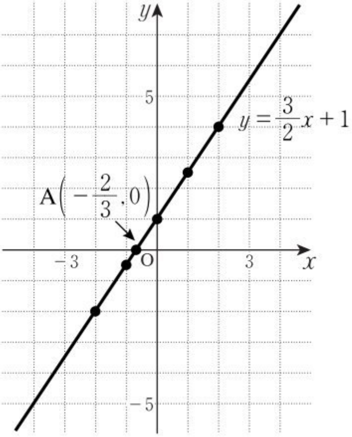
K I a KUMON

Review of Linear Functions

1. Given the function $y = \frac{3}{2}x + 1$, find the value of y for each value of x. First, complete the table below, then plot the $(x,\ y)$ coordinates to graph the function $y = \frac{3}{2}x + 1$.

x	y
-2	$\mathbf{-2}$
-1	$\mathbf{-\frac{1}{2}}$
0	$\mathbf{1}$
1	$\mathbf{\frac{5}{2}}$
2	$\mathbf{4}$



2. Find the value of x at the point where $y = 0$ in the above exercise. Then, plot and label the point as A, with its $(x,\ y)$ coordinates, on the graph above.

[Sol] Substituting $y = 0$ into $y = \frac{3}{2}x + 1$,

$$0 = \frac{3}{2}x + 1$$

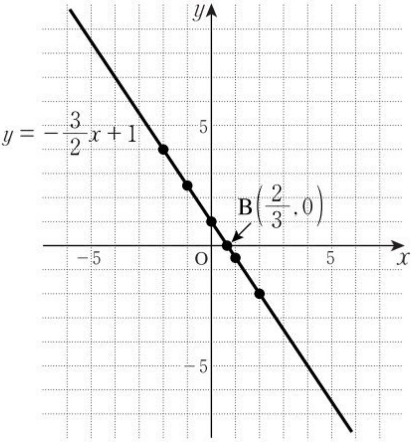
$$x = \boxed{-\frac{2}{3}}$$

---

y is called "a ***function*** of x" when the value of y is determined by the value of x.

3. Given the function $y=-\frac{3}{2}x+1$, find the value of y for each value of x. First, complete the table below, then plot the $(x,\ y)$ coordinates to graph the function $y=-\frac{3}{2}x+1$.

x	y
-2	$\mathbf{4}$
-1	$\mathbf{\frac{5}{2}}$
0	$\mathbf{1}$
1	$\mathbf{-\frac{1}{2}}$
2	$\mathbf{-2}$



4. Find the value of x at the point where $y=0$ in the above exercise. Then, plot and label the point as B, with its $(x,\ y)$ coordinates, on the graph above.

[Sol] Substituting $y=0$ into $y=-\frac{3}{2}x+1$,

$$0=-\frac{3}{2}x+1$$

$$\boldsymbol{x=\frac{2}{3}}$$

K 2a KUMON

Review of Linear Functions

1. Circle the letter(s) of the functions (A)～(F) that pass through the given points.

(1) point $(2,\ 3)$ … [✓ (A)] (B) (C) (D) [✓ (E)] [✓ (F)]

(2) point $(-4,\ 0)$ … (A) (B) [✓ (C)] (D) [✓ (E)] (F)

(A) $y=\frac{3}{2}x$

(B) $y=\frac{1}{2}x-2$

(C) $y=-\frac{1}{2}x-2$

(D) $y=-x+1$

(E) $y=\frac{1}{2}x+2$

(F) $y=-x+5$

K2b

2. Match each function with the corresponding line on the graph. Write the letter (A)～(E) of the line in the blank box provided.

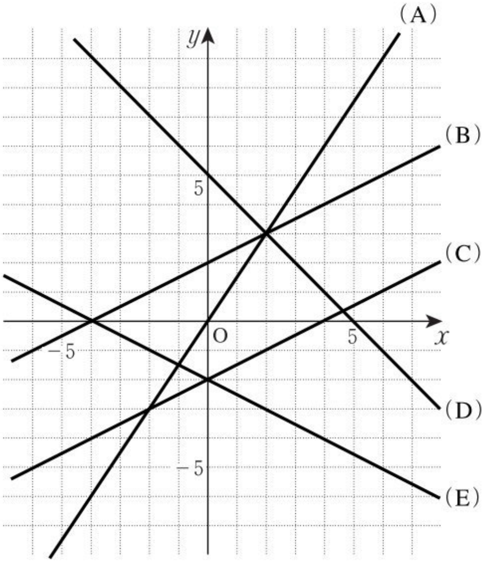
(1) $y=\frac{3}{2}x$ … **(A)**

(2) $y=\frac{1}{2}x-2$ … **(C)**

(3) $y=-\frac{1}{2}x-2$ … **(E)**

(4) $y=\frac{1}{2}x+2$ … **(B)**

(5) $y=-x+5$ … **(D)**



K 3a KUMON

Review of Linear Functions

1. Graph the following ***linear functions.***

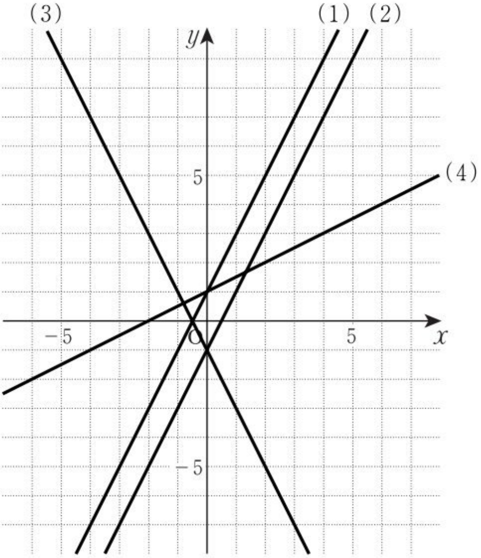
(1) $y=2x+1$

(3) $y=-2x-1$

(2) $y=2x-1$

(4) $y=\frac{1}{2}x+1$

Note: Be sure to label each line on the graph.



A function that can be written in the form $y=mx+b$ (where $m\neq 0$) is called a ***linear function***.
We can refer to "the ***line*** $y=mx+b$", and to $y=mx+b$ as "the ***equation of the line***".

2. Find the ***equations of the lines***, and then graph the lines.

Ex.

A line with gradient 2, and y-intercept 1.

[Sol] $y = 2x + 1$

(1) A line with gradient 2, and y-intercept -1.

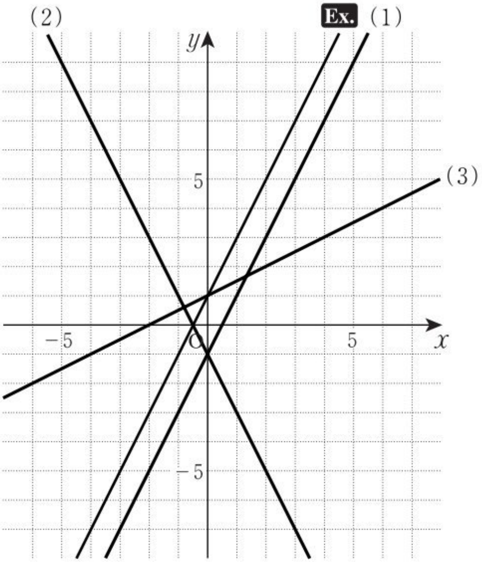
[Sol] $\boldsymbol{y = 2x - 1}$

(2) A line with gradient -2, and y-intercept -1.

[Sol] $\boldsymbol{y = -2x - 1}$

(3) A line with gradient $\frac{1}{2}$, and y-intercept 1.

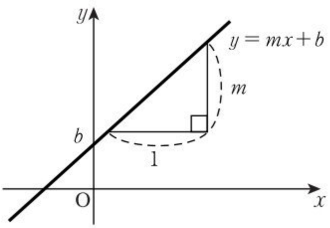
[Sol] $\boldsymbol{y = \frac{1}{2}x + 1}$



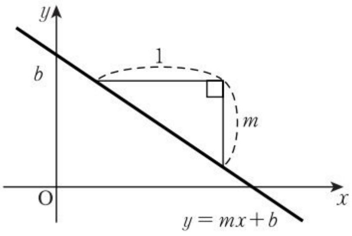
Note: Since the lines in the example and question (1) have the same gradient, the lines must be parallel. (They do not intersect.)
When two lines have equal gradients, the lines are parallel. When two lines are parallel, their gradients are equal. Parallel lines never intersect each other.

Given a line of the form $y = mx + b$, m is the ***gradient***, and b is the ***y-intercept***.

When $m > 0$,



When $m < 0$,



K 4a KUMON

Review of Linear Functions

1. Graph the following lines.

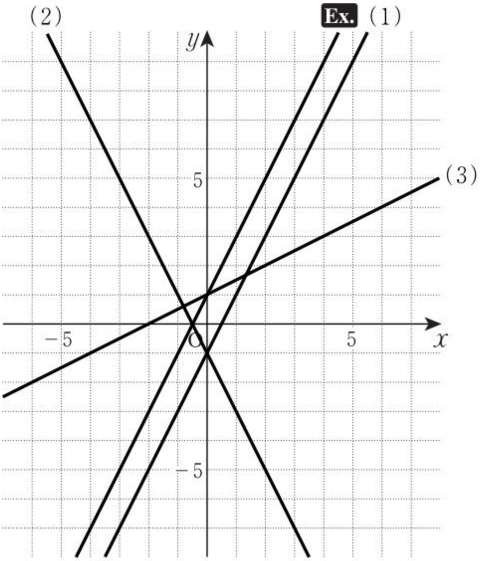
Ex.

