty$

... Diverges

When $r = 1$, $\lim_{n \rightarrow \infty} r^n = 1$

When $|r| < 1$, $\lim_{n \rightarrow \infty} r^n = 0$

... Converges

When $r \leq -1$, Oscillates (no limit) ... Diverges

Classify each given limit by indicating the appropriate number ① to ④.

Converges ... ①

Diverges to positive infinity ... ②

Diverges to negative infinity ... ③

Oscillates (no limit) ... ④

If the limit is classified as ①, state the limit value.

Ex. $\lim_{n \rightarrow \infty} \left(-\frac{1}{5}\right)^n$ $\left|-\frac{1}{5}\right| < 1$ (①, limit value 0)

(1) $\lim_{n \rightarrow \infty} \left(-\frac{2}{3}\right)^n$
(①, limit value 0)

(5) $\lim_{n \rightarrow \infty} \left(\frac{\sqrt{5}}{2}\right)^n$
(②)

Since $\sqrt{5} > 2$, $\frac{\sqrt{5}}{2} > 1$

(2) $\lim_{n \rightarrow \infty} \left(\frac{5}{4}\right)^n$
(②)

(6) $\lim_{n \rightarrow \infty} \left[1 + \left(\frac{\sqrt{3}}{2}\right)^n\right]$
(①, limit value 1)

(3) $\lim_{n \rightarrow \infty} \left[-\left(\frac{5}{3}\right)^n\right]$
(③)

(7) $\lim_{n \rightarrow \infty} \left[-\left(\frac{3}{2}\right)^{n+1}\right]$
(③)

$-\left(\frac{3}{2}\right)^{n+1} = -\frac{3}{2} \left(\frac{3}{2}\right)^n$

(4) $\lim_{n \rightarrow \infty} \left(-\frac{4}{3}\right)^n$
(④)

(8) $\lim_{n \rightarrow \infty} \left[\left(\frac{1}{2}\right)^n + (-\sqrt{2})^n\right]$
(④)

$\lim_{n \rightarrow \infty} \left(\frac{1}{2}\right)^n = 0$ and $\lim_{n \rightarrow \infty} (-\sqrt{2})^n$ oscillates.

Infinite Geometric Sequences

Find the following limits.

Ex. $\lim_{n \rightarrow \infty} \frac{1+3^n}{3^n+4^n} = \lim_{n \rightarrow \infty} \frac{\left(\frac{1}{4}\right)^n + \left(\frac{3}{4}\right)^n}{\left(\frac{3}{4}\right)^n + 1} = 0 \leftarrow$

Dividing the numerator and the denominator by the term with the largest absolute value of the base in the denominator (4^n in this case)

$$(1) \lim_{n \rightarrow \infty} \frac{1+2^n}{2^n+3^n} = \lim_{n \rightarrow \infty} \frac{\left(\frac{1}{3}\right)^n + \left(\frac{2}{3}\right)^n}{\left(\frac{2}{3}\right)^n + 1} = 0$$

$$(2) \lim_{n \rightarrow \infty} \frac{(-2)^n + 2 \cdot 3^n}{3^n + 1} = \lim_{n \rightarrow \infty} \frac{\left(-\frac{2}{3}\right)^n + 2}{1 + \left(\frac{1}{3}\right)^n} = 2$$

$$(3) \lim_{n \rightarrow \infty} \frac{3^{n+1}}{4^n + 2^n} = \lim_{n \rightarrow \infty} \frac{3\left(\frac{3}{4}\right)^n}{1 + \left(\frac{1}{2}\right)^n} = 0$$

$$(4) \lim_{n \rightarrow \infty} \frac{5^n - 3^{n-1}}{5^n + 3^n} = \lim_{n \rightarrow \infty} \frac{1 - \frac{1}{3}\left(\frac{3}{5}\right)^n}{1 + \left(\frac{3}{5}\right)^n} = 1$$

$$(5) \lim_{n \rightarrow \infty} \frac{3^{n-1} + 4^{n+1}}{3^n - 4^n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{3}\left(\frac{3}{4}\right)^n + 4}{\left(\frac{3}{4}\right)^n - 1} = -4$$

N72b

$$(6) \lim_{n \rightarrow \infty} \frac{4^n}{2^n + 3^n} = \lim_{n \rightarrow \infty} \frac{\left(\frac{4}{3}\right)^n}{\left(\frac{2}{3}\right)^n + 1} = \infty$$

$$(7) \lim_{n \rightarrow \infty} \frac{-5^n + 3^{n+1}}{3^n + 4^n} = \lim_{n \rightarrow \infty} \frac{-\left(\frac{5}{4}\right)^n + 3\left(\frac{3}{4}\right)^n}{\left(\frac{3}{4}\right)^n + 1} = -\infty$$

$$(8) \lim_{n \rightarrow \infty} \frac{1 + (-3)^n}{2^n - (-3)^n} = \lim_{n \rightarrow \infty} \frac{\left(-\frac{1}{3}\right)^n + 1}{\left(-\frac{2}{3}\right)^n - 1} = -1$$

$$(9) \lim_{n \rightarrow \infty} \frac{4^n + (-8)^n}{(-6)^n + 5^n} = \lim_{n \rightarrow \infty} \frac{\left(-\frac{2}{3}\right)^n + \left(\frac{4}{3}\right)^n}{1 + \left(-\frac{5}{6}\right)^n} = \infty$$

$$(10) \lim_{n \rightarrow \infty} \frac{9^{n+1} + 3^n}{3^{2n} - (-5)^n} = \lim_{n \rightarrow \infty} \frac{9^{n+1} + 3^n}{9^n - (-5)^n} = \lim_{n \rightarrow \infty} \frac{9 + \left(\frac{1}{3}\right)^n}{1 - \left(-\frac{5}{9}\right)^n} = 9$$

$$(11) \lim_{n \rightarrow \infty} \frac{7 + 4^{2n}}{6^n - 2} = \lim_{n \rightarrow \infty} \frac{7 + 16^n}{6^n - 2} = \lim_{n \rightarrow \infty} \frac{7\left(\frac{1}{6}\right)^n + \left(\frac{8}{3}\right)^n}{1 - 2\left(\frac{1}{6}\right)^n} = \infty$$

Infinite Geometric Sequences

Find the following limits.

Ex.

$$\lim_{n \rightarrow \infty} (3^n - 2^n) = \lim_{n \rightarrow \infty} 3^n \left[1 - \left(\frac{2}{3} \right)^n \right] = \infty \quad \leftarrow \dots$$

Taking out the term with the largest absolute value of the base

$$(1) \quad \lim_{n \rightarrow \infty} (4^n - 3^n) = \lim_{n \rightarrow \infty} 4^n \left[1 - \left(\frac{3}{4} \right)^n \right] = \infty$$

$$(2) \quad \lim_{n \rightarrow \infty} (3^n - 5^n) = \lim_{n \rightarrow \infty} 5^n \left[\left(\frac{3}{5} \right)^n - 1 \right] = -\infty$$

$$(3) \quad \lim_{n \rightarrow \infty} [4^n + (-2)^n] = \lim_{n \rightarrow \infty} 4^n \left[1 + \left(-\frac{1}{2} \right)^n \right] = \infty$$

$$(4) \quad \lim_{n \rightarrow \infty} (2^{3n} - 3^{2n}) = \lim_{n \rightarrow \infty} (8^n - 9^n) = \lim_{n \rightarrow \infty} 9^n \left[\left(\frac{8}{9} \right)^n - 1 \right] = -\infty$$

$$(5) \quad \lim_{n \rightarrow \infty} \frac{5^n - 6^n}{2^{2n}} = \lim_{n \rightarrow \infty} \frac{6^n \left[\left(\frac{5}{6} \right)^n - 1 \right]}{4^n} = \lim_{n \rightarrow \infty} \left(\frac{3}{2} \right)^n \left[\left(\frac{5}{6} \right)^n - 1 \right] = -\infty$$

Alternative Solution

$$\lim_{n \rightarrow \infty} \frac{5^n - 6^n}{2^{2n}} = \lim_{n \rightarrow \infty} \left[\left(\frac{5}{4} \right)^n - \left(\frac{3}{2} \right)^n \right] = \lim_{n \rightarrow \infty} \left(\frac{3}{2} \right)^n \left[\left(\frac{5}{6} \right)^n - 1 \right] = -\infty$$

$$(6) \quad \lim_{n \rightarrow \infty} \frac{4^n - 3^{n+1}}{2^n} = \lim_{n \rightarrow \infty} \frac{4^n \left[1 - 3 \left(\frac{3}{4} \right)^n \right]}{2^n} = \lim_{n \rightarrow \infty} 2^n \left[1 - 3 \left(\frac{3}{4} \right)^n \right] = \infty$$

Alternative Solution

$$\lim_{n \rightarrow \infty} \frac{4^n - 3^{n+1}}{2^n} = \lim_{n \rightarrow \infty} \left[2^n - 3 \left(\frac{3}{2} \right)^n \right] = \lim_{n \rightarrow \infty} 2^n \left[1 - 3 \left(\frac{3}{4} \right)^n \right] = \infty$$

Ex.

$$\lim_{n \rightarrow \infty} \frac{2 \cdot 2^2 \cdots 2^n}{3^{n(n+1)}} = \lim_{n \rightarrow \infty} \frac{2^{1+2+\cdots+n}}{3^{n(n+1)}}$$

$$= \lim_{n \rightarrow \infty} \frac{2^{\frac{1}{2}n(n+1)}}{9^{\frac{1}{2}n(n+1)}}$$

$$= \lim_{n \rightarrow \infty} \left(\frac{2}{9} \right)^{\frac{1}{2}n(n+1)}$$

$$= 0$$

Rearranging into the form $\frac{A^\bullet}{B^\bullet}$ so that the value $\bullet$ is the same

$$3^{n(n+1)}$$

$$= 3^{2 \cdot \frac{1}{2}n(n+1)}$$

$$= 9^{\frac{1}{2}n(n+1)}$$

$$(7) \lim_{n \rightarrow \infty} \frac{2 \cdot 2^2 \cdots 2^n}{8^{n(n+1)}} = \lim_{n \rightarrow \infty} \frac{2^{1+2+\cdots+n}}{8^{n(n+1)}}$$

$$= \lim_{n \rightarrow \infty} \frac{2^{\frac{1}{2}n(n+1)}}{64^{\frac{1}{2}n(n+1)}}$$

$$= \lim_{n \rightarrow \infty} \left(\frac{1}{32} \right)^{\frac{1}{2}n(n+1)}$$

$$= 0$$

$$(8) \lim_{n \rightarrow \infty} \frac{2^{2n(n+1)}}{16 \cdot 16^2 \cdots 16^n} = \lim_{n \rightarrow \infty} \frac{2^{2n(n+1)}}{16^{1+2+\cdots+n}}$$

$$= \lim_{n \rightarrow \infty} \frac{16^{\frac{1}{2}n(n+1)}}{16^{\frac{1}{2}n(n+1)}}$$

$$= \lim_{n \rightarrow \infty} 1$$

$$= 1$$

$$2^{2n(n+1)}$$

$$= 4^{n(n+1)}$$

$$= 4^{2 \cdot \frac{1}{2}n(n+1)}$$

$$= 16^{\frac{1}{2}n(n+1)}$$

$\lim_{n \rightarrow \infty} c = c$ (c is a constant)

Alternative Solution

$$\lim_{n \rightarrow \infty} \frac{2^{2n(n+1)}}{16 \cdot 16^2 \cdots 16^n} = \lim_{n \rightarrow \infty} \frac{2^{2n(n+1)}}{16^{1+2+\cdots+n}} = \lim_{n \rightarrow \infty} \frac{2^{2n(n+1)}}{16^{\frac{1}{2}n(n+1)}} = \lim_{n \rightarrow \infty} \frac{2^{2n(n+1)}}{2^{2n(n+1)}} = \lim_{n \rightarrow \infty} 1 = 1$$

Infinite Geometric Sequences

Find the following limits.

Ex.

$$\lim_{n \rightarrow \infty} \frac{r^n}{1+r^n} \quad (r \neq -1)$$

[Sol] (i) When $|r| < 1$,

$$\lim_{n \rightarrow \infty} \frac{r^n}{1+r^n} = \frac{0}{1+0} = 0$$



$$\lim_{n \rightarrow \infty} r^n = 0$$

(ii) When $r = 1$,

$$\lim_{n \rightarrow \infty} \frac{r^n}{1+r^n} = \frac{1}{1+1} = \frac{1}{2}$$



$$\lim_{n \rightarrow \infty} r^n = 1$$

(iii) When $|r| > 1$,

$$\lim_{n \rightarrow \infty} \frac{r^n}{1+r^n} = \lim_{n \rightarrow \infty} \frac{1}{\left(\frac{1}{r}\right)^n + 1} = \frac{1}{0+1} = 1$$



$$\lim_{n \rightarrow \infty} \left(\frac{1}{r}\right)^n = 0$$

$$(1) \lim_{n \rightarrow \infty} \frac{1-r^n}{1+r^n} \quad (r \neq -1)$$

[Sol] (i) When $|r| < 1$,

$$\lim_{n \rightarrow \infty} \frac{1-r^n}{1+r^n} = \frac{1-0}{1+0} = 1$$

(ii) When $r = 1$,

$$\lim_{n \rightarrow \infty} \frac{1-r^n}{1+r^n} = \frac{1-1}{1+1} = 0$$

(iii) When $|r| > 1$,

$$\lim_{n \rightarrow \infty} \frac{1-r^n}{1+r^n} = \lim_{n \rightarrow \infty} \frac{\left(\frac{1}{r}\right)^n - 1}{\left(\frac{1}{r}\right)^n + 1} = \frac{0-1}{0+1} = -1$$

In **Ex.** above, when $r = -1$, the denominator becomes 0 if n is an odd number. Therefore, the condition $r \neq -1$ is added.

N74b

$$(2) \lim_{n \rightarrow \infty} \frac{r^{n+1} - 1}{r^n + 1} \quad (r \neq -1)$$

[Sol] (i) When $|r| < 1$,

$$\lim_{n \rightarrow \infty} \frac{r^{n+1} - 1}{r^n + 1} = \frac{0 - 1}{0 + 1} = -1$$

(ii) When $r = 1$,

$$\lim_{n \rightarrow \infty} \frac{r^{n+1} - 1}{r^n + 1} = \frac{1 - 1}{1 + 1} = 0$$

(iii) When $|r| > 1$,

$$\lim_{n \rightarrow \infty} \frac{r^{n+1} - 1}{r^n + 1} = \lim_{n \rightarrow \infty} \frac{r - \left(\frac{1}{r}\right)^n}{1 + \left(\frac{1}{r}\right)^n} = \frac{r - 0}{1 + 0} = r$$

$$(3) \lim_{n \rightarrow \infty} \frac{r^{2n+1} - 1}{r^{2n} + 1}$$

(Solving for four different cases: $|r| < 1$, $r = 1$, $r = -1$ and $|r| > 1$)

[Sol] (i) When $|r| < 1$,

$$\lim_{n \rightarrow \infty} \frac{r^{2n+1} - 1}{r^{2n} + 1} = \frac{0 - 1}{0 + 1} = -1$$

(ii) When $r = 1$,

$$\lim_{n \rightarrow \infty} \frac{r^{2n+1} - 1}{r^{2n} + 1} = \frac{1 - 1}{1 + 1} = 0$$

(iii) When $r = -1$,

$$\lim_{n \rightarrow \infty} \frac{r^{2n+1} - 1}{r^{2n} + 1} = \frac{-1 - 1}{1 + 1} = -1$$

When m is a natural number,
 $(-1)^{2m} = 1$
 $(-1)^{2m+1} = -1$

(iv) When $|r| > 1$,

$$\lim_{n \rightarrow \infty} \frac{r^{2n+1} - 1}{r^{2n} + 1} = \lim_{n \rightarrow \infty} \frac{r - \left(\frac{1}{r}\right)^{2n}}{1 + \left(\frac{1}{r}\right)^{2n}} = \frac{r - 0}{1 + 0} = r$$

Infinite Geometric Sequences

Find the following limits.