A line passing through point $(-2, -3)$ with gradient 2.

(1) A line passing through point $(-2, -5)$ with gradient 2.

(2) A line passing through point $(-3, 5)$ with gradient -2.

(3) A line passing through point $(-4, -1)$ with gradient $\frac{1}{2}$.



K 4b

2. Find the equations of the lines described.

Ex.

A line passing through point $(-2, -3)$ with gradient 2.

[Sol] Let $y = 2x + b$. ☜ Since the gradient is 2.

Substituting $x = -2$ and $y = -3$, ☜ Since the line passes through $(-2, -3)$.

$$-3 = 2 \times (-2) + b$$

$$b = 1$$

Therefore, $y = 2x + 1$

(1) A line passing through point $(-2, -5)$ with gradient 2.

[Sol] Let $y = 2x + b$. ☜ Since the gradient is 2.

Substituting $x = -2$ and $y = -5$, ☜ Since the line passes through $(-2, -5)$.

$$-5 = 2 \times (-2) + b$$

$$b = -1$$

Therefore, $\boldsymbol{y = 2x - 1}$

(2) A line passing through point $(2, -3)$ with gradient $-\frac{1}{2}$.

[Sol] Let $y = -\frac{1}{2}x + b$. ☜ Since the gradient is $-\frac{1}{2}$.

Substituting $x = 2$ and $y = -3$, ☜ Since the line passes through $(2, -3)$.

$$-3 = -\frac{1}{2} \times 2 + b$$

$$b = -2$$

Therefore, $\boldsymbol{y = -\frac{1}{2}x - 2}$

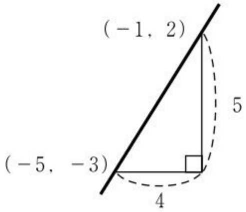
K 5a

Review of Linear Functions

1. Find the equation of the line passing through the two given points.

Ex.

(−5, −3), (−1, 2)



[Sol] Calculating the gradient of the line,

$$\frac{2-(-3)}{-1-(-5)}=\frac{5}{4}$$ ☞ $\frac{\textit{The change in } y}{\textit{The change in } x}$

Therefore, let $y=\frac{5}{4}x+b$.

Substituting $x=-1$ and $y=2$, ☞ Since the line passes through (−1, 2).

from $2=\frac{5}{4}\times(-1)+b$, $b=\frac{13}{4}$

Thus, $y=\frac{5}{4}x+\frac{13}{4}$

Note: Alternatively, since the line also passes through point (−5, −3), we can substitute $x=-5$ and $y=-3$ into the equation.

(1) (−3, 1), (3, 3)

[Sol] Calculating the gradient of the line,

$$\frac{3-1}{3-(-3)}=\frac{1}{3}$$ ☞ $\frac{\text{The change in } y}{\text{The change in } x}$

Therefore, let $y=\frac{1}{3}x+b$.

Substituting $x=3$ and $y=3$, ☞ Since the line passes through (3, 3).

from $3=\frac{1}{3}\times 3+b$, $b=2$

Thus, $\boldsymbol{y=\frac{1}{3}x+2}$

K5b

2. Find the equations of the lines passing through the two given points, and then graph the lines.

(1) (6, 0), (0, 3)

[Sol]

Calculating the gradient of the line,

$$\frac{3-0}{0-6}=-\frac{1}{2}$$

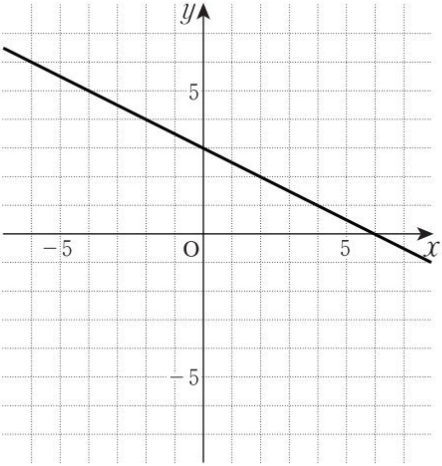
Therefore, let $y=-\frac{1}{2}x+b$.

Substituting $x=0$ and $y=3$,

☜ Since the line passes through (0, 3).

from $3=-\frac{1}{2}\times0+b$, $b=3$

Thus, $\boldsymbol{y=-\frac{1}{2}x+3}$



(2) (2, −1), (4, 2)

[Sol]

Calculating the gradient of the line,

$$\frac{2-(-1)}{4-2}=\frac{3}{2}$$

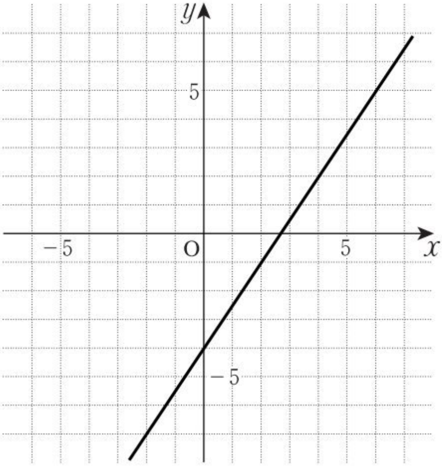
Therefore, let $y=\frac{3}{2}x+b$.

Substituting $x=2$ and $y=-1$,

☜ Since the line passes through (2, −1).

from $-1=\frac{3}{2}\times2+b$, $b=-4$

Thus, $\boldsymbol{y=\frac{3}{2}x-4}$



K 6a KUMON

Review of Linear Functions

1. Match each line with the corresponding line on the graphs below. Write the letter (A)～(H) of the line in the blank box provided.

(1) $y=x+3$ … $\boxed{\textbf{(A)}}$

(2) $y=-x+3$ … $\boxed{\textbf{(C)}}$

(3) $y=x-3$ … $\boxed{\textbf{(B)}}$

(4) $y=-x-3$ … $\boxed{\textbf{(D)}}$

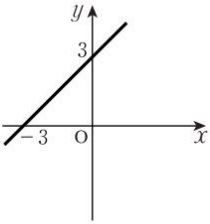
(5) $y=2x+2$ … $\boxed{\textbf{(G)}}$

(6) $y=-2x+2$ … $\boxed{\textbf{(E)}}$

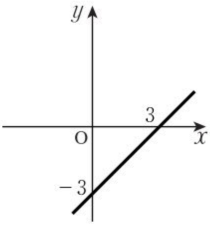
(7) $y=2$ … $\boxed{\textbf{(F)}}$

(8) $x=3$ … $\boxed{\textbf{(H)}}$

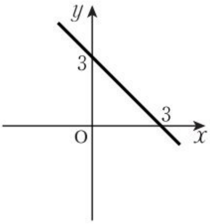
(A)



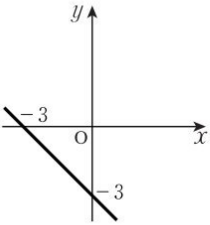
(B)



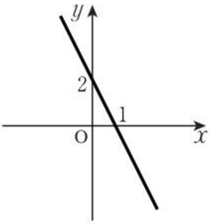
(C)



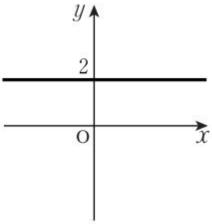
(D)



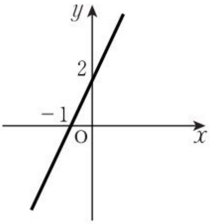
(E)



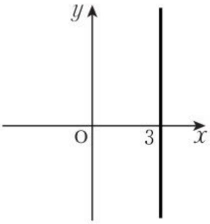
(F)



(G)



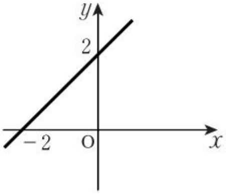
(H)



K6b

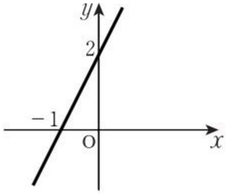
2. Find the equations of the graphed lines.

(1)



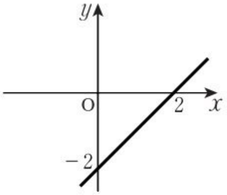
[Sol] $\boldsymbol{y = x+2}$

(5)



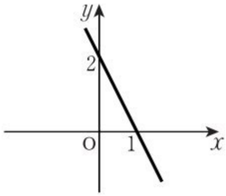
[Sol] $\boldsymbol{y = 2x+2}$

(2)



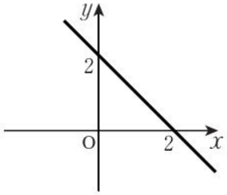
[Sol] $\boldsymbol{y = x-2}$

(6)



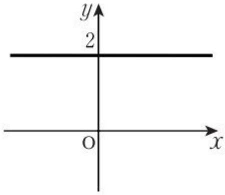
[Sol] $\boldsymbol{y = -2x+2}$

(3)



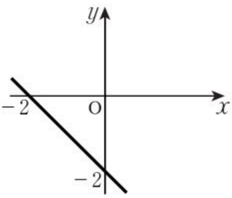
[Sol] $\boldsymbol{y = -x+2}$

(7)



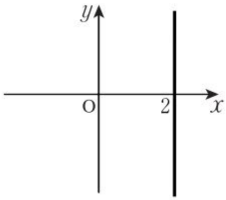
[Sol] $\boldsymbol{y = 2}$

(4)



[Sol] $\boldsymbol{y = -x-2}$

(8)



[Sol] $\boldsymbol{x = 2}$

K 7a KUMON

Review of Linear Functions

For each question, find the coordinates of the point of intersection of the two given lines by drawing their graphs.

Ex.

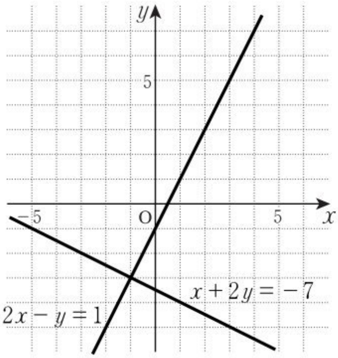
$$\begin{cases} 2x-y=1 & \cdots① \\ x+2y=-7 & \cdots② \end{cases}$$

[Sol]

From ①, $y=2x-1$

From ②, $y=-\frac{1}{2}x-\frac{7}{2}$

Therefore, graphing ① and ②,



From the graph, the point of intersection is $(-1,\ -3)$.

(Check your answer.)

In ①, LHS $=2\times(-1)-(-3)=1$

RHS $=1$

In ②, LHS $=-1+2\times(-3)=-7$

RHS $=-7$

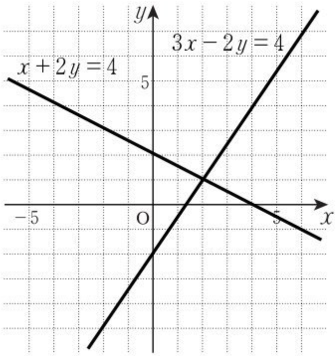
(1) $$\begin{cases} 3x-2y=4 & \cdots① \\ x+2y=4 & \cdots② \end{cases}$$

[Sol]

From ①, $y=\frac{3}{2}x-2$

From ②, $y=-\frac{1}{2}x+2$

Therefore, graphing ① and ②,



From the graph, the point of intersection is **(2, 1)**

(Check your answer.)

In ①, LHS $=3\times2-2\times1=4$

RHS $=4$

In ②, LHS $=2+2\times1=4$

RHS $=4$

Note: We first re-write each equation in the general form of a line, $y=mx+b$, so that we can easily graph it, using the gradient, m, and the y-intercept, b.

K7b

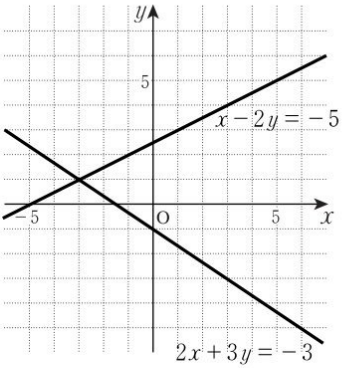
(2) $\begin{cases} x-2y=-5 & \cdots ① \\ 2x+3y=-3 & \cdots ② \end{cases}$

[Sol]

From ①, $y=\frac{1}{2}x+\frac{5}{2}$

From ②, $y=-\frac{2}{3}x-1$

Therefore, graphing ① and ②,



From the graph, the point of intersection is **(−3, 1)**.

(Check your answer.)

In ①, LHS $=-3-2\times1=-5$

RHS $=-5$

In ②, LHS $=2\times(-3)+3\times1=-3$

RHS $=-3$

(3) $\begin{cases} 3x-2y=-2 & \cdots ① \\ y=4 & \cdots ② \end{cases}$

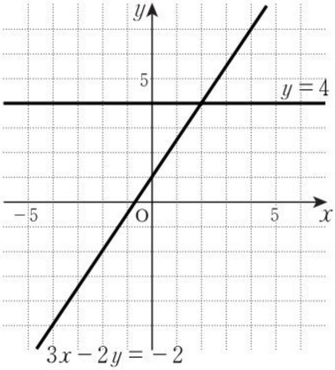
Hint

[Sol]

From ①, $y=\frac{3}{2}x+1$

and ② remains as $y=4$.

Therefore, graphing ① and ②,



From the graph, the point of intersection is **(2, 4)**.

(Check your answer.)

In ①, LHS $=3\times2-2\times4=-2$

RHS $=-2$

In ②, LHS $=4$

RHS $=4$

Hint

The line $y=0$ is the x-axis.
The line $y=4$ is parallel to the x-axis, and passes through point (0, 4).

K 8a KUMON

Review of Linear Functions

For each question, calculate the point of intersection of the two given lines, and check your answer by graphing.

Ex.

$$\begin{cases}2x-y=1 & \cdots① \\ x-y=-1 & \cdots②\end{cases}$$

[Sol]

(Calculation)

From ①−②,

$x=2$ ⋯③

Substituting ③ into ②,

$2-y=-1$

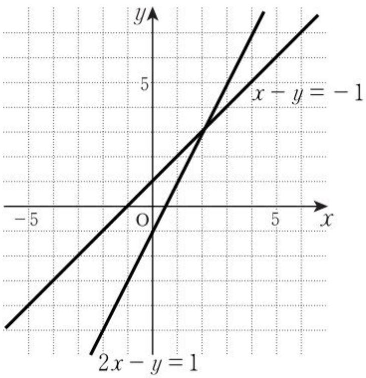
$y=3$

Therefore, the point of intersection is (2, 3).

(Graph)

From ①, $y=2x-1$

From ②, $y=x+1$



(1) $$\begin{cases}2x-y=-2 & \cdots① \\ x+y=5 & \cdots②\end{cases}$$

[Sol]

(Calculation)

From ①+②,

$3x=3$

$x=1$ ⋯③

Substituting ③ into ②,

$1+y=5$

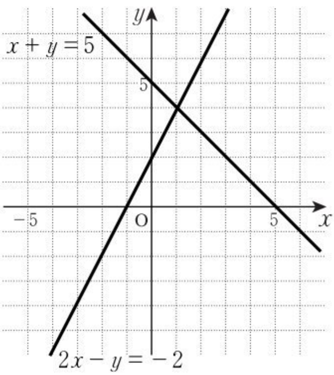
$y=4$

Therefore, the point of intersection is **(1, 4)**.

(Graph)

From ①, $y=2x+2$

From ②, $y=-x+5$



K8b

(2) $\begin{cases} 2x-3y=3 & \cdots① \\ x+y=4 & \cdots② \end{cases}$

[Sol]

(Calculation)

From ①$+3\times$②,

$5x=15$

$x=3 \quad \cdots③$

Substituting ③ into ②,

$3+y=4$

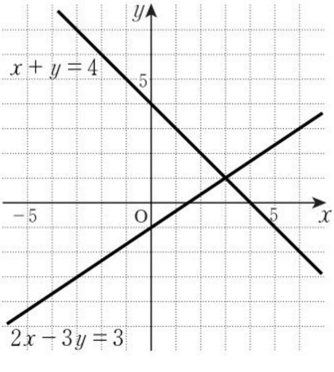
$y=1$

Therefore, the point of intersection is **(3, 1)**.

(Graph)

From ①, $y=\frac{2}{3}x-1$

From ②, $y=-x+4$



(3) $\begin{cases} 3x+y=3 & \cdots① \\ x+2y=-4 & \cdots② \end{cases}$

[Sol]

(Calculation)

From $2\times$①$-$②,

$5x=10$

$x=2 \quad \cdots③$

Substituting ③ into ①,

$6+y=3$

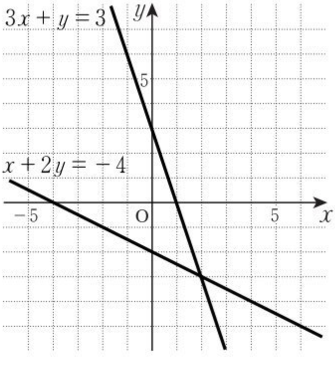
$y=-3$

Therefore, the point of intersection is **(2, −3)**.

(Graph)

From ①, $y=-3x+3$

From ②, $y=-\frac{1}{2}x-2$



Note: The solution of two simultaneous equations is a pair of (x, y) coordinates, which corresponds to the point of intersection of the lines of the two equations.

K9a KUMON

Review of Linear Functions

For each question, calculate the point of intersection of the two given lines, and check your answer by graphing.

Ex.

$$\begin{cases} 2x-y=1 & \cdots ① \\ y=0 & \cdots ② \end{cases}$$

[Sol]

(Calculation)

Substituting ② into ①,

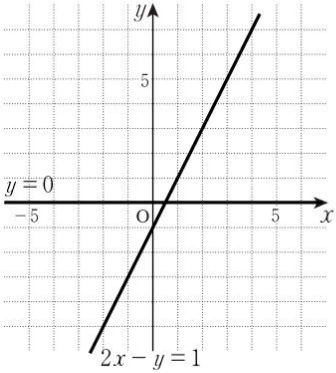
$2x=1$

$x=\dfrac{1}{2}$

Therefore, the point of intersection is $\left(\dfrac{1}{2},\ 0\right)$.