Ex. $\lim_{n \rightarrow \infty} \frac{r^n + 2^n}{r^{n+1} + 4^n} \quad (r \neq -4)$

[Sol] (i) When $|r| < 4$,

$$\lim_{n \rightarrow \infty} \frac{r^n + 2^n}{r^{n+1} + 4^n} = \lim_{n \rightarrow \infty} \frac{\left(\frac{r}{4}\right)^n + \left(\frac{1}{2}\right)^n}{r\left(\frac{r}{4}\right)^n + 1} = \frac{0+0}{r \cdot 0 + 1} = 0$$

(ii) When $r = 4$,

$$\lim_{n \rightarrow \infty} \frac{r^n + 2^n}{r^{n+1} + 4^n} = \lim_{n \rightarrow \infty} \frac{4^n + 2^n}{4^{n+1} + 4^n} = \lim_{n \rightarrow \infty} \frac{4^n \left[1 + \left(\frac{1}{2}\right)^n\right]}{5 \cdot 4^n} = \frac{1+0}{5} = \frac{1}{5}$$

(iii) When $|r| > 4$,

$$\lim_{n \rightarrow \infty} \frac{r^n + 2^n}{r^{n+1} + 4^n} = \lim_{n \rightarrow \infty} \frac{1 + \left(\frac{2}{r}\right)^n}{r + \left(\frac{4}{r}\right)^n} = \frac{1+0}{r+0} = \frac{1}{r}$$

(1) $\lim_{n \rightarrow \infty} \frac{r^{n+1} + 3^n}{r^n + 6^n} \quad (r \neq -6)$

[Sol] (i) When $|r| < 6$,

$$\lim_{n \rightarrow \infty} \frac{r^{n+1} + 3^n}{r^n + 6^n} = \lim_{n \rightarrow \infty} \frac{r\left(\frac{r}{6}\right)^n + \left(\frac{1}{2}\right)^n}{\left(\frac{r}{6}\right)^n + 1} = \frac{r \cdot 0 + 0}{0 + 1} = 0$$

(ii) When $r = 6$,

$$\lim_{n \rightarrow \infty} \frac{r^{n+1} + 3^n}{r^n + 6^n} = \lim_{n \rightarrow \infty} \frac{6^{n+1} + 3^n}{6^n + 6^n} = \lim_{n \rightarrow \infty} \frac{6^n \left[6 + \left(\frac{1}{2}\right)^n\right]}{2 \cdot 6^n} = \frac{6+0}{2} = 3$$

(iii) When $|r| > 6$,

$$\lim_{n \rightarrow \infty} \frac{r^{n+1} + 3^n}{r^n + 6^n} = \lim_{n \rightarrow \infty} \frac{r + \left(\frac{3}{r}\right)^n}{1 + \left(\frac{6}{r}\right)^n} = \frac{r+0}{1+0} = r$$

N75b

$$(2) \lim_{n \rightarrow \infty} \frac{r^{2n+1} - 5^{2n}}{r^{2n} + 5^{2n}}$$

[Sol] (i) When $|r| < 5$,

$$\lim_{n \rightarrow \infty} \frac{r^{2n+1} - 5^{2n}}{r^{2n} + 5^{2n}} = \lim_{n \rightarrow \infty} \frac{r \left(\frac{r}{5} \right)^{2n} - 1}{\left(\frac{r}{5} \right)^{2n} + 1} = \frac{r \cdot 0 - 1}{0 + 1} = -1$$

(ii) When $r = 5$,

$$\lim_{n \rightarrow \infty} \frac{r^{2n+1} - 5^{2n}}{r^{2n} + 5^{2n}} = \lim_{n \rightarrow \infty} \frac{5^{2n+1} - 5^{2n}}{5^{2n} + 5^{2n}} = \lim_{n \rightarrow \infty} \frac{4 \cdot 5^{2n}}{2 \cdot 5^{2n}} = 2$$

(iii) When $r = -5$,

$$\lim_{n \rightarrow \infty} \frac{r^{2n+1} - 5^{2n}}{r^{2n} + 5^{2n}} = \lim_{n \rightarrow \infty} \frac{(-5)^{2n+1} - 5^{2n}}{(-5)^{2n} + 5^{2n}} = \lim_{n \rightarrow \infty} \frac{-6 \cdot 5^{2n}}{2 \cdot 5^{2n}} = -3 \leftarrow$$

$$(-5)^{2n} = 5^{2n}$$

(iv) When $|r| > 5$,

$$\lim_{n \rightarrow \infty} \frac{r^{2n+1} - 5^{2n}}{r^{2n} + 5^{2n}} = \lim_{n \rightarrow \infty} \frac{r - \left(\frac{5}{r} \right)^{2n}}{1 + \left(\frac{5}{r} \right)^{2n}} = \frac{r - 0}{1 + 0} = r$$

from Kumon Hub

Infinite Geometric Sequences

Ex. Find the range of values of x for which the sequence $\{(x-1)^n\}$ converges. Then, state the limit values.

[Sol] Since the common ratio is $x-1$, $-1 < x-1 \leq 1$

$$\therefore 0 < x \leq 2$$

Also, the limit values are,

when $0 < x < 2$, $\lim_{n \rightarrow \infty} (x-1)^n = 0$ and

when $x=2$, $\lim_{n \rightarrow \infty} (x-1)^n = 1$

The sequence $\{r^n\}$ converges when $-1 < r \leq 1$.

When $|r| < 1$, $\lim_{n \rightarrow \infty} r^n = 0$

When $r=1$, $\lim_{n \rightarrow \infty} r^n = 1$

1. Find the range of values of x for which the sequence $\{(x+2)^n\}$ converges. Then, state the limit values.

[Sol] Since the common ratio is $x+2$, $-1 < x+2 \leq 1$

$$\therefore -3 < x \leq -1$$

Also, the limit values are,

when $-3 < x < -1$, $\lim_{n \rightarrow \infty} (x+2)^n = 0$ and

when $x=-1$, $\lim_{n \rightarrow \infty} (x+2)^n = 1$

2. Find the range of values of x for which the sequence $\left\{\left(\frac{x+1}{3}\right)^n\right\}$ converges. Then, state the limit values.

[Sol] Since the common ratio is $\frac{x+1}{3}$, $-1 < \frac{x+1}{3} \leq 1$

$$\therefore -4 < x \leq 2$$

Also, the limit values are,

when $-4 < x < 2$, $\lim_{n \rightarrow \infty} \left(\frac{x+1}{3}\right)^n = 0$ and

when $x=2$, $\lim_{n \rightarrow \infty} \left(\frac{x+1}{3}\right)^n = 1$

N76b

3. Find the range of values of x for which the sequence $\{(x^2-x-1)^n\}$ converges. Then, state the limit values.

[Sol] Since the common ratio is x^2-x-1 , $-1 < x^2-x-1 \leq 1$
 So, $-1 < x^2-x-1 \dots \textcircled{1}$ and also $x^2-x-1 \leq 1 \dots \textcircled{2}$

From $\textcircled{1}$, $x^2-x > \boxed{0}$; therefore, $x(x-1) > \boxed{0}$

$$\therefore x < \boxed{0}, \boxed{1} < x \dots \textcircled{3}$$

From $\textcircled{2}$, $x^2-x-2 \leq \boxed{0}$; therefore, $(x+1)(x-2) \leq \boxed{0}$

$$\therefore \boxed{-1} \leq x \leq \boxed{2} \dots \textcircled{4}$$

From $\textcircled{3}$ and $\textcircled{4}$, the range of values of x is

$$\boxed{-1} \leq x < \boxed{0}, \boxed{1} < x \leq \boxed{2}$$

Also, the limit values are,

when $-1 < x < 0$, $1 < x < 2$, $\lim_{n \rightarrow \infty} (x^2-x-1)^n = 0$ and

when $x = -1, 2$, $\lim_{n \rightarrow \infty} (x^2-x-1)^n = 1$

4. Find the range of values of x for which the sequence $\{(x^2-3x+1)^n\}$ converges. Then, state the limit values.

[Sol] Since the common ratio is x^2-3x+1 , $-1 < x^2-3x+1 \leq 1$

So, $-1 < x^2-3x+1 \dots \textcircled{1}$ and also $x^2-3x+1 \leq 1 \dots \textcircled{2}$

From $\textcircled{1}$, $x^2-3x+2 > 0$; therefore, $(x-1)(x-2) > 0$

$$\therefore x < 1, 2 < x \dots \textcircled{3}$$

From $\textcircled{2}$, $x^2-3x \leq 0$; therefore, $x(x-3) \leq 0$

$$\therefore 0 \leq x \leq 3 \dots \textcircled{4}$$

From $\textcircled{3}$ and $\textcircled{4}$, the range of values of x is

$$0 \leq x < 1, 2 < x \leq 3$$

Also, the limit values are,

when $0 < x < 1$, $2 < x < 3$, $\lim_{n \rightarrow \infty} (x^2-3x+1)^n = 0$ and

when $x = 0, 3$, $\lim_{n \rightarrow \infty} (x^2-3x+1)^n = 1$

1. Find the range of values of x for which the sequence $\left\{ \left(\frac{x^2+2x-5}{x^2-x+2} \right)^n \right\}$ converges. Then, state the limit values.

[Sol] Since the common ratio is $\frac{x^2+2x-5}{x^2-x+2}$, $-1 < \frac{x^2+2x-5}{x^2-x+2} \leq 1$

Since $x^2-x+2 > 0$, rearranging the inequality, $\leftarrow$

$$\begin{aligned} x^2-x+2 &= \left(x-\frac{1}{2}\right)^2 + \frac{7}{4} > 0 \end{aligned}$$

$$-(x^2-x+2) < x^2+2x-5 \leq x^2-x+2$$

So,

$$-(x^2-x+2) < x^2+2x-5 \quad \dots \textcircled{1} \quad \text{and also} \quad x^2+2x-5 \leq x^2-x+2 \quad \dots \textcircled{2}$$

From $\textcircled{1}$, $2x^2+x-3 > 0$; therefore, $(2x+3)(x-1) > 0$

$$\therefore x < -\frac{3}{2}, 1 < x \quad \dots \textcircled{3}$$

From $\textcircled{2}$, $3x \leq 7$

$$\therefore x \leq \frac{7}{3} \quad \dots \textcircled{4}$$

From $\textcircled{3}$ and $\textcircled{4}$, the range of values of x is

$$x < -\frac{3}{2}, 1 < x \leq \frac{7}{3}$$

Also, the limit values are,

$$\text{when } x < -\frac{3}{2}, 1 < x < \frac{7}{3}, \quad \lim_{n \rightarrow \infty} \left(\frac{x^2+2x-5}{x^2-x+2} \right)^n = 0 \text{ and}$$

$$\text{when } x = \frac{7}{3}, \quad \lim_{n \rightarrow \infty} \left(\frac{x^2+2x-5}{x^2-x+2} \right)^n = 1$$

N77b

2. Given $a_n = (4\sin^2\theta + 2\cos\theta - 3)^n$, find the range of values of θ for which the sequence $\{a_n\}$ converges. ($0 \leq \theta \leq \pi$)

[Sol] Since the common ratio is $4\sin^2\theta + 2\cos\theta - 3$, $-1 < 4\sin^2\theta + 2\cos\theta - 3 \leq 1$
 $4\sin^2\theta + 2\cos\theta - 3 = 4(1 - \cos^2\theta) + 2\cos\theta - 3 \leftarrow \sin^2\theta + \cos^2\theta = 1$
 $= -4\cos^2\theta + 2\cos\theta + 1$

Therefore, rearranging the inequality,

$$-1 < -4\cos^2\theta + 2\cos\theta + 1 \leq 1$$

So,

$$-1 < -4\cos^2\theta + 2\cos\theta + 1 \dots \textcircled{1} \text{ and also } -4\cos^2\theta + 2\cos\theta + 1 \leq 1 \dots \textcircled{2}$$

From $\textcircled{1}$, $2\cos^2\theta - \cos\theta - 1 < 0$; therefore, $(2\cos\theta + 1)(\cos\theta - 1) < 0$

$$\therefore -\frac{1}{2} < \cos\theta < 1 \dots \textcircled{3}$$

From $\textcircled{2}$, $2\cos^2\theta - \cos\theta \geq 0$; therefore, $\cos\theta(2\cos\theta - 1) \geq 0$

$$\therefore \cos\theta \leq 0, \frac{1}{2} \leq \cos\theta \dots \textcircled{4}$$

From $\textcircled{3}$ and $\textcircled{4}$, $-\frac{1}{2} < \cos\theta \leq 0, \frac{1}{2} \leq \cos\theta < 1$

Since $0 \leq \theta \leq \pi$,

$$\text{when } \cos\theta = -\frac{1}{2}, \theta = \frac{2}{3}\pi$$

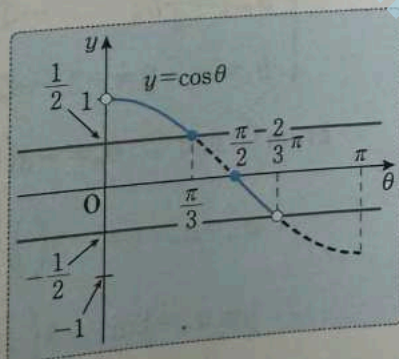
$$\text{when } \cos\theta = 0, \theta = \frac{\pi}{2}$$

$$\text{when } \cos\theta = \frac{1}{2}, \theta = \frac{\pi}{3}$$

$$\text{when } \cos\theta = 1, \theta = 0$$

Therefore, the range of values of θ is

$$0 < \theta \leq \frac{\pi}{3}, \frac{\pi}{2} \leq \theta < \frac{2}{3}\pi$$



Infinite Geometric Sequences

Find the limits of each given sequence $\{a_n\}$ defined by the following conditions.

Ex $a_1=1, a_{n+1}=\frac{2}{3}a_n-1 \quad (n=1, 2, 3, \dots)$

[Sol] Since $a_{n+1}=\frac{2}{3}a_n-1, a_{n+1}+3=\frac{2}{3}(a_n+3)$ $\leftarrow$

Let $b_n=a_n+3$.

$$\begin{cases} b_{n+1}=\frac{2}{3}b_n & \dots \textcircled{1} \\ b_1=a_1+3=4 & \dots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}, b_n=4\left(\frac{2}{3}\right)^{n-1}$

$$\therefore a_n=b_n-3=4\left(\frac{2}{3}\right)^{n-1}-3$$

$$\therefore \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left[4\left(\frac{2}{3}\right)^{n-1} - 3 \right] = -3$$

Replacing both a_{n+1} and a_n with $x, x=\frac{2}{3}x-1$
 $\therefore x=-3$ (N44)

$\{b_n\}$ is the geometric sequence with 1st term 4 and common ratio $\frac{2}{3}$.

(1) $a_1=3, a_{n+1}=\frac{1}{2}a_n+3 \quad (n=1, 2, 3, \dots)$

[Sol] Since $a_{n+1}=\frac{1}{2}a_n+3, a_{n+1}-6=\frac{1}{2}(a_n-6)$ $\leftarrow$

Let $b_n=a_n-6$.

$$\begin{cases} b_{n+1}=\frac{1}{2}b_n & \dots \textcircled{1} \\ b_1=a_1-6=-3 & \dots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}, b_n=-3\left(\frac{1}{2}\right)^{n-1}$

$$\therefore a_n=b_n+6=-3\left(\frac{1}{2}\right)^{n-1}+6$$

$$\therefore \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left[-3\left(\frac{1}{2}\right)^{n-1} + 6 \right] = 6$$

Replacing both a_{n+1} and a_n with $x, x=\frac{1}{2}x+3$
 $\therefore x=6$

N78b

(2) $a_1 = 2, a_{n+1} = \frac{a_n}{a_n + 3} \quad (n = 1, 2, 3, \dots)$

[Sol] Since $a_{n+1} = \frac{a_n}{a_n + 3}, \frac{1}{a_{n+1}} = \frac{a_n + 3}{a_n} = \frac{3}{a_n} + 1$

Let $b_n = \frac{1}{a_n}$.

Since $b_{n+1} = 3b_n + 1, b_{n+1} + \frac{1}{2} = 3\left(b_n + \frac{1}{2}\right)$

Replacing both b_{n+1} and b_n with $x, x = 3x + 1$
 $\therefore x = -\frac{1}{2}$

Let $c_n = b_n + \frac{1}{2}$.

$$\begin{cases} c_{n+1} = 3c_n & \dots \textcircled{1} \\ c_1 = b_1 + \frac{1}{2} = \frac{1}{a_1} + \frac{1}{2} = 1 & \dots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}, c_n = 3^{n-1}$

$\therefore b_n = c_n - \frac{1}{2} = 3^{n-1} - \frac{1}{2}$

$\therefore a_n = \frac{1}{b_n} = \frac{1}{3^{n-1} - \frac{1}{2}} = \frac{2}{2 \cdot 3^{n-1} - 1}$

$\therefore \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{2}{2 \cdot 3^{n-1} - 1} = 0$

Replacing both b_{n+1} and b_n with $x, x = 3x + 1$
 $\therefore x = -\frac{1}{2}$

Replacing both b_{n+1} and b_n with $x, x = 3x + 1$
 $\therefore x = -\frac{1}{2}$

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Infinite Geometric Sequences

1. Find the range of values of θ which satisfies $\lim_{n \rightarrow \infty} \frac{\tan^n \theta + 2}{2 \tan^n \theta + 2} = \frac{1}{2}$.

$$\left(0 \leq \theta < \frac{\pi}{2}\right)$$

[Sol] (i) When $0 \leq \tan \theta < 1$, i.e. $0 \leq \theta < \frac{\pi}{4}$,

$$\lim_{n \rightarrow \infty} \tan^n \theta = 0$$

$$\therefore \lim_{n \rightarrow \infty} \frac{\tan^n \theta + 2}{2 \tan^n \theta + 2} = \frac{0 + 2}{2 \cdot 0 + 2} = 1$$

(ii) When $\tan \theta = 1$, i.e. $\theta = \frac{\pi}{4}$,

$$\lim_{n \rightarrow \infty} \tan^n \theta = 1$$

$$\therefore \lim_{n \rightarrow \infty} \frac{\tan^n \theta + 2}{2 \tan^n \theta + 2} = \frac{1 + 2}{2 \cdot 1 + 2} = \frac{3}{4}$$

(iii) When $\tan \theta > 1$, i.e. $\frac{\pi}{4} < \theta < \frac{\pi}{2}$,

$$\lim_{n \rightarrow \infty} \frac{1}{\tan^n \theta} = 0$$

$$\therefore \lim_{n \rightarrow \infty} \frac{\tan^n \theta + 2}{2 \tan^n \theta + 2} = \lim_{n \rightarrow \infty} \frac{1 + \frac{2}{\tan^n \theta}}{2 + \frac{2}{\tan^n \theta}} = \frac{1 + 2 \cdot 0}{2 + 2 \cdot 0} = \frac{1}{2}$$

From (i) ~ (iii), the range of values of θ is

$$\frac{\pi}{4} < \theta < \frac{\pi}{2}$$

N79b

2. Given the sequence $\{a_n\}$ which is defined by $a_1=0$, $a_2=1$ and $a_{n+2}=\frac{1}{4}(a_{n+1}+3a_n)$ ($n=1, 2, 3, \dots$), solve the following questions.

(1) Let $b_n=a_{n+1}-a_n$ ($n=1, 2, 3, \dots$). Express b_n in terms of n .

[Sol] Since $a_{n+2}=\frac{1}{4}(a_{n+1}+3a_n)$,

$$a_{n+2}-\frac{1}{4}a_{n+1}-\frac{3}{4}a_n=0 \dots \textcircled{1}$$

Rearranging $\textcircled{1}$,

$$a_{n+2}-a_{n+1}=-\frac{3}{4}(a_{n+1}-a_n)$$

Let $b_n=a_{n+1}-a_n$.

$$\begin{cases} b_{n+1}=-\frac{3}{4}b_n \dots \textcircled{2} \end{cases}$$

$$\begin{cases} b_1=a_2-a_1=1 \dots \textcircled{3} \end{cases}$$

From $\textcircled{2}$ and $\textcircled{3}$, $b_n=\left(-\frac{3}{4}\right)^{n-1}$

Replacing a_{n+2} , a_{n+1} , a_n with x^2 , x , 1 respectively,

$$x^2-\frac{1}{4}x-\frac{3}{4}=0$$

$$4x^2-x-3=0$$

$$(4x+3)(x-1)=0$$

$$\therefore x=-\frac{3}{4}, 1$$

(N47)

$\{b_n\}$ is the geometric

sequence with 1st term

$a_2-a_1=1$ and common

ratio $-\frac{3}{4}$.

(2) Express a_n in terms of n .

[Sol] From (1), since the general term of the sequence of differences of $\{a_n\}$ is

$$\left(-\frac{3}{4}\right)^{n-1},$$

when $n \geq 2$,

$$a_n=a_1+\sum_{k=1}^{n-1}\left(-\frac{3}{4}\right)^{k-1}=0+\frac{1-\left(-\frac{3}{4}\right)^{n-1}}{1-\left(-\frac{3}{4}\right)} \\ =\frac{4}{7}\left[1-\left(-\frac{3}{4}\right)^{n-1}\right] \dots \textcircled{4}$$

When $n \geq 2$,

$$a_n=a_1+\sum_{k=1}^{n-1}b_k$$

(N32)

Since $a_1=0$, $\textcircled{4}$ is also true when $n=1$.

$$\therefore a_n=\frac{4}{7}\left[1-\left(-\frac{3}{4}\right)^{n-1}\right]$$

(3) Find the limit value $\lim_{n \rightarrow \infty} a_n$.

[Sol] From (2), $\lim_{n \rightarrow \infty} a_n=\lim_{n \rightarrow \infty} \frac{4}{7}\left[1-\left(-\frac{3}{4}\right)^{n-1}\right]=\frac{4}{7}$

Infinite Geometric Sequences

1. Find the following limits.

$$(1) \lim_{n \rightarrow \infty} \frac{3^n + 2^n}{2^{2n} - 3^n} = \lim_{n \rightarrow \infty} \frac{3^n + 2^n}{4^n - 3^n} = \lim_{n \rightarrow \infty} \frac{\left(\frac{3}{4}\right)^n + \left(\frac{1}{2}\right)^n}{1 - \left(\frac{3}{4}\right)^n} = 0 \quad \Rightarrow \text{N72}$$

$$(2) \lim_{n \rightarrow \infty} \frac{4^n + 2^n}{3^n} = \lim_{n \rightarrow \infty} \frac{4^n \left[1 + \left(\frac{1}{2}\right)^n\right]}{3^n} = \lim_{n \rightarrow \infty} \left(\frac{4}{3}\right)^n \left[1 + \left(\frac{1}{2}\right)^n\right] = \infty \quad \Rightarrow \text{N73}$$

$$\left[\begin{array}{l} \text{Alternative Solution} \\ \lim_{n \rightarrow \infty} \frac{4^n + 2^n}{3^n} = \lim_{n \rightarrow \infty} \left[\left(\frac{4}{3}\right)^n + \left(\frac{2}{3}\right)^n \right] = \lim_{n \rightarrow \infty} \left(\frac{4}{3}\right)^n \left[1 + \left(\frac{1}{2}\right)^n\right] = \infty \end{array} \right]$$

$$(3) \lim_{n \rightarrow \infty} \frac{r^{2n-1}}{r^{2n} + 1}$$

 $\Rightarrow \text{N74}$ [Sol] (i) When $|r| < 1$,

$$\lim_{n \rightarrow \infty} \frac{r^{2n-1}}{r^{2n} + 1} = \frac{0}{0 + 1} = 0$$

(ii) When $r = 1$,

$$\lim_{n \rightarrow \infty} \frac{r^{2n-1}}{r^{2n} + 1} = \frac{1}{1 + 1} = \frac{1}{2}$$

(iii) When $r = -1$,

$$\lim_{n \rightarrow \infty} \frac{r^{2n-1}}{r^{2n} + 1} = \frac{-1}{1 + 1} = -\frac{1}{2}$$

(iv) When $|r| > 1$,

$$\lim_{n \rightarrow \infty} \frac{r^{2n-1}}{r^{2n} + 1} = \lim_{n \rightarrow \infty} \frac{\frac{1}{r}}{1 + \left(\frac{1}{r}\right)^{2n}} = \frac{\frac{1}{r}}{1 + 0} = \frac{1}{r}$$

N80b

2. Find the range of values of x for which the sequence $\left\{\left(\frac{x+3}{4}\right)^n\right\}$ converges.
Then, state the limit values.

➡ N76

[Sol] Since the common ratio is $\frac{x+3}{4}$, $-1 < \frac{x+3}{4} \leq 1$

$$\therefore -7 < x \leq 1$$

Also, the limit values are,

when $-7 < x < 1$, $\lim_{n \rightarrow \infty} \left(\frac{x+3}{4}\right)^n = 0$ and

when $x = 1$, $\lim_{n \rightarrow \infty} \left(\frac{x+3}{4}\right)^n = 1$

3. Find the limit of the sequence $\{a_n\}$ defined by the following conditions.

$$a_1 = 1, a_{n+1} = 3 - \frac{1}{2}a_n \quad (n = 1, 2, 3, \dots)$$

➡ N78

[Sol] Since $a_{n+1} = 3 - \frac{1}{2}a_n$, $a_{n+1} - 2 = -\frac{1}{2}(a_n - 2)$

Let $b_n = a_n - 2$.

$$\begin{cases} b_{n+1} = -\frac{1}{2}b_n & \dots \textcircled{1} \end{cases}$$

$$\begin{cases} b_1 = a_1 - 2 = -1 & \dots \textcircled{2} \end{cases}$$

From ① and ②, $b_n = -\left(-\frac{1}{2}\right)^{n-1}$

$$\therefore a_n = b_n + 2 = -\left(-\frac{1}{2}\right)^{n-1} + 2$$

$$\therefore \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left[-\left(-\frac{1}{2}\right)^{n-1} + 2 \right] = 2$$

Infinite Geometric Series

Given an infinite sequence $\{a_n\}$, the expression $a_1 + a_2 + a_3 + \dots + a_n + \dots$ ① is called the **infinite series**, where a_1 and a_n are called the **1st term** and the **n^{th} term** respectively.

Also, given an infinite sequence $\{a_n\}$, let S_n be the sum of the first n terms. When the infinite sequence $\{S_n\}$ converges, the infinite series ① is said to converge. When the infinite sequence $\{S_n\}$ diverges, the infinite series ① is said to diverge. Likewise, $a + ar + ar^2 + \dots + ar^{n-1} + \dots$ which is the infinite series that is derived from the infinite geometric sequence with 1st term a and common ratio r is called the **infinite geometric series** with 1st term a and common ratio r .

Determine whether the following infinite geometric series converges or diverges.
 $a + ar + ar^2 + \dots + ar^{n-1} + \dots$

[Sol] When $a=0$, all terms become 0; therefore, the infinite geometric series converges to 0.

When $a \neq 0$, let S_n be the sum of the first n terms.

$$\text{When } r \neq 1, S_n = \frac{a(1-r^n)}{1-r} \dots \text{①}$$

$$\text{When } r=1, S_n = na$$

$$(i) \text{ When } |r| < 1, \text{ since } \lim_{n \rightarrow \infty} r^n = 0, \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \frac{a(1-r^n)}{1-r} = \boxed{\frac{a}{1-r}}$$

Therefore, the infinite geometric series converges and the sum

$$\text{is } \boxed{\frac{a}{1-r}}.$$

(ii) When $r=1$,

since $S_n = na$ and $a \neq 0$, the sequence $\{S_n\}$ diverges.

Therefore, the infinite geometric series diverges.

(iii) When $r \leq -1$ and $r > 1$,

since the sequence $\{r^n\}$ diverges,

from ①, the sequence $\{S_n\}$ also **diverges**.

Therefore, the infinite geometric series **diverges**.

When $r \leq -1$,
 $\{r^n\}$ oscillates
 (no limit).
 When $r > 1$,
 $\lim_{n \rightarrow \infty} r^n = \infty$.
 Therefore, in
 either case, it
 diverges. (N71)

Answers: in order $\frac{1-r}{a}$, $\frac{1-r}{a}$, diverges, diverges

Convergence and Divergence of an Infinite Geometric Series

Given an infinite geometric series $a + ar + ar^2 + \dots + ar^{n-1} + \dots$, the following is true.

When $a \neq 0$,

if $|r| < 1$, then the series converges and the sum is $\frac{a}{1-r}$;

if $|r| \geq 1$, then the series diverges.

When $a = 0$, the series converges and the sum is 0.

Determine whether each given infinite geometric series converges or diverges. If it converges, find the sum S .

Ex.

$$2 + 1 + \frac{1}{2} + \frac{1}{4} + \dots$$

[Sol] The 1st term is 2 and the common ratio is $\frac{1}{2}$. Since $\left|\frac{1}{2}\right| < 1$, the series converges.

$$\therefore S = \frac{2}{1 - \frac{1}{2}} = 4$$

$$(1) \quad 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \dots$$

[Sol] The 1st term is 1 and the common ratio is $\frac{1}{3}$. Since $\left|\frac{1}{3}\right| < 1$, the series **converges**.

$$\therefore S = \frac{1}{1 - \frac{1}{3}} = \frac{3}{2}$$

$$(2) \quad 3 - \frac{3}{10} + \frac{3}{100} - \frac{3}{1000} + \dots$$

[Sol] The 1st term is 3 and the common ratio is $-\frac{1}{10}$. Since $\left|-\frac{1}{10}\right| < 1$, the series **converges**.

$$\therefore S = \frac{3}{1 - \left(-\frac{1}{10}\right)} = \frac{30}{11}$$

$$(3) \quad -1 + 2 - 4 + 8 - \dots$$

[Sol] The 1st term is -1 and the common ratio is -2 . Since $|-2| > 1$, the series **diverges**.