(Graph)

From ①, $y=2x-1$



(1) $$\begin{cases} x-y=-1 & \cdots ① \\ y=0 & \cdots ② \end{cases}$$

[Sol]

(Calculation)

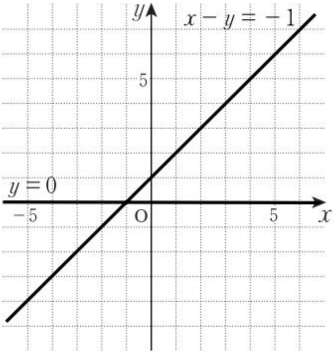
Substituting ② into ①,

$x=-1$

Therefore, the point of intersection is **(−1, 0)**.

(Graph)

From ①, $y=x+1$



K9b

(2) $\begin{cases} 2x+y=-2 & \cdots ① \\ y=2 & \cdots ② \end{cases}$

[Sol]

(Calculation)

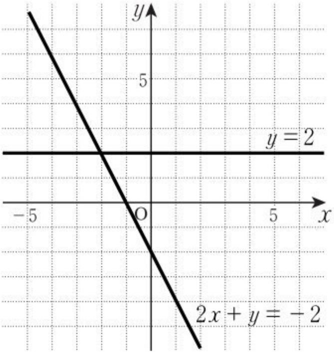
Substituting ② into ①,

$$2x+2=-2$$
$$x=-2$$

Therefore, the point of intersection is **$(-2,\ 2)$**.

(Graph)

From ①, $y=-2x-2$



(3) $\begin{cases} x+2y=2 & \cdots ① \\ x=4 & \cdots ② \end{cases}$ *Hint*

[Sol]

(Calculation)

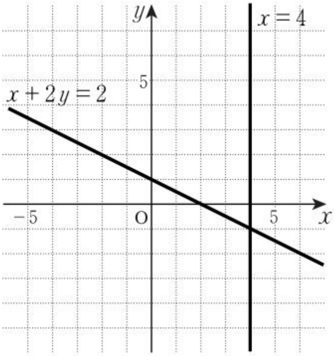
Substituting ② into ①,

$$4+2y=2$$
$$y=-1$$

Therefore, the point of intersection is **$(4,\ -1)$**.

(Graph)

From ①, $y=-\frac{1}{2}x+1$



Hint

- The line $y=0$ is the x-axis.
  The line $y=2$ is parallel to the x-axis, and passes through point $(0,\ 2)$.
- The line $x=0$ is the y-axis.
  The line $x=4$ is parallel to the y-axis, and passes through point $(4,\ 0)$.

K 10a KUMON

Review of Linear Functions

1. Find the equations of the lines described.

(1) A line with gradient -2, and y-intercept 3.

[Sol] $\boldsymbol{y=-2x+3}$

(2) A line passing through point $(2, -1)$ with gradient -2.

[Sol] Let $y=-2x+b$. ☜ Since the gradient is -2.

Substituting $x=2$ and $y=-1$, ☜ Since the line passes through $(2, -1)$.

$-1=-4+b$

$b=3$

Therefore, $\boldsymbol{y=-2x+3}$

(3) A line passing through two points, $(-1, 5)$ and $(3, -3)$.

[Sol] Calculating the gradient of the line,

$$\frac{-3-5}{3-(-1)}=\frac{-8}{4}=-2 \quad ☜ \quad \frac{\text{The change in } y}{\text{The change in } x}$$

Therefore, let $y=-2x+b$.

Substituting $x=-1$ and $y=5$, ☜ Since the line passes through $(-1, 5)$.

$5=-2\times(-1)+b$

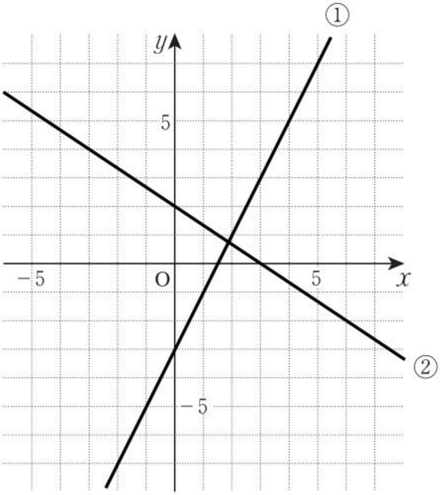
$b=3$

Thus, $\boldsymbol{y=-2x+3}$

Note: All three questions above give the same equation.

K 10b

2. Find the equations of lines ① and ② shown on the graph below.



① The gradient is 2,
the y-intercept is -3.
Therefore, $\boldsymbol{y=2x-3}$

② The gradient is $-\frac{2}{3}$,
the y-intercept is 2.
Therefore, $\boldsymbol{y=-\frac{2}{3}x+2}$

3. Calculate the point of intersection of the two given lines, and check your answer by graphing.

$$\begin{cases} 2x-3y=-6 & \cdots① \\ x+2y=11 & \cdots② \end{cases}$$

[Sol]

(Calculation)

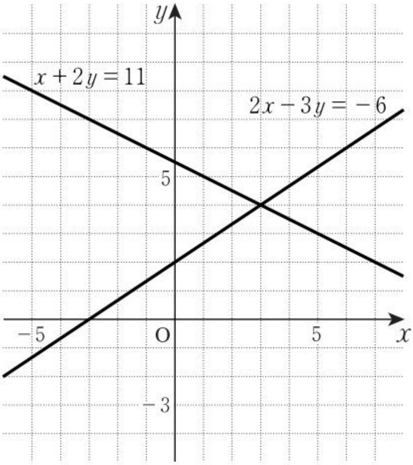
From ①$-2\times$②,

$$-7y=-28$$
$$y=4 \quad \cdots③$$

Substituting ③ into ②,

$$x+8=11$$
$$x=3$$

Therefore, the point of intersection is **(3, 4)**.



(Graph)

From ①, $y=\frac{2}{3}x+2$

From ②, $y=-\frac{1}{2}x+\frac{11}{2}$

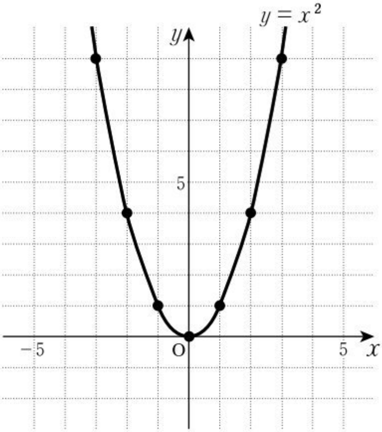
K II a KUMON

Review of Quadratic Functions

Graph each of the following quadratic functions.

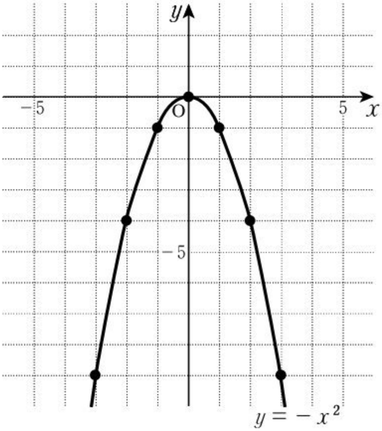
(1) $y = x^2$

x	y
-3	**9**
-2	**4**
-1	**1**
0	**0**
1	**1**
2	**4**
3	**9**



(2) $y = -x^2$

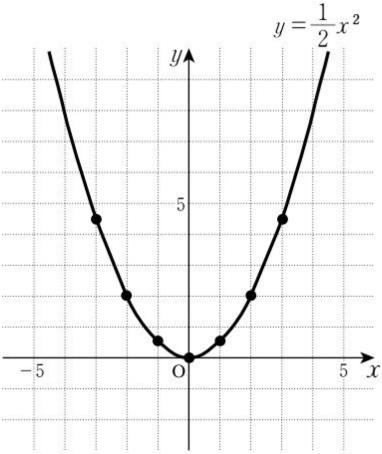
x	y
-3	**−9**
-2	**−4**
-1	**−1**
0	**0**
1	**−1**
2	**−4**
3	**−9**



A function that can be written in the form $y = ax^2 + bx + c$ (where $a \neq 0$) is called a ***quadratic function***.

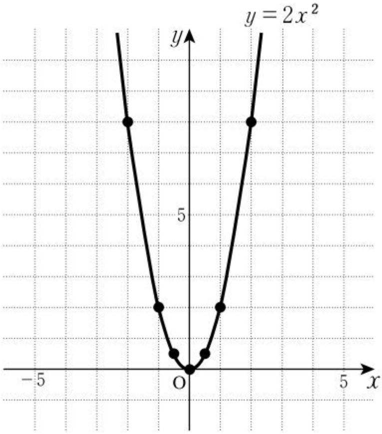
(3) $y=\frac{1}{2}x^2$

x	y
-3	$\frac{9}{2}$
-2	2
-1	$\frac{1}{2}$
0	0
1	$\frac{1}{2}$
2	2
3	$\frac{9}{2}$

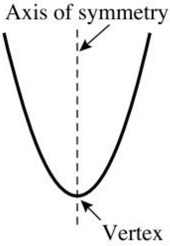


(4) $y=2x^2$

x	y
-2	8
-1	2
$-\frac{1}{2}$	$\frac{1}{2}$
0	0
$\frac{1}{2}$	$\frac{1}{2}$
1	2
2	8



The graph of a quadratic function is called a ***parabola***. Each parabola has a symmetrical axis called the ***axis of the parabola***, or the ***axis of symmetry***. The point of intersection of the axis of symmetry and the parabola is called the ***vertex*** of the parabola.