Infinite Geometric Series

Determine whether each given infinite geometric series converges or diverges. If it converges, find the sum S .

$$(1) \quad 1 - \frac{\sqrt{2}}{2} + \frac{1}{2} - \frac{\sqrt{2}}{4} + \dots$$

[Sol] The 1st term is 1 and the common ratio is $-\frac{\sqrt{2}}{2}$. Since $\left| -\frac{\sqrt{2}}{2} \right| < 1$, the series **converges**.

$$\begin{aligned} \therefore S &= \frac{1}{1 - \left(-\frac{\sqrt{2}}{2} \right)} \\ &= \frac{2}{2 + \sqrt{2}} \\ &= \frac{2(2 - \sqrt{2})}{(2 + \sqrt{2})(2 - \sqrt{2})} \\ &= 2 - \sqrt{2} \end{aligned}$$

Multiplying the numerator and the denominator by $2 - \sqrt{2}$

$$(2) \quad 6 - 2\sqrt{3} + 2 - \frac{2\sqrt{3}}{3} + \dots$$

[Sol] The 1st term is 6 and the common ratio is $-\frac{2\sqrt{3}}{6} = -\frac{\sqrt{3}}{3}$. Since $\left| -\frac{\sqrt{3}}{3} \right| < 1$, the series **converges**.

$$\begin{aligned} \therefore S &= \frac{6}{1 - \left(-\frac{\sqrt{3}}{3} \right)} \\ &= \frac{18}{3 + \sqrt{3}} \\ &= \frac{18(3 - \sqrt{3})}{(3 + \sqrt{3})(3 - \sqrt{3})} \\ &= 3(3 - \sqrt{3}) \end{aligned}$$

N82b

(3) $\sqrt{2} + (2 - \sqrt{2}) + (3\sqrt{2} - 4) + (10 - 7\sqrt{2}) + \dots$

[Sol] The 1st term is $\sqrt{2}$ and the common ratio is $\frac{2 - \sqrt{2}}{\sqrt{2}} = \sqrt{2} - 1$.
 Since $|\sqrt{2} - 1| < 1$, the series converges.

$$\begin{aligned} \therefore S &= \frac{\sqrt{2}}{1 - (\sqrt{2} - 1)} \\ &= \frac{\sqrt{2}}{2 - \sqrt{2}} \\ &= \frac{\sqrt{2}(\sqrt{2} + 1)}{(2 - \sqrt{2})(2 + \sqrt{2})} \\ &= \sqrt{2} + 1 \end{aligned}$$

(4) $(\sqrt{3} - 1) - 2(2 - \sqrt{3}) + 2(3\sqrt{3} - 5) - 4(7 - 4\sqrt{3}) + \dots$

[Sol] The 1st term is $\sqrt{3} - 1$ and the common ratio is

$$\frac{-2(2 - \sqrt{3})}{\sqrt{3} - 1} = \frac{-2(2 - \sqrt{3})(\sqrt{3} + 1)}{(\sqrt{3} - 1)(\sqrt{3} + 1)} = -\sqrt{3} + 1.$$

Since $|\sqrt{3} - 1| < 1$, the series converges.

$$\begin{aligned} \therefore S &= \frac{\sqrt{3} - 1}{1 - (-\sqrt{3} + 1)} \\ &= \frac{\sqrt{3} - 1}{\sqrt{3}} \\ &= \frac{3 - \sqrt{3}}{3} \end{aligned}$$

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Find the range of values of a real number x for which each given infinite geometric series converges. Then, find the sum S .

Ex. $1 + 2x + 4x^2 + \dots$

[Sol] Since the common ratio is $2x$, $-1 < 2x < 1$

$$\therefore -\frac{1}{2} < x < \frac{1}{2}$$

Also, $S = \frac{1}{1-2x}$

$S = \frac{a}{1-r}$

Let a be the 1st term and r be the common ratio. If $|r| < 1$ when $a \neq 0$, then the infinite geometric series converges.

(1) $3 + x + \frac{x^2}{3} + \dots$

[Sol] Since the common ratio is $\frac{x}{3}$, $-1 < \frac{x}{3} < 1$

$$\therefore -3 < x < 3$$

Also, $S = \frac{3}{1-\frac{x}{3}} = \frac{9}{3-x}$

(2) $1 + (2-x^2) + (2-x^2)^2 + \dots$

[Sol] Since the common ratio is $2-x^2$, $-1 < 2-x^2 < 1$

So, $-1 < 2-x^2 \dots \textcircled{1}$ and also $2-x^2 < 1 \dots \textcircled{2}$

From $\textcircled{1}$, $x^2 < 3$; therefore, $-\sqrt{3} < x < \sqrt{3} \dots \textcircled{3}$

From $\textcircled{2}$, $x^2 > 1$; therefore, $x < -1$, $1 < x \dots \textcircled{4}$

From $\textcircled{3}$ and $\textcircled{4}$, the range of values of x is

$$-\sqrt{3} < x < -1, \quad 1 < x < \sqrt{3}$$

Also, $S = \frac{1}{1-(2-x^2)} = \frac{1}{x^2-1}$

N83b

(3) $x + x(1-x^2) + x(1-x^2)^2 + \dots$

[Sol] (i) When $x \neq 0$, since the common ratio is $1-x^2$, $-1 < 1-x^2 < 1$
 So, $-1 < 1-x^2 \dots \textcircled{1}$ and also $1-x^2 < 1 \dots \textcircled{2}$

From $\textcircled{1}$, $x^2 < \boxed{2}$; therefore, $\boxed{-\sqrt{2}} < x < \boxed{\sqrt{2}} \dots \textcircled{3}$

From $\textcircled{2}$, $x^2 > \boxed{0}$; therefore, $x < \boxed{0}$, $\boxed{0} < x \dots \textcircled{4}$

From $\textcircled{3}$ and $\textcircled{4}$, $\boxed{-\sqrt{2}} < x < \boxed{0}$, $\boxed{0} < x < \boxed{\sqrt{2}}$

Also, $S = \frac{\boxed{x}}{1 - (1-x^2)} = \frac{\boxed{1}}{\boxed{x}}$

(ii) When $x=0$, all terms become 0; therefore, the series converges.

$\therefore S = \boxed{0}$

From (i) and (ii), the range of values of x is $\boxed{-\sqrt{2}} < x < \boxed{\sqrt{2}}$.

Also, when $\boxed{-\sqrt{2}} < x < \boxed{0}$, $\boxed{0} < x < \boxed{\sqrt{2}}$, $S = \frac{\boxed{1}}{\boxed{x}}$ and

when $x = \boxed{0}$, $S = \boxed{0}$

(4) $x + x(1-x)^2 + x(1-x)^4 + \dots$

[Sol] (i) When $x \neq 0$, since the common ratio is $(1-x)^2$, $-1 < (1-x)^2 < 1$

So, $-1 < (1-x)^2 \dots \textcircled{1}$ and also $(1-x)^2 < 1 \dots \textcircled{2}$

From $\textcircled{1}$, $(1-x)^2 + 1 > 0$; therefore, it is true for all real numbers x .

From $\textcircled{2}$, $x(x-2) < 0$; therefore, $0 < x < 2 \dots \textcircled{3}$

From $\textcircled{1}$ and $\textcircled{3}$, $0 < x < 2$

Also, $S = \frac{x}{1 - (1-x)^2} = -\frac{1}{x-2}$

(ii) When $x=0$, all terms become 0; therefore, the series converges.

$\therefore S = 0$

From (i) and (ii), the range of values of x is $0 \leq x < 2$.

Also, when $0 < x < 2$, $S = -\frac{1}{x-2}$ and

when $x=0$, $S=0$

Ex. Given the infinite geometric series whose sum is 9 and 2nd term is -4, find the 1st term a and the common ratio r .

[Sol] Since the sum is 9, $a \neq 0$, $-1 < r < 1$ ←

Since the sum is 9, the infinite geometric series converges.

$$\begin{cases} \frac{a}{1-r} = 9 \quad \dots \textcircled{1} \\ ar = -4 \quad \dots \textcircled{2} \end{cases}$$

From ① and ②,

$$9(1-r) \cdot r = -4$$

$$9r^2 - 9r - 4 = 0$$

$$(3r+1)(3r-4) = 0$$

$$r = -\frac{1}{3}, \frac{4}{3}$$

Since $-1 < r < 1$, $r = -\frac{1}{3}$

From ②, $a = 12$

From ①, $a = 9(1-r)$
Substituting this into ②

1. Given the infinite geometric series whose sum is $\frac{8}{3}$ and 2nd term is -2, find the 1st term a and the common ratio r .

[Sol] Since the sum is $\frac{8}{3}$, $a \neq 0$, $-1 < r < 1$

$$\begin{cases} \frac{a}{1-r} = \frac{8}{3} \quad \dots \textcircled{1} \\ ar = -2 \quad \dots \textcircled{2} \end{cases}$$

From ① and ②,

$$\frac{8}{3}(1-r) \cdot r = -2$$

$$4r^2 - 4r - 3 = 0$$

$$(2r+1)(2r-3) = 0$$

$$r = -\frac{1}{2}, \frac{3}{2}$$

Since $-1 < r < 1$, $r = -\frac{1}{2}$

From ②, $a = 4$

N84b

2. Given the infinite geometric series whose sum is 3, squaring each term of the series gives another infinite geometric series whose sum is 6. Find the 1st term a and the common ratio r of the original infinite geometric series.

[Sol] Since the sum is 3, $a \neq 0$, $-1 < r < 1$

$$\begin{cases} \frac{a}{1-r} = 3 & \dots \textcircled{1} \\ \frac{a^2}{1-r^2} = 6 & \dots \textcircled{2} \end{cases}$$

1st term: a^2 , common ratio: r^2 , sum: 6

From ① and ②,

$$\frac{[3(1-r)]^2}{1-r^2} = 6$$

$$\frac{9(1-r)^2}{(1+r)(1-r)} = 6$$

$$3(1-r) = 2(1+r)$$

$$\therefore r = \frac{1}{5}$$

$$\text{From ①, } a = \frac{12}{5}$$

From ①, $a = 3(1-r)$
Substituting this into ②

Alternative Solution

Since the sum is 3,
 $a \neq 0$, $-1 < r < 1$

$$\begin{cases} \frac{a}{1-r} = 3 & \dots \textcircled{1} \\ \frac{a^2}{1-r^2} = 6 & \dots \textcircled{2} \end{cases}$$

$$\text{From ①, } a = 3(1-r) \dots \textcircled{3}$$

$$\text{From ②, } a^2 = 6(1-r^2) \dots \textcircled{4}$$

From ③ and ④,

$$[3(1-r)]^2 = 6(1-r^2)$$

$$5r^2 - 6r + 1 = 0$$

$$(5r-1)(r-1) = 0$$

$$r = \frac{1}{5}, 1$$

$$\text{Since } -1 < r < 1, r = \frac{1}{5}$$

$$\text{From ③, } a = \frac{12}{5}$$

3. Given the infinite geometric series whose sum is 2, cubing each term of the series gives another infinite geometric series whose sum is $\frac{32}{13}$. Find the 1st term a and the common ratio r of the original infinite geometric series.

[Sol] Since the sum is 2, $a \neq 0$, $-1 < r < 1$

$$\begin{cases} \frac{a}{1-r} = 2 & \dots \textcircled{1} \\ \frac{a^3}{1-r^3} = \frac{32}{13} & \dots \textcircled{2} \end{cases}$$

From ① and ②,

$$\frac{[2(1-r)]^3}{1-r^3} = \frac{32}{13}$$

$$\frac{8(1-r)^3}{(1-r)(1+r+r^2)} = \frac{32}{13}$$

$$13(1-r)^2 = 4(1+r+r^2)$$

$$3r^2 - 10r + 3 = 0$$

$$(3r-1)(r-3) = 0$$

$$r = \frac{1}{3}, 3$$

$$\text{Since } -1 < r < 1, r = \frac{1}{3}$$

$$\text{From ①, } a = \frac{4}{3}$$

$$\frac{a^3 - b^3}{1 - r^3} = \frac{(a-b)(a^2 + ab + b^2)}{1 - r^3}$$

Alternative Solution

Since the sum is 2,
 $a \neq 0$, $-1 < r < 1$

$$\begin{cases} \frac{a}{1-r} = 2 & \dots \textcircled{1} \\ \frac{a^3}{1-r^3} = \frac{32}{13} & \dots \textcircled{2} \end{cases}$$

$$\text{From ①, } a = 2(1-r) \dots \textcircled{3}$$

$$\text{From ②, } a^3 = \frac{32}{13}(1-r^3) \dots \textcircled{4}$$

From ③ and ④,

$$[2(1-r)]^3 = \frac{32}{13}(1-r^3)$$

$$3r^3 - 13r^2 + 13r - 3 = 0$$

$$(r-1)(3r^2 - 10r + 3) = 0$$

$$(r-1)(3r-1)(r-3) = 0$$

$$r = \frac{1}{3}, 1, 3$$

$$\text{Since } -1 < r < 1, r = \frac{1}{3}$$

$$\text{From ③, } a = \frac{4}{3}$$

Infinite Geometric Series

A decimal which contains a digit or block of digits that repeats without end in its decimal part is called a **repeating decimal**. The repeating decimal is expressed by placing "." over the first and the last repeating digits as shown below:

$$0.33333... = 0.\dot{3}, \quad 0.454545... = 0.\dot{4}\dot{5}, \quad 0.123123123... = 0.\dot{1}\dot{2}\dot{3}$$

Also, the repeating decimal can be expressed by a fraction using the sum of an infinite geometric series.

Convert the following repeating decimals into fractions.

Ex. $0.\dot{6}$

[Sol] $0.\dot{6} = 0.6 + 0.06 + 0.006 + \dots$

$0.\dot{6}$ is the infinite geometric series with 1st term 0.6 and common ratio 0.1. Since $|0.1| < 1$, it converges.

$$\therefore 0.\dot{6} = \frac{0.6}{1-0.1} = \frac{2}{3}$$

(1) $0.\dot{1}\dot{2}$

[Sol] $0.\dot{1}\dot{2} = 0.12 + 0.0012 + 0.000012 + \dots$

$0.\dot{1}\dot{2}$ is the infinite geometric series with 1st term 0.12 and common ratio 0.01. Since $|0.01| < 1$, it converges.

$$\therefore 0.\dot{1}\dot{2} = \frac{0.12}{1-0.01} = \frac{4}{33}$$

(2) $0.\dot{3}0\dot{6}$

[Sol] $0.\dot{3}0\dot{6} = 0.306 + 0.000306 + 0.000000306 + \dots$

$0.\dot{3}0\dot{6}$ is the infinite geometric series with 1st term 0.306 and common ratio 0.001. Since $|0.001| < 1$, it converges.

$$\therefore 0.\dot{3}0\dot{6} = \frac{0.306}{1-0.001} = \frac{34}{111}$$

N85b

(3) $0.3\dot{2}\dot{4}$

[Sol] $0.3\dot{2}\dot{4} = 0.3 + 0.024 + 0.00024 + 0.0000024 + \dots$

The 2nd and subsequent terms of the RHS is the infinite geometric series with 1st term 0.024 and common ratio 0.01.

Since $|0.01| < 1$, it converges.

$$\therefore 0.3\dot{2}\dot{4} = \boxed{0.3} + \frac{\boxed{0.024}}{1 - \boxed{0.01}} = \frac{3}{10} + \frac{4}{165} = \frac{107}{330}$$

(4) $3.5\dot{2}$

[Sol] $3.5\dot{2} = 3.5 + 0.02 + 0.002 + 0.0002 + \dots$

The 2nd and subsequent terms of the RHS is the infinite geometric series with 1st term 0.02 and common ratio 0.1.

Since $|0.1| < 1$, it converges.

$$\therefore 3.5\dot{2} = 3.5 + \frac{0.02}{1 - 0.1} = \frac{7}{2} + \frac{1}{45} = \frac{317}{90}$$

(5) $5.4\dot{7}\dot{7}$

[Sol] $5.4\dot{7}\dot{7} = 5 + 0.477 + 0.000477 + 0.000000477 + \dots$

The 2nd and subsequent terms of the RHS is the infinite geometric series with 1st term 0.477 and common ratio 0.001.

Since $|0.001| < 1$, it converges.

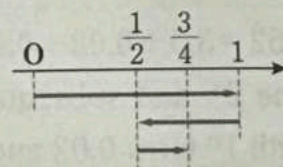
$$\therefore 5.4\dot{7}\dot{7} = 5 + \frac{0.477}{1 - 0.001} = 5 + \frac{53}{111} = \frac{608}{111}$$

Infinite Geometric Series

Ex. On a number line, starting from origin O, point P moves 1 unit in the positive direction and then moves $\frac{1}{2}$ units in the negative direction. Then, it moves $\frac{1}{2^2}$ units in the positive direction and then moves $\frac{1}{2^3}$ units in the negative direction. When this motion continues infinitely, find the coordinate of the point which point P approaches.

[Sol] The coordinates of point P are as follows.

$$\begin{aligned}
 &1 \\
 &1 - \frac{1}{2} \\
 &1 - \frac{1}{2} + \frac{1}{2^2} \\
 &1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} \\
 &\vdots
 \end{aligned}$$



Let x be the coordinate of the point which point P approaches.

$$x = 1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \dots$$

x is the infinite geometric series with 1st term 1 and common ratio

$-\frac{1}{2}$. Since $\left| -\frac{1}{2} \right| < 1$, it converges.

$$\therefore x = \frac{1}{1 - \left(-\frac{1}{2} \right)} = \frac{2}{3}$$

Thus, the coordinate of the point which point P approaches

is $\frac{2}{3}$.

Answers: in order $-\frac{1}{2}, \frac{2}{3}, -\frac{1}{2}, \frac{2}{3}, -\frac{1}{2}, \frac{2}{3}, \dots$

N86b

1. On a number line, starting from origin O, point P moves 1 unit in the positive direction and then moves $\frac{1}{3}$ units in the negative direction. Then, it moves $\frac{1}{3^2}$ units in the positive direction and then moves $\frac{1}{3^3}$ units in the negative direction. When this motion continues infinitely, find the coordinate of the point which point P approaches.

[Sol] The coordinates of point P are as follows.

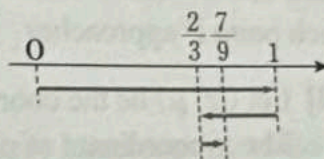
1

$$1 - \frac{1}{3}$$

$$1 - \frac{1}{3} + \frac{1}{3^2}$$

$$1 - \frac{1}{3} + \frac{1}{3^2} - \frac{1}{3^3}$$

$\vdots$



Let x be the coordinate of the point which point P approaches.

$$x = 1 - \frac{1}{3} + \frac{1}{3^2} - \frac{1}{3^3} + \dots$$

x is the infinite geometric series with 1st term 1 and common ratio $-\frac{1}{3}$.

Since $\left| -\frac{1}{3} \right| < 1$, it converges.

$$\therefore x = \frac{1}{1 - \left(-\frac{1}{3} \right)} = \frac{3}{4}$$

Thus, the coordinate of the point which point P approaches is $\frac{3}{4}$.

[Sol] Let (x, y) be the coordinates of the point which point P approaches.
The x -coordinate of point P is

$$x = 1 - \frac{1}{2^2} + \frac{1}{2^4} - \frac{1}{2^6} + \dots$$

$-\frac{1}{4}$. Since $\left|-\frac{1}{4}\right| < 1$, it converges.

$$\therefore x = \frac{1}{1 - \left(-\frac{1}{4}\right)} = \frac{4}{5}$$

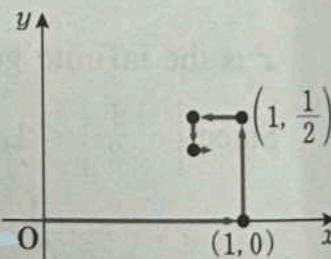
$$y = \frac{1}{2} - \frac{1}{2^3} + \frac{1}{2^5} - \frac{1}{2^7} + \dots$$

$-\frac{1}{4}$. Since $\left| -\frac{1}{4} \right| < 1$, it converges.

$$\therefore y = \frac{\frac{1}{2}}{1 - \left(-\frac{1}{4}\right)} = \frac{2}{5}$$

Thus, the coordinates of the point which point P approaches
 (1 , 2)

are $\left(\frac{4}{5}, \frac{2}{5} \right)$.



N87b

1. On a coordinate plane, starting from origin O , point P moves 1 unit in the positive direction to the x -axis and then moves $\frac{2}{3}$ units in the positive direction to the y -axis. Then, it moves $\left(\frac{2}{3}\right)^2$ units in the positive direction to the x -axis and then moves $\left(\frac{2}{3}\right)^3$ units in the positive direction to the y -axis. When this motion continues infinitely, find the coordinates of the point which point P approaches.

[Sol] Let (x, y) be the coordinates of the point which point P approaches.

The x -coordinate of point P is

$$x = 1 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^4 + \left(\frac{2}{3}\right)^6 + \dots$$

x is the infinite geometric series with 1st term 1 and common ratio $\frac{4}{9}$.

Since $\left|\frac{4}{9}\right| < 1$, it converges.

$$\therefore x = \frac{1}{1 - \frac{4}{9}} = \frac{9}{5}$$

The y -coordinate of point P is

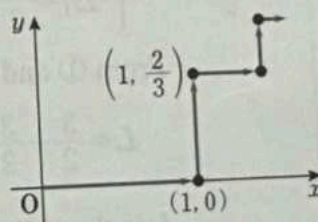
$$y = \frac{2}{3} + \left(\frac{2}{3}\right)^3 + \left(\frac{2}{3}\right)^5 + \left(\frac{2}{3}\right)^7 + \dots$$

y is the infinite geometric series with 1st term $\frac{2}{3}$ and common ratio $\frac{4}{9}$.

Since $\left|\frac{4}{9}\right| < 1$, it converges.

$$\therefore y = \frac{\frac{2}{3}}{1 - \frac{4}{9}} = \frac{6}{5}$$

Thus, the coordinates of the point which point P approaches are $\left(\frac{9}{5}, \frac{6}{5}\right)$.



$$\begin{cases} L_{n+1} = \frac{1}{2}L_n & \dots \textcircled{1} \\ L_1 = \frac{1}{2} \cdot 3 = \frac{3}{2} & \dots \textcircled{2} \end{cases}$$
$$A_{n+1}B_{n+1} + B_{n+1}C_{n+1} + C_{n+1}A_{n+1} = \frac{1}{2}(A_nB_n + B_nC_n + C_nA_n)$$
$$L = \frac{3}{2} + \frac{3}{2} \cdot \frac{1}{2} + \frac{3}{2} \left(\frac{1}{2} \right)^2 + \frac{3}{2} \left(\frac{1}{2} \right)^3 + \dots$$

$\frac{1}{2}$. Since $\left| \frac{1}{2} \right| < 1$, it converges.

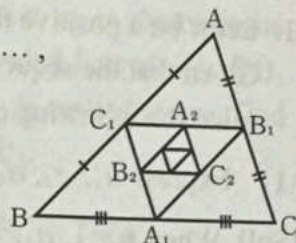
$$\therefore L = \frac{\frac{3}{2}}{1 - \frac{1}{2}} = 3$$

Answers: in order $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$

MN and BC are parallel, $MN = \frac{1}{2}BC$

N88b

1. Given that $\triangle ABC$ has a perimeter of 10, form $\triangle A_1B_1C_1$ whose vertices are the midpoints of the sides of $\triangle ABC$ and then form $\triangle A_2B_2C_2$ whose vertices are the midpoints of the sides of $\triangle A_1B_1C_1$. When triangles are formed infinitely as $\triangle A_1B_1C_1, \triangle A_2B_2C_2, \triangle A_3B_3C_3, \dots, \triangle A_nB_nC_n, \dots$, find the sum L of the perimeters.



[Sol] Let L_n be the perimeter of $\triangle A_nB_nC_n$.

$$\begin{cases} L_{n+1} = \frac{1}{2}L_n & \dots \textcircled{1} \\ L_1 = \frac{1}{2} \cdot 10 = 5 & \dots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}$,

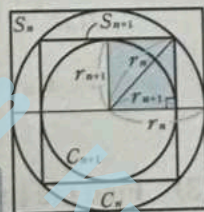
$$L = 5 + 5 \cdot \frac{1}{2} + 5 \left(\frac{1}{2}\right)^2 + 5 \left(\frac{1}{2}\right)^3 + \dots$$

L is the infinite geometric series with 1st term 5 and common ratio $\frac{1}{2}$.

Since $\left|\frac{1}{2}\right| < 1$, it converges.

$$\therefore L = \frac{5}{1 - \frac{1}{2}} = 10$$

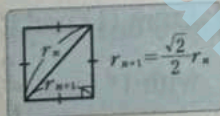
2. Given square S_n and circle C_n ($n=1, 2, \dots$), C_n is inscribed in S_n and S_{n+1} is inscribed in C_n . Let r_n be the radius of C_n and a be the length of each side of S_1 . Find the sum L of the circumferences.



Circumference $= 2 \times \text{radius} \times \pi$;
therefore, $L_n = 2\pi r_n$

[Sol] Let L_n be the circumference of C_n .

$$\begin{cases} L_{n+1} = 2\pi r_{n+1} = 2\pi \cdot \frac{\sqrt{2}}{2} r_n = \frac{\sqrt{2}}{2} L_n & \dots \textcircled{1} \\ L_1 = 2\pi r_1 = \pi a & \dots \textcircled{2} \end{cases}$$



The diameter of C_1 = the length of each side of S_1 ;
therefore, $2r_1 = a$

From $\textcircled{1}$ and $\textcircled{2}$,

$$L = \pi a + \pi a \cdot \frac{\sqrt{2}}{2} + \pi a \left(\frac{\sqrt{2}}{2}\right)^2 + \dots$$

L is the infinite geometric series with 1st term πa and common ratio $\frac{\sqrt{2}}{2}$.

Since $\left|\frac{\sqrt{2}}{2}\right| < 1$, it converges.

$$\therefore L = \frac{\pi a}{1 - \frac{\sqrt{2}}{2}} = \frac{2\pi a}{2 - \sqrt{2}} = \frac{2\pi a(2 + \sqrt{2})}{(2 - \sqrt{2})(2 + \sqrt{2})} = (2 + \sqrt{2})\pi a$$

Infinite Geometric Series

1. Let n be a positive integer and r be a real number that satisfies $r > 1$.
Given that the sequence $\{a_n\}$ satisfies $a_1r + a_2r^2 + \dots + a_nr^n = (-1)^n$,
solve the following questions.

- (1) Express a_1, a_2, a_3 in terms of r .

[Sol] When $n=1$, $a_1r = -1$; therefore, $a_1 = -\frac{1}{r}$

When $n=2$, $a_1r + a_2r^2 = 1$; therefore, $a_2 = \frac{2}{r^2}$ $-1 + a_2r^2 = 1$

When $n=3$, $a_1r + a_2r^2 + a_3r^3 = -1$; therefore, $a_3 = -\frac{2}{r^3}$ $1 + a_3r^3 = -1$

- (2) When $n \geq 2$, express a_n in terms of r .

[Sol] When $n \geq 2$,

$$\begin{cases} a_1r + a_2r^2 + \dots + a_{n-1}r^{n-1} = (-1)^{n-1} & \dots \textcircled{1} \\ a_1r + a_2r^2 + \dots + a_{n-1}r^{n-1} + a_nr^n = (-1)^n & \dots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}$,

$$a_nr^n = (-1)^n - (-1)^{n-1}$$

$$\therefore a_n = \frac{(-1)^n - (-1)^{n-1}}{r^n} = 2\left(-\frac{1}{r}\right)^n$$

$$\begin{aligned} & \frac{(-1)^n - (-1)^{n-1}}{r^n} \\ &= \frac{(-1)^n + (-1)^n}{r^n} \\ &= \frac{2(-1)^n}{r^n} \end{aligned}$$

- (3) Find the sum of the infinite series $a_1 + a_2 + \dots + a_n + \dots$.

[Sol] From (1) and (2), $a_2 + a_3 + \dots + a_n + \dots$ is the infinite geometric series
with 1st term $\frac{2}{r^2}$ and common ratio $-\frac{1}{r}$.