For example, on the graph of $y=x^2$, the axis of symmetry is the line $x=0$, i.e. the y-axis, and the vertex is $(0, 0)$.

Note: An axis of symmetry divides a figure into two halves in such a way that when the figure is folded along the axis of symmetry, the two halves fit together perfectly.

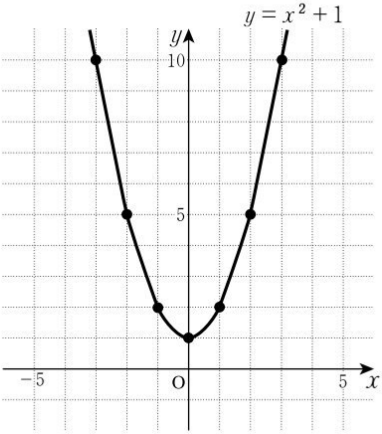
K 12a KUMON

Review of Quadratic Functions

Graph each of the following quadratic functions.

(1) $y = x^2 + 1$

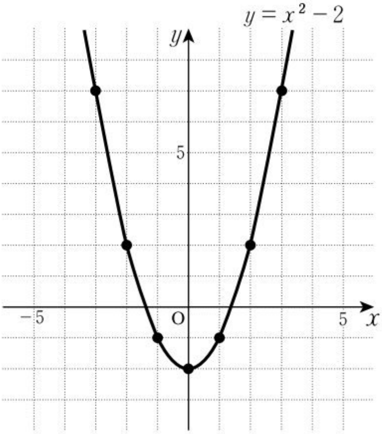
x	y
-3	**10**
-2	**5**
-1	**2**
0	**1**
1	**2**
2	**5**
3	**10**



The axis of symmetry is $x = 0$, and the vertex is $(0, 1)$.

(2) $y = x^2 - 2$

x	y
-3	**7**
-2	**2**
-1	**−1**
0	**−2**
1	**−1**
2	**2**
3	**7**

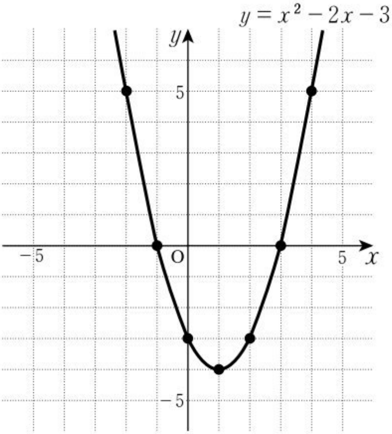


The axis of symmetry is $x = \boxed{\mathbf{0}}$, and the vertex is $(\boxed{\mathbf{0}}, \boxed{\mathbf{-2}})$.

K 12b

(3) $y = x^2 - 2x - 3$

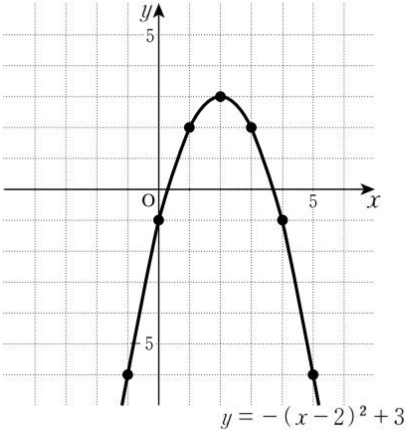
x	y
-2	**5**
-1	**0**
0	**-3**
1	**-4**
2	**-3**
3	**0**
4	**5**



The axis of symmetry is $x = 1$, and the vertex is $(1, \boxed{-4})$.

(4) $y = -(x-2)^2 + 3$

x	y
-1	**-6**
0	**-1**
1	**2**
2	**3**
3	**2**
4	**-1**
5	**-6**



The axis of symmetry is $x = 2$, and the vertex is $(2, \boxed{3})$.

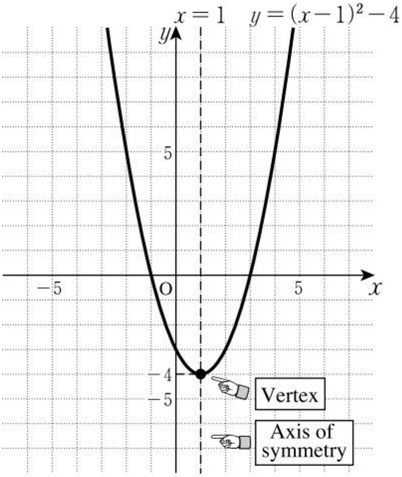
Review of Quadratic Functions

Graph each of the following quadratic functions. For each graph, find the axis of symmetry and the vertex.

Ex.

$y=(x-1)^2-4$

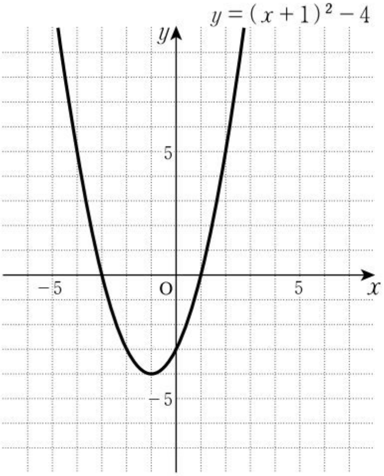
x	y
-2	5
-1	0
0	-3
1	-4
2	-3
3	0
4	5



The axis of symmetry is $x=1$, and the vertex is $(1,\ -4)$.

(1) $y=(x+1)^2-4$

x	y
-4	$\mathbf{5}$
-3	$\mathbf{0}$
-2	$\mathbf{-3}$
-1	$\mathbf{-4}$
0	$\mathbf{-3}$
1	$\mathbf{0}$
2	$\mathbf{5}$

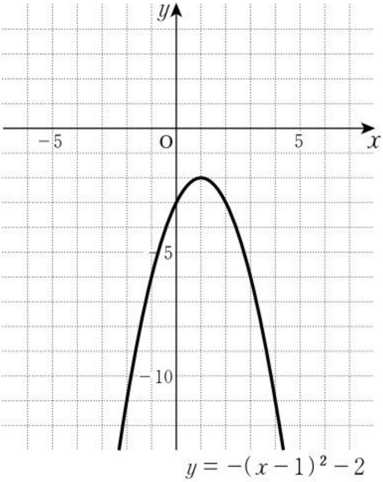


The axis of symmetry is $x=\boxed{\mathbf{-1}}$, and the vertex is ($\mathbf{-1}$, $\mathbf{-4}$).

K13b

(2) $y = -(x-1)^2 - 2$

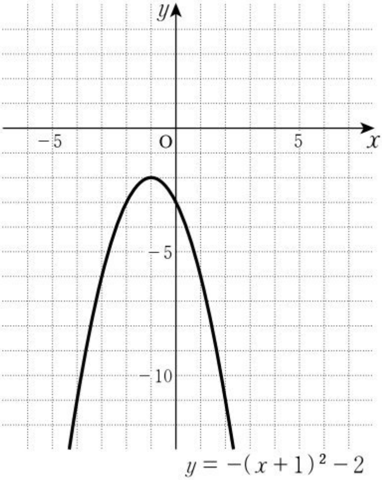
x	y
-2	$\mathbf{-11}$
-1	$\mathbf{-6}$
0	$\mathbf{-3}$
1	$\mathbf{-2}$
2	$\mathbf{-3}$
3	$\mathbf{-6}$
4	$\mathbf{-11}$



The axis of symmetry is $\boxed{\boldsymbol{x = 1}}$, and the vertex is (**1** , **-2**).

(3) $y = -(x+1)^2 - 2$

x	y
-4	$\mathbf{-11}$
-3	$\mathbf{-6}$
-2	$\mathbf{-3}$
-1	$\mathbf{-2}$
0	$\mathbf{-3}$
1	$\mathbf{-6}$
2	$\mathbf{-11}$



The axis of symmetry is $\boxed{\boldsymbol{x = -1}}$, and the vertex is (**$-1$** , **$-2$**).

Note: Given a quadratic function of the form $y = a(x-p)^2 + q$, where a, p and q are constants (with $a \neq 0$), the axis of symmetry is $x = p$ and the vertex is $(p,\ q)$.

K 14a KUMON

Review of Quadratic Functions

Change each of the following quadratic functions into the form $y=a(x-p)^2+q$, and find the axis of symmetry and the vertex. Then draw the graph of the quadratic function.