Since $\left| -\frac{1}{r} \right| < 1$, it converges.

$$\begin{aligned} \therefore a_1 + a_2 + \dots + a_n + \dots \\ = a_1 + (a_2 + a_3 + \dots + a_n + \dots) \end{aligned}$$

$$= -\frac{1}{r} + \frac{\frac{2}{r^2}}{1 - \left(-\frac{1}{r}\right)}$$

$$= -\frac{r-1}{r(r+1)}$$

From (1),
 $a_1 = -\frac{1}{r}$

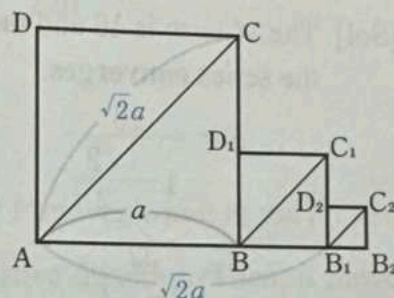
N89b

2. Given square ABCD with side length a , extend the line of side AB through point B in the opposite direction to point A. Then, place point B_1 on this extended line of side AB such that $AB_1 = AC$. Make a square $BB_1C_1D_1$ using BB_1 as a side and extend the line of side BB_1 from point B_1 in the opposite direction to point B. Then, place point B_2 on this extended line of side BB_1 such that $BB_2 = BC_1$. Given points $B_1, B_2, B_3, \dots$ as described above, find the sum of the lengths of line segments AB, $BB_1, B_1B_2, \dots$

[Sol] Since $AB = a$,

$$\begin{aligned} BB_1 &= AB_1 - AB \\ &= \sqrt{2}AB - AB \\ &= \sqrt{2}a - a \\ &= (\sqrt{2} - 1)a \end{aligned}$$

$$\begin{aligned} AB_1 &= AC \\ &= \sqrt{2}AB \end{aligned}$$



$$\begin{aligned} B_1B_2 &= BB_2 - BB_1 \\ &= \sqrt{2}BB_1 - BB_1 \\ &= (\sqrt{2} - 1) \cdot (\sqrt{2} - 1)a \\ &= (\sqrt{2} - 1)^2 a \end{aligned}$$

$$\begin{aligned} B_2B_3 &= B_1B_3 - B_1B_2 \\ &= \sqrt{2}B_1B_2 - B_1B_2 \\ &= (\sqrt{2} - 1) \cdot (\sqrt{2} - 1)^2 a \\ &= (\sqrt{2} - 1)^3 a \end{aligned}$$

$\vdots$

Let L be the sum of the lengths of line segments AB, $BB_1, B_1B_2, \dots$

$$L = a + (\sqrt{2} - 1)a + (\sqrt{2} - 1)^2 a + (\sqrt{2} - 1)^3 a + \dots$$

Therefore, L is the infinite geometric series with 1st term a and common ratio $\sqrt{2} - 1$. Since $|\sqrt{2} - 1| < 1$, it converges.

$$\therefore L = \frac{a}{1 - (\sqrt{2} - 1)} = \frac{a}{2 - \sqrt{2}} = \frac{a(2 + \sqrt{2})}{(2 - \sqrt{2})(2 + \sqrt{2})} = \frac{2 + \sqrt{2}}{2} a$$

Thus, the sum of the lengths of line segments AB, $BB_1, B_1B_2, \dots$

$$\text{is } \frac{2 + \sqrt{2}}{2} a.$$

Infinite Geometric Series

1. Determine whether the following infinite geometric series converges or diverges. If it converges, find the sum S . ➡ N81

$$16 + 8\sqrt{2} + 8 + 4\sqrt{2} + \dots$$

[Sol] The 1st term is 16 and the common ratio is $\frac{8\sqrt{2}}{16} = \frac{\sqrt{2}}{2}$. Since $\left| \frac{\sqrt{2}}{2} \right| < 1$, the series **converges**.

$$\begin{aligned} \therefore S &= \frac{16}{1 - \frac{\sqrt{2}}{2}} \\ &= \frac{32}{2 - \sqrt{2}} \\ &= \frac{32(2 + \sqrt{2})}{(2 - \sqrt{2})(2 + \sqrt{2})} \\ &= 16(2 + \sqrt{2}) \end{aligned}$$

2. Find the range of values of a real number x for which the following infinite geometric series converges. Then, find the sum S . ➡ N83

$$2 + 2(x^2 - 3) + 2(x^2 - 3)^2 + \dots$$

[Sol] Since the common ratio is $x^2 - 3$, $-1 < x^2 - 3 < 1$

So, $-1 < x^2 - 3 \dots \textcircled{1}$ and also $x^2 - 3 < 1 \dots \textcircled{2}$

From $\textcircled{1}$, $x^2 > 2$; therefore, $x < -\sqrt{2}$, $\sqrt{2} < x \dots \textcircled{3}$

From $\textcircled{2}$, $x^2 < 4$; therefore, $-2 < x < 2 \dots \textcircled{4}$

From $\textcircled{3}$ and $\textcircled{4}$, the range of values of x is

$$-2 < x < -\sqrt{2}, \quad \sqrt{2} < x < 2$$

$$\text{Also, } S = \frac{2}{1 - (x^2 - 3)} = \frac{2}{4 - x^2}$$

N90b

3. Convert the following repeating decimal into fraction.
0.0318

➡ N85

[Sol] $0.0318 = 0.03 + 0.0018 + 0.000018 + 0.00000018 + \dots$

The 2nd and subsequent terms of the RHS is the infinite geometric series with 1st term 0.0018 and common ratio 0.01.

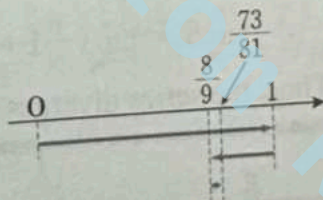
Since $|0.01| < 1$, it converges.

$$\therefore 0.0318 = 0.03 + \frac{0.0018}{1 - 0.01} = \frac{3}{100} + \frac{1}{550} = \frac{7}{220}$$

4. On a number line, starting from origin O, point P moves 1 unit in the positive direction and then moves $\frac{1}{3^2}$ units in the negative direction. Then, it moves $\frac{1}{3^4}$ units in the positive direction and then moves $\frac{1}{3^6}$ units in the negative direction. When this motion continues infinitely, find the coordinate of the point which point P approaches. ➡ N86

[Sol] The coordinates of point P are as follows.

$$\begin{aligned} &1 \\ &1 - \frac{1}{3^2} \\ &1 - \frac{1}{3^2} + \frac{1}{3^4} \\ &1 - \frac{1}{3^2} + \frac{1}{3^4} - \frac{1}{3^6} \\ &\vdots \end{aligned}$$



Let x be the coordinate of the point which point P approaches.

$$x = 1 - \frac{1}{3^2} + \frac{1}{3^4} - \frac{1}{3^6} + \dots$$

x is the infinite geometric series with 1st term 1 and common ratio $-\frac{1}{9}$.

Since $\left| -\frac{1}{9} \right| < 1$, it converges.

$$\therefore x = \frac{1}{1 - \left(-\frac{1}{9} \right)} = \frac{9}{10}$$

Thus, the coordinate of the point which point P approaches is $\frac{9}{10}$.

Infinite Series

Given the infinite series $a_1 + a_2 + a_3 + \cdots + a_n + \cdots$ ①, the sum of the first n terms

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n$$

is called the **partial sum** of the first n terms of the infinite series.

The infinite series ① can be written as $\sum_{n=1}^{\infty} a_n$.

Determine whether each given infinite series converges or diverges.

If it converges, find the sum.

Ex.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

$$\begin{aligned} \text{[Sol]} \quad \frac{1}{\sqrt{n+1} + \sqrt{n}} &= \frac{\sqrt{n+1} - \sqrt{n}}{(\sqrt{n+1} + \sqrt{n})(\sqrt{n+1} - \sqrt{n})} \\ &= \sqrt{n+1} - \sqrt{n} \end{aligned}$$

Therefore, let S_n be the partial sum of the first n terms.

$$\begin{aligned} S_n &= (\sqrt{2} - \sqrt{1}) + (\sqrt{3} - \sqrt{2}) + (\sqrt{4} - \sqrt{3}) + \cdots + (\sqrt{n+1} - \sqrt{n}) \\ &= -1 + \sqrt{n+1} \end{aligned}$$

$$\therefore \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} (-1 + \sqrt{n+1}) = \infty$$

Thus, the series diverges.

$$(1) \quad \sum_{n=1}^{\infty} \frac{3}{\sqrt{3n+2} + \sqrt{3n-1}}$$

$$\begin{aligned} \text{[Sol]} \quad \frac{3}{\sqrt{3n+2} + \sqrt{3n-1}} &= \frac{3(\sqrt{3n+2} - \sqrt{3n-1})}{(\sqrt{3n+2} + \sqrt{3n-1})(\sqrt{3n+2} - \sqrt{3n-1})} \\ &= \sqrt{3n+2} - \sqrt{3n-1} \end{aligned}$$

Therefore, let S_n be the partial sum of the first n terms.

$$\begin{aligned} S_n &= (\sqrt{5} - \sqrt{2}) + (\sqrt{8} - \sqrt{5}) + (\sqrt{11} - \sqrt{8}) + \cdots + (\sqrt{3n+2} - \sqrt{3n-1}) \\ &= -\sqrt{2} + \sqrt{3n+2} \end{aligned}$$

$$\therefore \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} (-\sqrt{2} + \sqrt{3n+2}) = \infty$$

Thus, the series **diverges**.

$$(2) \sum_{n=1}^{\infty} \frac{1}{\sqrt{2n+1} + \sqrt{2n-1}}$$

$$\begin{aligned} [\text{Sol}] \quad \frac{1}{\sqrt{2n+1} + \sqrt{2n-1}} &= \frac{\sqrt{2n+1} - \sqrt{2n-1}}{(\sqrt{2n+1} + \sqrt{2n-1})(\sqrt{2n+1} - \sqrt{2n-1})} \\ &= \frac{1}{2}(\sqrt{2n+1} - \sqrt{2n-1}) \end{aligned}$$

Therefore, let S_n be the partial sum of the first n terms.

$$\begin{aligned} S_n &= \frac{1}{2}(\sqrt{3} - \sqrt{1}) + \frac{1}{2}(\sqrt{5} - \sqrt{3}) + \frac{1}{2}(\sqrt{7} - \sqrt{5}) + \dots + \frac{1}{2}(\sqrt{2n+1} - \sqrt{2n-1}) \\ &= \frac{1}{2}(-1 + \sqrt{2n+1}) \end{aligned}$$

$$\therefore \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \frac{1}{2}(-1 + \sqrt{2n+1}) = \infty$$

Thus, the series **diverges**.

$$(3) \sum_{n=1}^{\infty} \frac{1}{(n+1)\sqrt{n} + n\sqrt{n+1}}$$

$$\begin{aligned} [\text{Sol}] \quad \frac{1}{(n+1)\sqrt{n} + n\sqrt{n+1}} &= \frac{(n+1)\sqrt{n} - n\sqrt{n+1}}{[(n+1)\sqrt{n} + n\sqrt{n+1}][(n+1)\sqrt{n} - n\sqrt{n+1}]} \\ &= \frac{(n+1)\sqrt{n} - n\sqrt{n+1}}{n(n+1)} \\ &= \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \end{aligned}$$

Therefore, let S_n be the partial sum of the first n terms.

$$\begin{aligned} S_n &= \left(\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} \right) + \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} \right) + \left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} \right) + \dots + \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \right) \\ &= 1 - \frac{1}{\sqrt{n+1}} \end{aligned}$$

$$\therefore \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{\sqrt{n+1}} \right) = 1$$

Thus, the series **converges** and the sum is 1.

Infinite Series

Determine whether each given infinite series converges or diverges.
If it converges, find the sum.

Ex. $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{n(n+1)} + \dots$

[Sol] Let $\frac{1}{n(n+1)} = \frac{a}{n} - \frac{b}{n+1}$.

$$1 = a(n+1) - bn$$

$$= (a-b)n + a$$

$$\therefore a=1, b=1$$

Multiplying both sides
by $n(n+1)$

Since $1 = (a-b)n + a$,

$$\begin{cases} a-b=0 \\ a=1 \end{cases}$$

$$a=1$$

Therefore, let S_n be the partial sum of the first n terms.

$$\begin{aligned} S_n &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1} \end{aligned}$$

$$\therefore \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n+1}\right) = 1$$

Thus, the series converges and the sum is 1.

(1) $\frac{3}{1 \cdot 4} + \frac{3}{4 \cdot 7} + \frac{3}{7 \cdot 10} + \dots + \frac{3}{(3n-2)(3n+1)} + \dots$

[Sol] Let $\frac{3}{(3n-2)(3n+1)} = \frac{a}{3n-2} - \frac{b}{3n+1}$.

$$3 = a(3n+1) - b(3n-2)$$

$$= 3(a-b)n + (a+2b)$$

$$\therefore a=1, b=1$$

Therefore, let S_n be the partial sum of the first n terms.

$$\begin{aligned} S_n &= \left(\frac{1}{1} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{7}\right) + \left(\frac{1}{7} - \frac{1}{10}\right) + \dots + \left(\frac{1}{3n-2} - \frac{1}{3n+1}\right) \\ &= 1 - \frac{1}{3n+1} \end{aligned}$$

$$\therefore \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{3n+1}\right) = 1$$

Thus, the series converges and the sum is 1.

N92b

$$(2) \frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \dots + \frac{1}{(2n-1)(2n+1)} + \dots$$

[Sol] Let $\frac{1}{(2n-1)(2n+1)} = \frac{a}{2n-1} - \frac{b}{2n+1}$.

$$1 = a(2n+1) - b(2n-1)$$

$$= 2(a-b)n + (a+b)$$

$$\therefore a = \frac{1}{2}, b = \frac{1}{2}$$

Therefore, let S_n be the partial sum of the first n terms.

$$S_n = \frac{1}{2} \left(\frac{1}{1} - \frac{1}{3} \right) + \frac{1}{2} \left(\frac{1}{3} - \frac{1}{5} \right) + \frac{1}{2} \left(\frac{1}{5} - \frac{1}{7} \right) + \dots + \frac{1}{2} \left(\frac{1}{2n-1} - \frac{1}{2n+1} \right)$$

$$= \frac{1}{2} \left(1 - \frac{1}{2n+1} \right)$$

$$\therefore \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \frac{1}{2} \left(1 - \frac{1}{2n+1} \right) = \frac{1}{2}$$

Thus, the series **converges** and the sum is $\frac{1}{2}$.

$$(3) \frac{1}{1} + \frac{1}{1+2} + \frac{1}{1+2+3} + \dots + \frac{1}{1+2+\dots+n} + \dots$$

[Sol] $\frac{1}{1+2+\dots+n} = \frac{1}{\frac{1}{2}n(n+1)} = \frac{2}{n(n+1)}$ ← $1+2+\dots+n = \sum_{k=1}^n k$

Let $\frac{2}{n(n+1)} = \frac{a}{n} - \frac{b}{n+1}$.

$$2 = a(n+1) - bn$$

$$= (a-b)n + a$$

$$\therefore a = 2, b = 2$$

Therefore, let S_n be the partial sum of the first n terms.

$$S_n = 2 \left(\frac{1}{1} - \frac{1}{2} \right) + 2 \left(\frac{1}{2} - \frac{1}{3} \right) + 2 \left(\frac{1}{3} - \frac{1}{4} \right) + \dots + 2 \left(\frac{1}{n} - \frac{1}{n+1} \right)$$

$$= 2 \left(1 - \frac{1}{n+1} \right)$$

$$\therefore \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} 2 \left(1 - \frac{1}{n+1} \right) = 2$$

Thus, the series **converges** and the sum is 2.

Infinite Series

Determine whether each given infinite series converges or diverges.
If it converges, find the sum.

$$(1) \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{3 \cdot 4 \cdot 5} + \cdots + \frac{1}{n(n+1)(n+2)} + \cdots$$

[Sol] Let $\frac{1}{n(n+1)(n+2)} = \frac{a}{n(n+1)} - \frac{b}{(n+1)(n+2)}$.

$$1 = a(n+2) - bn$$

$$= (a-b)n + 2a$$

$$\therefore a = \frac{1}{2}, \quad b = \frac{1}{2}$$

Therefore, let S_n be the partial sum of the first n terms.

$$\begin{aligned} S_n &= \frac{1}{2} \left(\frac{1}{2} - \frac{1}{6} \right) + \frac{1}{2} \left(\frac{1}{6} - \frac{1}{12} \right) + \frac{1}{2} \left(\frac{1}{12} - \frac{1}{20} \right) + \cdots + \frac{1}{2} \left[\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)} \right] \\ &= \frac{1}{2} \left[\frac{1}{2} - \frac{1}{(n+1)(n+2)} \right] \end{aligned}$$

$$\therefore \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \frac{1}{2} \left[\frac{1}{2} - \frac{1}{(n+1)(n+2)} \right] = \frac{1}{4}$$

Thus, the series **converges** and the sum is $\frac{1}{4}$.