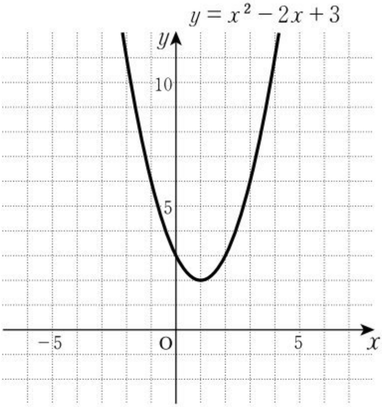
Ex.

$y=x^2-2x+3$

[Sol] $y=(x^2-2x)+3$

$=(x^2-2x+1)-1+3$

$=(x-1)^2+2$



Axis of symmetry: $x=1$

Vertex: $(1, 2)$

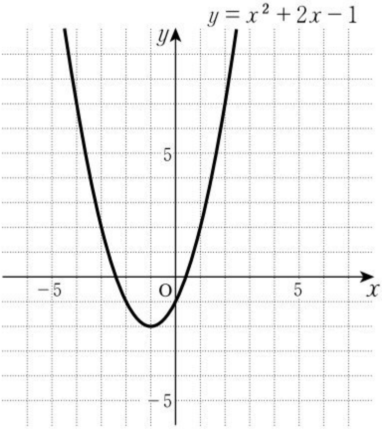
(1) $y=x^2+2x-1$

[Sol] $y=(x^2+2x)-1$

$=(x^2+2x+1)-1-1$

$=(x+1)^2-2$

Note:
To change the form of the quadratic equation, we use the method of "completing the square".



Axis of symmetry: $\boldsymbol{x=-1}$

Vertex: $\boldsymbol{(-1, -2)}$

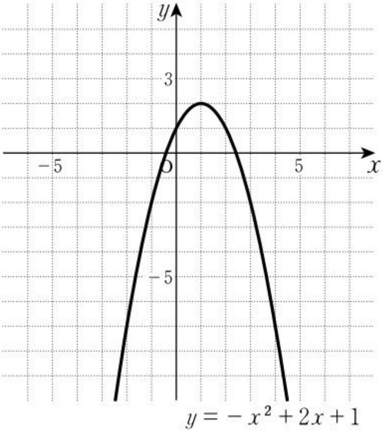
K 14b

(2) $y=-x^2+2x+1$

[Sol] $y=-(x^2-2x)+1$

$=-(x^2-2x+1)+1+1$

$=-(x-1)^2+2$



Axis of symmetry: $\boldsymbol{x=1}$

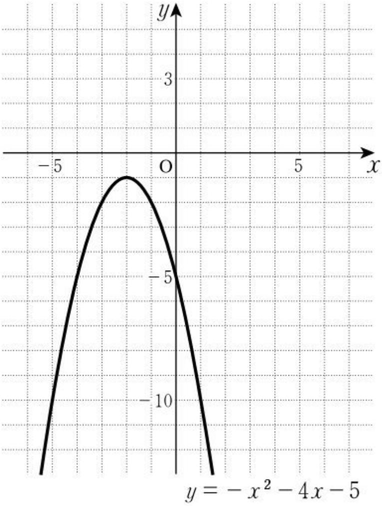
Vertex: $\boldsymbol{(1,\ 2)}$

(3) $y=-x^2-4x-5$

[Sol] $y=-(x^2+4x)-5$

$=-(x^2+4x+4)+4-5$

$=-(x+2)^2-1$



Axis of symmetry: $\boldsymbol{x=-2}$

Vertex: $\boldsymbol{(-2,\ -1)}$

K 15a KUMON

Review of Quadratic Functions

Change each of the following quadratic functions into the form $y=a(x-p)^2+q$, and find the axis of symmetry and the vertex. Then draw the graph of the quadratic function.

Ex.

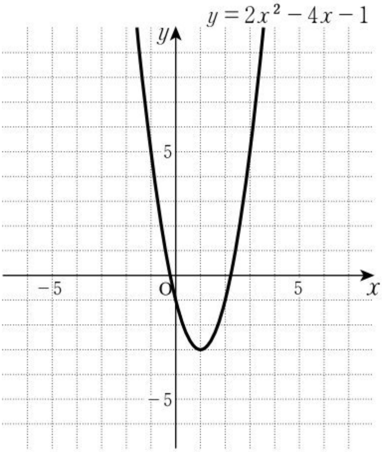
$y=2x^2-4x-1$

[Sol] $y=2(x^2-2x)-1$

$=2[(x^2-2x+1)-1]-1$

$=2(x-1)^2-2-1$

$=2(x-1)^2-3$



Axis of symmetry: $x=1$

Vertex: $(1, -3)$

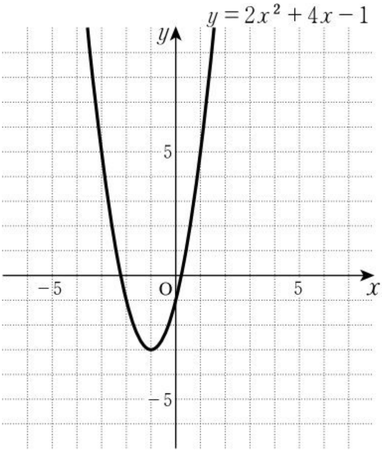
(1) $y=2x^2+4x-1$

[Sol] $y=2(x^2+2x)-1$

$=2[(x^2+2x+1)-1]-1$

$=2(x+1)^2-2-1$

$=2(x+1)^2-3$



Axis of symmetry: $\boldsymbol{x=-1}$

Vertex: $\boldsymbol{(-1, -3)}$

K15b

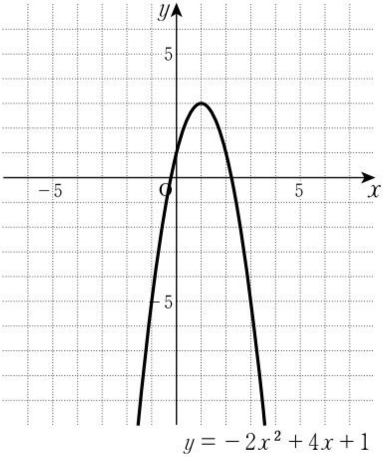
（2）　$y=-2x^2+4x+1$

[Sol] $y=-2(x^2-2x)+1$

$=-2[(x^2-2x+1)-1]+1$

$=-2(x-1)^2+2+1$

$=-2(x-1)^2+3$



$y=-2x^2+4x+1$

Axis of symmetry: $\boldsymbol{x=1}$

Vertex: **(1, 3)**

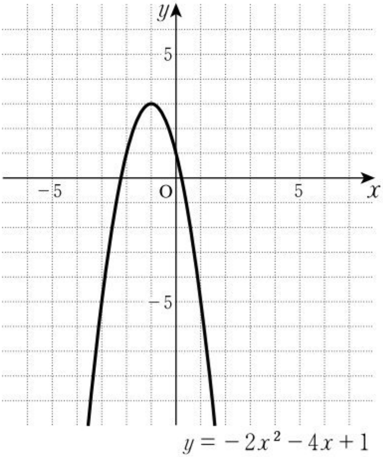
（3）　$y=-2x^2-4x+1$

[Sol] $y=-2(x^2+2x)+1$

$=-2[(x^2+2x+1)-1]+1$

$=-2(x+1)^2+2+1$

$=-2(x+1)^2+3$



$y=-2x^2-4x+1$

Axis of symmetry: $\boldsymbol{x=-1}$

Vertex: **(−1, 3)**

K 16a KUMON

Review of Quadratic Functions

1. Change each of the following quadratic functions into the form $y=a(x-p)^2+q$, and find the axis of symmetry and the vertex. Then choose a graph for the quadratic function from (A)～(D) on the figure below.

(1) $y=x^2$

[Sol] $y=(x-0)^2+0$

Axis of symmetry: $\boldsymbol{x=0}$

Vertex: (**0** , **0**)

Graph: $\boxed{\textbf{(A)}}$

(2) $y=x^2-3$

[Sol] $y=(x-0)^2-3$

Axis of symmetry: $\boldsymbol{x=0}$

Vertex: (**0** , **−3**)

Graph: $\boxed{\textbf{(B)}}$

(3) $y=x^2-4x$

[Sol] $y=(x^2-4x+4)-4$
$=(x-2)^2-4$

Axis of symmetry: $\boldsymbol{x=2}$

Vertex: (**2** , **−4**)

Graph: $\boxed{\textbf{(D)}}$

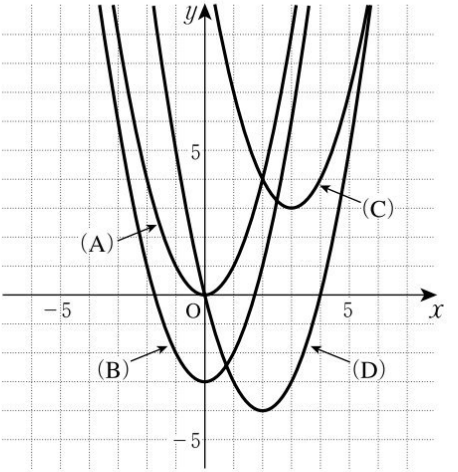
(4) $y=x^2-6x+12$

[Sol] $y=(x^2-6x+9)-9+12$
$=(x-3)^2+3$

Axis of symmetry: $\boldsymbol{x=3}$

Vertex: (**3** , **3**)

Graph: $\boxed{\textbf{(C)}}$



2. Change each of the following quadratic functions into the form $y=a(x-p)^2+q$, and find the axis of symmetry and the vertex. Then choose a graph for the quadratic function from (A)〜(D) on the figure below.