N93b

$$(2) \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 + 2 \cdot 3} + \frac{1}{1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4} + \dots + \frac{1}{1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \dots + n(n+1)} + \dots$$

$$[\text{Sol}] \frac{1}{1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \dots + n(n+1)} = \frac{1}{\sum_{k=1}^n k(k+1)}$$

$$= \frac{1}{\frac{1}{6}n(n+1)(2n+1) + \frac{1}{2}n(n+1)}$$

$$= \frac{1}{\frac{1}{6}n(n+1)[(2n+1)+3]}$$

$$= \frac{3}{n(n+1)(n+2)}$$

$$\text{Let } \frac{3}{n(n+1)(n+2)} = \frac{a}{n(n+1)} - \frac{b}{(n+1)(n+2)}.$$

$$3 = a(n+2) - bn$$

$$= (a-b)n + 2a$$

$$\therefore a = \frac{3}{2}, \quad b = \frac{3}{2}$$

Therefore, let S_n be the partial sum of the first n terms.

$$S_n = \frac{3}{2} \left(\frac{1}{2} - \frac{1}{6} \right) + \frac{3}{2} \left(\frac{1}{6} - \frac{1}{12} \right) + \frac{3}{2} \left(\frac{1}{12} - \frac{1}{20} \right) + \dots + \frac{3}{2} \left[\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)} \right]$$

$$= \frac{3}{2} \left[\frac{1}{2} - \frac{1}{(n+1)(n+2)} \right]$$

$$\therefore \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \frac{3}{2} \left[\frac{1}{2} - \frac{1}{(n+1)(n+2)} \right] = \frac{3}{4}$$

Thus, the series **converges** and the sum is $\frac{3}{4}$.

$$\sum_{k=1}^n k^2 = \frac{1}{6}n(n+1)(2n+1)$$

Find the sum of the following infinite series.

Ex. $\sum_{k=1}^{\infty} \left(\cos \frac{\pi}{k} - \cos \frac{\pi}{k+1} \right)$

[Sol] Let S_n be the partial sum of the first n terms.

$$S_n = \left(\cos \pi - \cos \frac{\pi}{2} \right) + \left(\cos \frac{\pi}{2} - \cos \frac{\pi}{3} \right) + \cdots + \left(\cos \frac{\pi}{n} - \cos \frac{\pi}{n+1} \right)$$

$$= \cos \pi - \cos \frac{\pi}{n+1}$$

$$= -1 - \cos \frac{\pi}{n+1}$$

$$\therefore \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left(-1 - \cos \frac{\pi}{n+1} \right) = -2$$

$$\therefore \sum_{k=1}^{\infty} \left(\cos \frac{\pi}{k} - \cos \frac{\pi}{k+1} \right) = -2$$

$$\lim_{n \rightarrow \infty} \cos \frac{\pi}{n+1} = \cos 0 = 1$$

(1) $\sum_{k=1}^{\infty} \left(\sin \frac{\pi}{2k} - \sin \frac{\pi}{2k+2} \right)$

[Sol] Let S_n be the partial sum of the first n terms.

$$S_n = \left(\sin \frac{\pi}{2} - \sin \frac{\pi}{4} \right) + \left(\sin \frac{\pi}{4} - \sin \frac{\pi}{6} \right) + \cdots + \left(\sin \frac{\pi}{2n} - \sin \frac{\pi}{2n+2} \right)$$

$$= \sin \frac{\pi}{2} - \sin \frac{\pi}{2n+2}$$

$$= 1 - \sin \frac{\pi}{2n+2}$$

$$\therefore \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left(1 - \sin \frac{\pi}{2n+2} \right) = 1$$

$$\therefore \sum_{k=1}^{\infty} \left(\sin \frac{\pi}{2k} - \sin \frac{\pi}{2k+2} \right) = 1$$

N94b

$$(2) \sum_{k=1}^{\infty} 2 \cos \frac{3\pi}{2^{k+1}} \sin \frac{\pi}{2^{k+1}}$$

$$[\text{Sol}] \quad 2 \cos \frac{3\pi}{2^{k+1}} \sin \frac{\pi}{2^{k+1}}$$

$$= 2 \cdot \frac{1}{2} \left[\sin \left(\frac{3\pi}{2^{k+1}} + \frac{\pi}{2^{k+1}} \right) - \sin \left(\frac{3\pi}{2^{k+1}} - \frac{\pi}{2^{k+1}} \right) \right]$$

$$\cos \alpha \sin \beta = \frac{1}{2} [\sin(\alpha + \beta) - \sin(\alpha - \beta)]$$

$$= \sin \frac{\pi}{2^{k-1}} - \sin \frac{\pi}{2^k}$$

Therefore, let S_n be the partial sum of the first n terms.

$$S_n = \left(\sin \pi - \sin \frac{\pi}{2} \right) + \left(\sin \frac{\pi}{2} - \sin \frac{\pi}{4} \right) + \cdots + \left(\sin \frac{\pi}{2^{n-1}} - \sin \frac{\pi}{2^n} \right)$$

$$= \sin \pi - \sin \frac{\pi}{2^n}$$

$$= -\sin \frac{\pi}{2^n}$$

$$\therefore \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left(-\sin \frac{\pi}{2^n} \right) = 0$$

$$\therefore \sum_{k=1}^{\infty} 2 \cos \frac{3\pi}{2^{k+1}} \sin \frac{\pi}{2^{k+1}} = 0$$

from Kumon Hub

Properties of Infinite Series

When the infinite series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge, where $\sum_{n=1}^{\infty} a_n = S$ and $\sum_{n=1}^{\infty} b_n = T$, the following properties are true:

$$\sum_{n=1}^{\infty} k a_n = k S \quad (k \text{ is a constant})$$

$$\sum_{n=1}^{\infty} (a_n + b_n) = S + T$$

$$\sum_{n=1}^{\infty} (a_n - b_n) = S - T$$

Find the sum of the following infinite series.

Ex.

$$\sum_{n=1}^{\infty} \left(\frac{1}{2^{n-1}} - \frac{5}{4^n} \right)$$

$$[\text{Sol}] \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = \frac{1}{1 - \frac{1}{2}} = 2$$

Since the series is the infinite geometric series with 1st term 1 and common ratio $\frac{1}{2}$, it converges.

$$\sum_{n=1}^{\infty} \frac{5}{4^n} = \frac{\frac{5}{4}}{1 - \frac{1}{4}} = \frac{5}{3}$$

Since the series is the infinite geometric series with 1st term $\frac{5}{4}$ and common ratio $\frac{1}{4}$, it converges.

$$\therefore \sum_{n=1}^{\infty} \left(\frac{1}{2^{n-1}} - \frac{5}{4^n} \right) = \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} - \sum_{n=1}^{\infty} \frac{5}{4^n} = 2 - \frac{5}{3} = \frac{1}{3}$$

$$(1) \sum_{n=1}^{\infty} \left(\frac{2}{3^n} - \frac{1}{6^{n-1}} \right)$$

$$[\text{Sol}] \sum_{n=1}^{\infty} \frac{2}{3^n} = \frac{\frac{2}{3}}{1 - \frac{1}{3}} = 1$$

$$\sum_{n=1}^{\infty} \frac{1}{6^{n-1}} = \frac{1}{1 - \frac{1}{6}} = \frac{6}{5}$$

$$\therefore \sum_{n=1}^{\infty} \left(\frac{2}{3^n} - \frac{1}{6^{n-1}} \right) = \sum_{n=1}^{\infty} \frac{2}{3^n} - \sum_{n=1}^{\infty} \frac{1}{6^{n-1}} = 1 - \frac{6}{5} = -\frac{1}{5}$$

N95b

$$(2) \sum_{n=1}^{\infty} \frac{2^n - 3^n}{4^n}$$

$$[\text{Sol}] \sum_{n=1}^{\infty} \frac{2^n}{4^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1$$

$$\sum_{n=1}^{\infty} \frac{3^n}{4^n} = \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n = \frac{\frac{3}{4}}{1 - \frac{3}{4}} = 3$$

$$\therefore \sum_{n=1}^{\infty} \frac{2^n - 3^n}{4^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n - \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n = 1 - 3 = -2$$

$$(3) \sum_{n=1}^{\infty} \frac{3^{n+1} + 4^n}{5^n}$$

$$[\text{Sol}] \sum_{n=1}^{\infty} \frac{3^{n+1}}{5^n} = \sum_{n=1}^{\infty} 3 \left(\frac{3}{5}\right)^n = \frac{\frac{9}{5}}{1 - \frac{3}{5}} = \frac{9}{2}$$

$$\sum_{n=1}^{\infty} \frac{4^n}{5^n} = \sum_{n=1}^{\infty} \left(\frac{4}{5}\right)^n = \frac{\frac{4}{5}}{1 - \frac{4}{5}} = 4$$

$$\therefore \sum_{n=1}^{\infty} \frac{3^{n+1} + 4^n}{5^n} = \sum_{n=1}^{\infty} 3 \left(\frac{3}{5}\right)^n + \sum_{n=1}^{\infty} \left(\frac{4}{5}\right)^n = \frac{9}{2} + 4 = \frac{17}{2}$$

$$(4) \sum_{n=1}^{\infty} \frac{3^{-n} - (-1)^n}{2^{3n}}$$

$$[\text{Sol}] \sum_{n=1}^{\infty} \frac{3^{-n}}{2^{3n}} = \sum_{n=1}^{\infty} \frac{(3^{-1})^n}{8^n} = \sum_{n=1}^{\infty} \left(\frac{1}{24}\right)^n = \frac{\frac{1}{24}}{1 - \frac{1}{24}} = \frac{1}{23}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2^{3n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{8^n} = \sum_{n=1}^{\infty} \left(-\frac{1}{8}\right)^n = \frac{-\frac{1}{8}}{1 - \left(-\frac{1}{8}\right)} = -\frac{1}{9}$$

$$\therefore \sum_{n=1}^{\infty} \frac{3^{-n} - (-1)^n}{2^{3n}} = \sum_{n=1}^{\infty} \left(\frac{1}{24}\right)^n - \sum_{n=1}^{\infty} \left(-\frac{1}{8}\right)^n = \frac{1}{23} + \frac{1}{9} = \frac{32}{207}$$

Infinite Series

Let S_n be the partial sum of the first n terms of infinite series $\sum_{n=1}^{\infty} a_n$.

When $n \geq 2$, $a_n = S_n - S_{n-1}$

When the infinite series $\sum_{n=1}^{\infty} a_n$ converges, let S be the sum. Then,

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} (S_n - S_{n-1}) = \lim_{n \rightarrow \infty} S_n - \lim_{n \rightarrow \infty} S_{n-1} = S - S = 0$$

Therefore, when $\sum_{n=1}^{\infty} a_n$ converges, $\lim_{n \rightarrow \infty} a_n = 0$

Also, when the sequence $\{a_n\}$ does not converge to 0, $\sum_{n=1}^{\infty} a_n$ diverges.

From the above, the following statement is true.

Convergence and Divergence of Infinite Series

If the infinite series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n \rightarrow \infty} a_n = 0$.

If the sequence $\{a_n\}$ does not converge to 0, then the infinite series $\sum_{n=1}^{\infty} a_n$ diverges.

Determine whether each given infinite series converges or diverges.

Ex. $\frac{1}{1} + \frac{2}{3} + \frac{3}{5} + \frac{4}{7} + \dots$

[Sol] The n^{th} term a_n is $a_n = \frac{n}{2n-1}$.

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{2 - \frac{1}{n}} = \frac{1}{2}$$

The sequence $\{a_n\}$ does not converge to 0.
Therefore, the series diverges.

(1) $\frac{1}{3} + \frac{2}{4} + \frac{3}{5} + \frac{4}{6} + \dots$

[Sol] The n^{th} term a_n is $a_n = \frac{n}{n+2}$.

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{2}{n}} = 1$$

The sequence $\{a_n\}$ does not converge to 0.
Therefore, the series **diverges**.

N96b

$$(2) \quad \frac{1}{3} + \frac{3}{6} + \frac{5}{9} + \frac{7}{12} + \dots$$

[Sol] The n^{th} term a_n is $a_n = \frac{2n-1}{3n}$.

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{2 - \frac{1}{n}}{3} = \frac{2}{3}$$

The sequence $\{a_n\}$ does not converge to 0.

Therefore, the series **diverges**.

$$(3) \quad \frac{1}{2} + \frac{4}{5} + \frac{9}{10} + \frac{16}{17} + \dots$$

[Sol] The n^{th} term a_n is $a_n = \frac{n^2}{n^2+1}$.

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{1}{n^2}} = 1$$

The sequence $\{a_n\}$ does not converge to 0.

Therefore, the series **diverges**.

$$(4) \quad 1 - 2 + 3 - 4 + \dots$$

[Sol] The n^{th} term a_n is $a_n = (-1)^{n-1}n$.

The sequence $\{a_n\}$ oscillates and does not converge to 0.

Therefore, the series **diverges**.

$$(5) \quad -1 + 3 - 5 + 7 - \dots$$

[Sol] The n^{th} term a_n is $a_n = (-1)^n(2n-1)$.

The sequence $\{a_n\}$ oscillates and does not converge to 0.

Therefore, the series **diverges**.

Infinite Series

Determine whether each given infinite series converges or diverges.

If it converges, find the sum.

Ex. $\frac{1}{2} - \frac{2}{3} + \frac{2}{3} - \frac{3}{4} + \frac{3}{4} - \dots - \frac{n}{n+1} + \frac{n}{n+1} - \frac{n+1}{n+2} + \dots$

[Sol] Let S_n be the partial sum of the first n terms and m be a natural number.

(i) When $n = 2m - 1$,

$$S_n = S_{2m-1}$$

$$= \frac{1}{2} + \left(-\frac{2}{3} + \frac{2}{3}\right) + \left(-\frac{3}{4} + \frac{3}{4}\right) + \dots + \left(-\frac{m}{m+1} + \frac{m}{m+1}\right)$$

$$= \frac{1}{2}$$

$$\therefore \lim_{m \rightarrow \infty} S_{2m-1} = \frac{1}{2}$$

$$S_1 = \frac{1}{2}$$

$$S_3 = \frac{1}{2} + \left(-\frac{2}{3} + \frac{2}{3}\right)$$

$$S_5 = \frac{1}{2} + \left(-\frac{2}{3} + \frac{2}{3}\right) + \left(-\frac{3}{4} + \frac{3}{4}\right)$$

⋮

(ii) When $n = 2m$,

$$S_n = S_{2m}$$

$$= S_{2m-1} + \left(-\frac{m+1}{m+2}\right)$$

$$S_{2m} = S_{2m-1} + (\text{the } 2m^{\text{th}} \text{ term})$$

$$\text{From (i), } \lim_{m \rightarrow \infty} S_{2m-1} = \frac{1}{2}$$

$$\therefore \lim_{m \rightarrow \infty} S_{2m} = \lim_{m \rightarrow \infty} \left(S_{2m-1} - \frac{1 + \frac{1}{m}}{1 + \frac{2}{m}} \right) = \frac{1}{2}$$

From (i) and (ii), $\lim_{m \rightarrow \infty} S_{2m-1} \neq \lim_{m \rightarrow \infty} S_{2m}$

Therefore, the series diverges.