(1) $y=x^2-4x+5$

[Sol] $y=(x^2-4x+4)-4+5$

$=(x-2)^2+1$

Axis of symmetry: $\boldsymbol{x=2}$

Vertex: (**2** , **1**)

Graph: $\boxed{\textbf{(C)}}$

(2) $y=x^2+4x+5$

[Sol] $y=(x^2+4x+4)-4+5$

$=(x+2)^2+1$

Axis of symmetry: $\boldsymbol{x=-2}$

Vertex: (**−2** , **1**)

Graph: $\boxed{\textbf{(A)}}$

(3) $y=-x^2-4x-5$

[Sol] $y=-(x^2+4x)-5$

$=-(x^2+4x+4)+4-5$

$=-(x+2)^2-1$

Axis of symmetry: $\boldsymbol{x=-2}$

Vertex: (**−2** , **−1**)

Graph: $\boxed{\textbf{(B)}}$

(4) $y=-x^2+4x-5$

[Sol] $y=-(x^2-4x)-5$

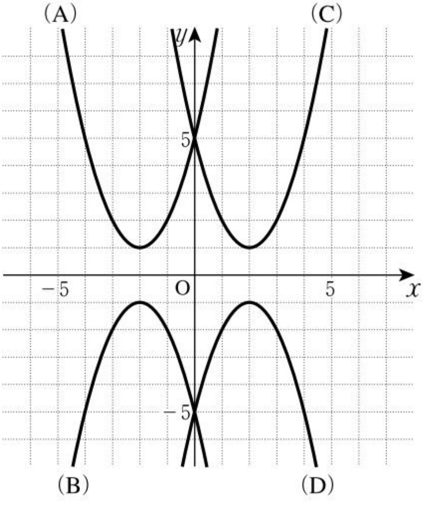
$=-(x^2-4x+4)+4-5$

$=-(x-2)^2-1$

Axis of symmetry: $\boldsymbol{x=2}$

Vertex: (**2** , **−1**)

Graph: $\boxed{\textbf{(D)}}$



K 17a KUMON

Review of Quadratic Functions

For each quadratic function, calculate the coordinates of the points where the parabola intersects the x-axis, and check your answer by drawing the graph.

Ex.

$y = x^2 - 2x - 3$

[Sol]

(Calculation)

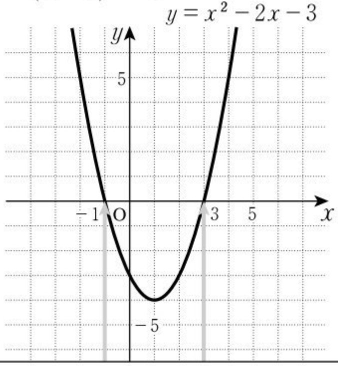
Substituting $y = 0$,

$$0 = x^2 - 2x - 3$$
$$(x-3)(x+1) = 0$$
$$x = 3,\ -1$$

Therefore, the parabola intersects the x-axis at $(3,\ 0)$, $(-1,\ 0)$.

(Graph)

$$y = x^2 - 2x - 3$$
$$= (x^2 - 2x + 1) - 1 - 3$$
$$= (x-1)^2 - 4$$



These two points are the x-intercepts, i.e. the points where the graph intersects the x-axis. These two points correspond to the solutions of the equation $x^2 - 2x - 3 = 0$.

(1) $y = x^2 + 2x - 3$

[Sol]

(Calculation)

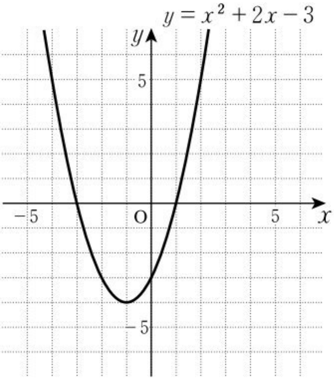
Substituting $y = 0$,

$$0 = x^2 + 2x - 3$$
$$(x+3)(x-1) = 0$$
$$x = -3,\ 1$$

Therefore, the parabola intersects the x-axis at $\mathbf{(-3,\ 0)}$, $\mathbf{(1,\ 0)}$.

(Graph)

$$y = x^2 + 2x - 3$$
$$= (x^2 + 2x + 1) - 1 - 3$$
$$= (x+1)^2 - 4$$



K 17b

(2) $y=x^2-4x+3$

[Sol]

(Calculation)

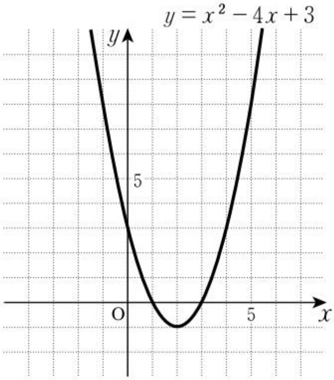
Substituting $y=0$,

$$0=x^2-4x+3$$
$$(x-3)(x-1)=0$$
$$x=3,\ 1$$

Therefore, the parabola intersects the x-axis at **(3, 0)**, **(1, 0)**.

(Graph)

$$\begin{aligned}y&=x^2-4x+3\\&=(x^2-4x+4)-4+3\\&=(x-2)^2-1\end{aligned}$$



(3) $y=-x^2-4x$

[Sol]

(Calculation)

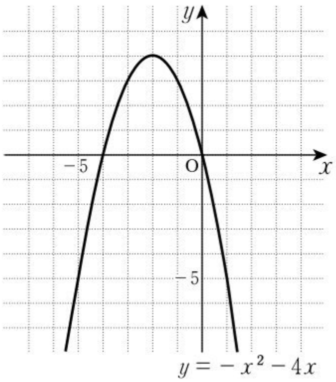
Substituting $y=0$,

$$0=-x^2-4x$$
$$-x(x+4)=0$$
$$x=0,\ -4$$

Therefore, the parabola intersects the x-axis at **(0, 0)**, **(−4, 0)**.

(Graph)

$$\begin{aligned}y&=-x^2-4x\\&=-(x^2+4x+4)+4\\&=-(x+2)^2+4\end{aligned}$$



Note: Given a quadratic function of the form $y=ax^2+bx+c$, the points where the graph intersects the x-axis correspond to the solutions of the quadratic equation $ax^2+bx+c=0$.

K 18a KUMON

Review of Quadratic Functions

For each quadratic function, calculate the coordinates of the points where the parabola intersects the x-axis, and check your answer by drawing the graph.

(1) $y=2x^2+4x-6$

[Sol]

(Calculation)

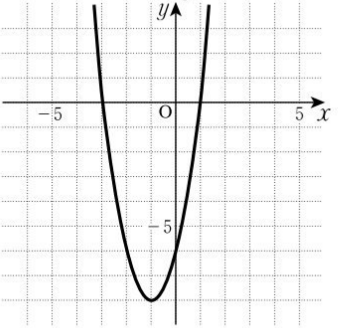
Substituting $y=0$,

$$0=2x^2+4x-6$$
$$x^2+2x-3=0$$
$$(x+3)(x-1)=0$$
$$x=-3,\ 1$$

Therefore, the parabola intersects the x-axis at $\mathbf{(-3,\ 0),\ (1,\ 0)}$.

(Graph)

$$\begin{aligned} y&=2x^2+4x-6\\ &=2(x^2+2x)-6\\ &=2(x^2+2x+1)-2-6\\ &=2(x+1)^2-8 \end{aligned}$$



(2) $y=2x^2+8x+6$

[Sol]

(Calculation)

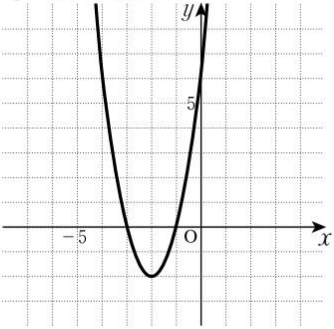
Substituting $y=0$,

$$0=2x^2+8x+6$$
$$x^2+4x+3=0$$
$$(x+3)(x+1)=0$$
$$x=-3,\ -1$$

Therefore, the parabola intersects the x-axis at $\mathbf{(-3,\ 0),\ (-1,\ 0)}$.

(Graph)

$$\begin{aligned} y&=2x^2+8x+6\\ &=2(x^2+4x)+6\\ &=2(x^2+4x+4)-8+6\\ &=2(x+2)^2-2 \end{aligned}$$



K18b

（3）　$y=2x^2-4x+2$

[Sol]

(Calculation)

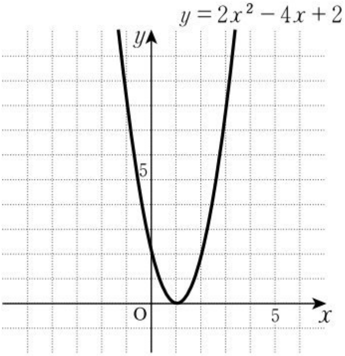
Substituting $y=0$,

$$0=2x^2-4x+2$$
$$x^2-2x+1=0$$
$$(x-1)^2=0$$
$$x=1$$

Therefore, the parabola intersects the x-axis at **(1, 0)**.