$\{S_n\}$ oscillates and diverges.

Answers: in order $\frac{1}{2}, \frac{2}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{2}$

N97b

$$(1) \quad 3 - 2 + 2 - \frac{5}{3} + \frac{5}{3} - \dots - \frac{n+2}{n} + \frac{n+2}{n} - \frac{n+3}{n+1} + \dots$$

[Sol] Let S_n be the partial sum of the first n terms and m be a natural number.

(i) When $n = 2m - 1$,

$$S_n = S_{2m-1}$$

$$= 3 + (-2 + 2) + \left(-\frac{5}{3} + \frac{5}{3}\right) + \dots + \left(-\frac{m+2}{m} + \frac{m+2}{m}\right)$$

$$= 3$$

$$\therefore \lim_{m \rightarrow \infty} S_{2m-1} = 3$$

(ii) When $n = 2m$,

$$S_n = S_{2m}$$

$$= S_{2m-1} + \left(-\frac{m+3}{m+1}\right)$$

$$\therefore \lim_{m \rightarrow \infty} S_{2m} = \lim_{m \rightarrow \infty} \left(S_{2m-1} - \frac{1 + \frac{3}{m}}{1 + \frac{1}{m}} \right) = 2$$

From (i) and (ii), $\lim_{m \rightarrow \infty} S_{2m-1} \neq \lim_{m \rightarrow \infty} S_{2m}$

Therefore, the series **diverges**.

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Infinite Series

Determine whether each given infinite series converges or diverges.

If it converges, find the sum.

Ex. $\frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{2^4} + \frac{1}{3^4} + \dots$

[Sol] Let S_n be the partial sum of the first n terms and m be a natural number.

(i) When $n = 2m$,

$$\begin{aligned} S_n &= S_{2m} \\ &= \left[\frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \dots + \left(\frac{1}{2}\right)^m \right] + \left[\frac{1}{3} + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \dots + \left(\frac{1}{3}\right)^m \right] \\ &= \frac{\frac{1}{2} \left[1 - \left(\frac{1}{2}\right)^m \right]}{1 - \frac{1}{2}} + \frac{\frac{1}{3} \left[1 - \left(\frac{1}{3}\right)^m \right]}{1 - \frac{1}{3}} \\ &= \left[1 - \left(\frac{1}{2}\right)^m \right] + \frac{1}{2} \left[1 - \left(\frac{1}{3}\right)^m \right] \\ \therefore \lim_{m \rightarrow \infty} S_{2m} &= \boxed{\frac{3}{2}} \end{aligned}$$

(ii) When $n = 2m - 1$,

$$\begin{aligned} S_n &= S_{2m-1} \\ &= S_{2m} - \left(\frac{1}{3}\right)^m \end{aligned}$$

$S_{2m-1} = S_{2m} - (\text{the } 2m^{\text{th}} \text{ term})$

$$\therefore \lim_{m \rightarrow \infty} S_{2m-1} = \lim_{m \rightarrow \infty} \left[S_{2m} - \left(\frac{1}{3}\right)^m \right] = \boxed{\frac{3}{2}}$$

From (i) and (ii), $\lim_{m \rightarrow \infty} S_{2m} = \lim_{m \rightarrow \infty} S_{2m-1}$

Therefore, the series converges and the sum is $\boxed{\frac{3}{2}}$.

From (i),
 $\lim_{m \rightarrow \infty} S_{2m} = \frac{3}{2}$

Answers: in order $\frac{2}{3}, \frac{2}{3}, \frac{3}{1}, \frac{3}{1}, \frac{2}{3}, \frac{2}{3}$

N98b

$$(1) \ 1 + 2 + \frac{1}{2} - \frac{2}{3} + \frac{1}{4} + \frac{2}{9} + \frac{1}{8} - \frac{2}{27} + \dots$$

[Sol] Let S_n be the partial sum of the first n terms and m be a natural number.

(i) When $n = 2m$,

$$S_n = S_{2m}$$

term $2m$
↓

$$= \left[1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \dots + \left(\frac{1}{2}\right)^{m-1} \right] + \left[2 + 2 \cdot \left(-\frac{1}{3}\right) + 2 \left(-\frac{1}{3}\right)^2 + \dots + 2 \left(-\frac{1}{3}\right)^{m-1} \right]$$

$$= \frac{1 - \left(\frac{1}{2}\right)^m}{1 - \frac{1}{2}} + \frac{2 \left[1 - \left(-\frac{1}{3}\right)^m \right]}{1 - \left(-\frac{1}{3}\right)}$$

$$= 2 \left[1 - \left(\frac{1}{2}\right)^m \right] + \frac{3}{2} \left[1 - \left(-\frac{1}{3}\right)^m \right]$$

$$\therefore \lim_{m \rightarrow \infty} S_{2m} = \frac{7}{2}$$

(ii) When $n = 2m - 1$,

$$S_n = S_{2m-1}$$

$$= S_{2m} - 2 \left(-\frac{1}{3}\right)^{m-1}$$

$$\therefore \lim_{m \rightarrow \infty} S_{2m-1} = \lim_{m \rightarrow \infty} \left[S_{2m} - 2 \left(-\frac{1}{3}\right)^{m-1} \right] = \frac{7}{2}$$

From (i) and (ii), $\lim_{m \rightarrow \infty} S_{2m} = \lim_{m \rightarrow \infty} S_{2m-1}$

Therefore, the series **converges** and the sum is $\frac{7}{2}$.

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1. Given the sequence $\{a_n\}$: $\frac{3}{1 \cdot 2 \cdot 2^1}, \frac{4}{2 \cdot 3 \cdot 2^2}, \frac{5}{3 \cdot 4 \cdot 2^3}, \frac{6}{4 \cdot 5 \cdot 2^4}, \dots$,
solve the following questions.

- (1) Find the general term of $\{a_n\}$.

[Sol] $a_n = \frac{n+2}{n(n+1) \cdot 2^n}$

- (2) Find the constants b and c which satisfy $2^n a_n = \frac{b}{n} + \frac{c}{n+1}$.

[Sol] From (1), $2^n a_n = \frac{n+2}{n(n+1)}$... ①

Since $2^n a_n = \frac{b}{n} + \frac{c}{n+1}$,

$2^n a_n = \frac{(b+c)n+b}{n(n+1)}$... ②

From ① and ②, $n+2 = (b+c)n+b$
 $\therefore b=2, c=-1$

Since $n+2 = (b+c)n+b$,
 $\begin{cases} b+c=1 \\ b=2 \end{cases}$

- (3) Find the sum S_n of the first n terms of $\{a_n\}$ and its limit value $\lim_{n \rightarrow \infty} S_n$.

[Sol] From (2), $a_n = \left(\frac{2}{n} - \frac{1}{n+1} \right) \frac{1}{2^n} = \frac{1}{n \cdot 2^{n-1}} - \frac{1}{(n+1) \cdot 2^n}$
 $\therefore S_n = \left(\frac{1}{1} - \frac{1}{4} \right) + \left(\frac{1}{4} - \frac{1}{12} \right) + \left(\frac{1}{12} - \frac{1}{32} \right) + \dots + \left[\frac{1}{n \cdot 2^{n-1}} - \frac{1}{(n+1) \cdot 2^n} \right]$
 $= 1 - \frac{1}{(n+1) \cdot 2^n}$

$\therefore \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left[1 - \frac{1}{(n+1) \cdot 2^n} \right] = 1$

N99b

2. Find the sum of the following infinite series.

$$1 + \frac{2}{\pi} + \frac{3}{\pi^2} + \frac{4}{\pi^3} + \dots + \frac{n}{\pi^{n-1}} + \dots$$

(Use the condition $\lim_{n \rightarrow \infty} nx^n = 0$ when $|x| < 1$.)

[Sol] Let S_n be the partial sum of the first n terms.

$$S_n = 1 + \frac{2}{\pi} + \frac{3}{\pi^2} + \frac{4}{\pi^3} + \dots + \frac{n}{\pi^{n-1}} \quad \dots \textcircled{1}$$

Multiplying both sides of $\textcircled{1}$ by $\frac{1}{\pi}$,

$$\frac{1}{\pi} S_n = \frac{1}{\pi} + \frac{2}{\pi^2} + \frac{3}{\pi^3} + \dots + \frac{n-1}{\pi^{n-1}} + \frac{n}{\pi^n} \quad \dots \textcircled{2}$$

From $\textcircled{1} - \textcircled{2}$,

$$\begin{aligned} \left(1 - \frac{1}{\pi}\right) S_n &= \left(1 + \frac{1}{\pi} + \frac{1}{\pi^2} + \frac{1}{\pi^3} + \dots + \frac{1}{\pi^{n-1}}\right) - \frac{n}{\pi^n} \\ &= \frac{1 - \left(\frac{1}{\pi}\right)^n}{1 - \frac{1}{\pi}} - \frac{n}{\pi^n} \end{aligned}$$

$$\begin{aligned} \therefore S_n &= \frac{1 - \left(\frac{1}{\pi}\right)^n}{\left(1 - \frac{1}{\pi}\right)^2} - \frac{1}{1 - \frac{1}{\pi}} \cdot \frac{n}{\pi^n} \\ &= \left(\frac{\pi}{\pi-1}\right)^2 \left[1 - \left(\frac{1}{\pi}\right)^n\right] - \frac{\pi}{\pi-1} \cdot \frac{n}{\pi^n} \end{aligned}$$

Therefore, the sum of the infinite series is

$$\begin{aligned} \lim_{n \rightarrow \infty} S_n &= \lim_{n \rightarrow \infty} \left\{ \left(\frac{\pi}{\pi-1}\right)^2 \left[1 - \left(\frac{1}{\pi}\right)^n\right] - \frac{\pi}{\pi-1} \cdot \frac{n}{\pi^n} \right\} \\ &= \left(\frac{\pi}{\pi-1}\right)^2 \end{aligned}$$

N35

Sum of the geometric sequence with 1st term 1, common ratio $\frac{1}{\pi}$ and number of terms n

From the condition, $\lim_{n \rightarrow \infty} nx^n = 0$ ($|x| < 1$)

Also, $\left|\frac{1}{\pi}\right| < 1$ from $\pi = 3.14\dots$

$$\therefore \lim_{n \rightarrow \infty} \frac{n}{\pi^n} = \lim_{n \rightarrow \infty} n \left(\frac{1}{\pi}\right)^n = 0$$

1. Determine whether each given infinite series converges or diverges.
If it converges, find the sum.

(1) $\sum_{n=1}^{\infty} \frac{2}{\sqrt{n+1} + \sqrt{n+2}}$

➡ N91

[Sol]
$$\frac{2}{\sqrt{n+1} + \sqrt{n+2}} = \frac{2(\sqrt{n+1} - \sqrt{n+2})}{(\sqrt{n+1} + \sqrt{n+2})(\sqrt{n+1} - \sqrt{n+2})}$$
$$= -2(\sqrt{n+1} - \sqrt{n+2})$$

Therefore, let S_n be the partial sum of the first n terms.

$$S_n = -2(\sqrt{2} - \sqrt{3}) - 2(\sqrt{3} - \sqrt{4}) - 2(\sqrt{4} - \sqrt{5}) - \cdots - 2(\sqrt{n+1} - \sqrt{n+2})$$
$$= -2(\sqrt{2} - \sqrt{n+2})$$

$$\therefore \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} [-2(\sqrt{2} - \sqrt{n+2})] = \infty$$

Thus, the series **diverges**.

(2) $\frac{4}{1 \cdot 5} + \frac{4}{5 \cdot 9} + \frac{4}{9 \cdot 13} + \cdots + \frac{4}{(4n-3)(4n+1)} + \cdots$

➡ N92

[Sol] Let $\frac{4}{(4n-3)(4n+1)} = \frac{a}{4n-3} - \frac{b}{4n+1}$.

$$4 = a(4n+1) - b(4n-3)$$

$$= 4(a-b)n + (a+3b)$$

$$\therefore a=1, b=1$$

Therefore, let S_n be the partial sum of the first n terms.

$$S_n = \left(\frac{1}{1} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{9}\right) + \left(\frac{1}{9} - \frac{1}{13}\right) + \cdots + \left(\frac{1}{4n-3} - \frac{1}{4n+1}\right)$$
$$= 1 - \frac{1}{4n+1}$$

$$\therefore \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{4n+1}\right) = 1$$

Thus, the series **converges** and the sum is 1.

2. Find the sum of the following infinite series.

➡ N95

$$\sum_{n=1}^{\infty} \frac{5^n - 2^n}{10^n}$$

$$[\text{Sol}] \sum_{n=1}^{\infty} \frac{5^n}{10^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1$$

$$\sum_{n=1}^{\infty} \frac{2^n}{10^n} = \sum_{n=1}^{\infty} \left(\frac{1}{5}\right)^n = \frac{\frac{1}{5}}{1 - \frac{1}{5}} = \frac{1}{4}$$

$$\therefore \sum_{n=1}^{\infty} \frac{5^n - 2^n}{10^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n - \sum_{n=1}^{\infty} \left(\frac{1}{5}\right)^n = 1 - \frac{1}{4} = \frac{3}{4}$$

3. Determine whether the following infinite series converges or diverges.

If it converges, find the sum.

➡ N97

$$\frac{2}{3} - \frac{4}{5} + \frac{4}{5} - \frac{6}{7} + \frac{6}{7} - \dots - \frac{2n}{2n+1} + \frac{2n}{2n+1} - \frac{2n+2}{2n+3} + \dots$$

[Sol] Let S_n be the partial sum of the first n terms and m be a natural number.

(i) When $n = 2m - 1$,

$$\begin{aligned} S_n &= S_{2m-1} \\ &= \frac{2}{3} + \left(-\frac{4}{5} + \frac{4}{5}\right) + \left(-\frac{6}{7} + \frac{6}{7}\right) + \dots + \left(-\frac{2m}{2m+1} + \frac{2m}{2m+1}\right) \\ &= \frac{2}{3} \end{aligned}$$

$$\therefore \lim_{m \rightarrow \infty} S_{2m-1} = \frac{2}{3}$$

(ii) When $n = 2m$,

$$\begin{aligned} S_n &= S_{2m} \\ &= S_{2m-1} + \left(-\frac{2m+2}{2m+3}\right) \end{aligned}$$

$$\therefore \lim_{m \rightarrow \infty} S_{2m} = \lim_{m \rightarrow \infty} \left(S_{2m-1} - \frac{2 + \frac{2}{m}}{2 + \frac{3}{m}} \right) = -\frac{1}{3}$$

From (i) and (ii), $\lim_{m \rightarrow \infty} S_{2m-1} \neq \lim_{m \rightarrow \infty} S_{2m}$

Therefore, the series **diverges**.

Mathematics **Solution Book N (Part One)**

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