(Graph)

$$\begin{aligned} y&=2x^2-4x+2\\ &=2(x^2-2x)+2\\ &=2(x^2-2x+1)-2+2\\ &=2(x-1)^2 \end{aligned}$$



（4）　$y=-2x^2+4x$

[Sol]

(Calculation)

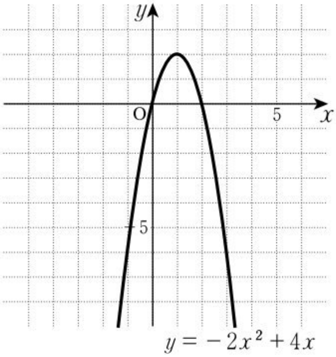
Substituting $y=0$,

$$0=-2x^2+4x$$
$$x^2-2x=0$$
$$x=0,\ 2$$

Therefore, the parabola intersects the x-axis at **(0, 0)**, **(2, 0)**.

(Graph)

$$\begin{aligned} y&=-2x^2+4x\\ &=-2(x^2-2x)\\ &=-2(x^2-2x+1)+2\\ &=-2(x-1)^2+2 \end{aligned}$$



Review of Quadratic Functions

Calculate the points of intersection of the given functions, and check your answer by drawing the graph.

Ex.

$$\begin{cases} y = x^2 - 4x + 1 & \cdots ① \\ y = 2x - 4 & \cdots ② \end{cases}$$

[Sol]

(Calculation)

Substituting ② into ①,

$$2x - 4 = x^2 - 4x + 1$$

$$x^2 - 6x + 5 = 0$$

$$(x - 5)(x - 1) = 0$$

Therefore, $x = 5,\ 1$

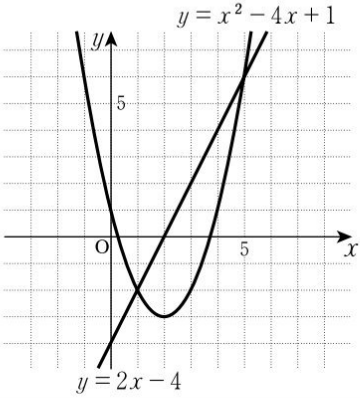
Substituting $x = 5$ into ②, $y = 6$.

Substituting $x = 1$ into ②, $y = -2$.

Therefore, the points of intersection are $(5,\ 6),\ (1,\ -2)$.

(Graph)

From ①, $y = (x - 2)^2 - 3$



(1)

$$\begin{cases} y = x^2 & \cdots ① \\ y = x + 2 & \cdots ② \end{cases}$$

[Sol]

(Calculation)

Substituting ② into ①,

$$x + 2 = x^2$$

$$x^2 - x - 2 = 0$$

$$(x - 2)(x + 1) = 0$$

Therefore, $x = 2,\ -1$

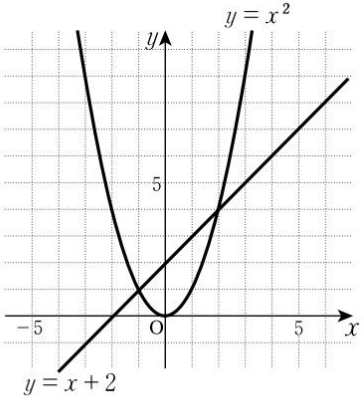
Substituting $x = 2$ into ②, $y = 4$.

Substituting $x = -1$ into ②, $y = 1$.

Therefore, the points of intersection are $(2,\ 4),\ (-1,\ 1)$.

(Graph)

From ①, $y = (x - 0)^2 + 0$



K 19b

(2) $$\begin{cases} y = x^2 + 2x - 3 & \cdots① \\ y = -x + 1 & \cdots② \end{cases}$$

[Sol]

(Calculation)

Substituting ② into ①,

$$-x + 1 = x^2 + 2x - 3$$
$$x^2 + 3x - 4 = 0$$
$$(x + 4)(x - 1) = 0$$

Therefore, $x = -4,\ 1$

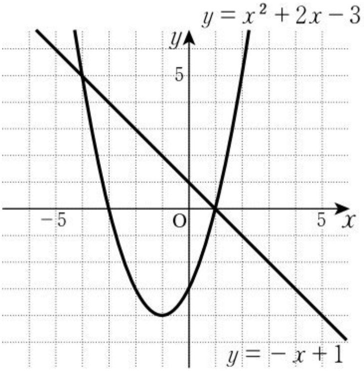
Substituting $x = -4$ into ②, $y = 5$.

Substituting $x = 1$ into ②, $y = 0$.

Therefore, the points of intersection are $\mathbf{(-4,\ 5),\ (1,\ 0)}$.

(Graph)

From ①, $y = (x^2 + 2x + 1) - 1 - 3$
$= (x + 1)^2 - 4$



(3) $$\begin{cases} y = -x^2 + 2x + 5 & \cdots① \\ y = 2x + 1 & \cdots② \end{cases}$$

[Sol]

(Calculation)

Substituting ② into ①,

$$2x + 1 = -x^2 + 2x + 5$$
$$x^2 - 4 = 0$$
$$(x + 2)(x - 2) = 0$$

Therefore, $x = -2,\ 2$

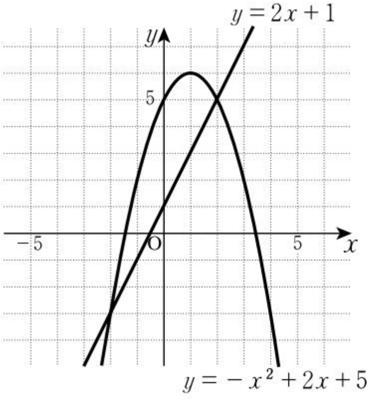
Substituting $x = -2$ into ②, $y = -3$.

Substituting $x = 2$ into ②, $y = 5$.

Therefore, the points of intersection are $\mathbf{(-2,\ -3),\ (2,\ 5)}$.

(Graph)

From ①, $y = -(x^2 - 2x) + 5$
$= -(x^2 - 2x + 1) + 1 + 5$
$= -(x - 1)^2 + 6$



Note: The solution(s) of x and y obtained from solving simultaneous equations correspond to the points of intersection of the graphs of the equations.

K 20a

KUMON

Review of Quadratic Functions

Calculate the points of intersection of the given functions, and check your answer by drawing the graph.

(1) $\begin{cases} y = -2x^2 + 8x - 2 & \cdots ① \\ y = -2x + 6 & \cdots ② \end{cases}$

[Sol]

(Calculation)

Substituting ② into ①,

$$-2x + 6 = -2x^2 + 8x - 2$$
$$2x^2 - 10x + 8 = 0$$
$$x^2 - 5x + 4 = 0$$
$$(x-4)(x-1) = 0$$

Therefore, $x = 4,\ 1$

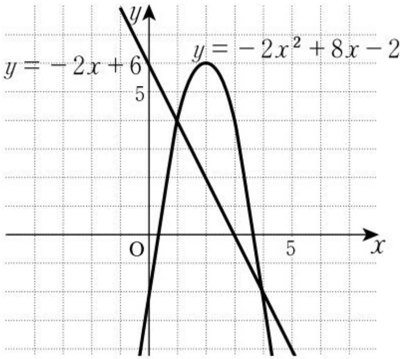
Substituting $x = 4$ into ②, $y = -2$.

Substituting $x = 1$ into ②, $y = 4$.

Therefore, the points of intersection are **(4, −2)**, **(1, 4)**.

(Graph)

From ①, $y = -2(x^2 - 4x) - 2$
$$= -2(x^2 - 4x + 4) + 8 - 2$$
$$= -2(x-2)^2 + 6$$



(2) $\begin{cases} y = 2x^2 + 4x - 1 & \cdots ① \\ y = 5 & \cdots ② \end{cases}$

[Sol]

(Calculation)

Substituting ② into ①,

$$5 = 2x^2 + 4x - 1$$
$$2x^2 + 4x - 6 = 0$$
$$x^2 + 2x - 3 = 0$$
$$(x+3)(x-1) = 0$$

Therefore, $x = -3,\ 1$

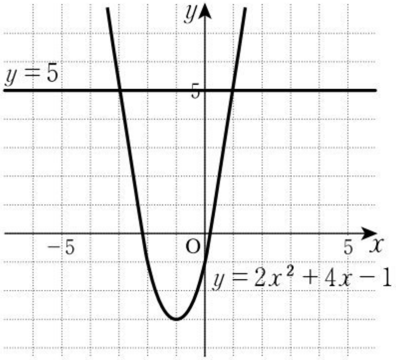
When $x = -3$, from ②, $y = 5$.

When $x = 1$, from ②, $y = 5$.

Therefore, the points of intersection are **(−3, 5)**, **(1, 5)**.

(Graph)

From ①, $y = 2(x^2 + 2x) - 1$
$$= 2(x^2 + 2x + 1) - 2 - 1$$
$$= 2(x+1)^2 - 3$$



K 20b

(3) $\begin{cases} y = x^2 - 2x & \cdots① \\ y = 2x - 4 & \cdots② \end{cases}$

[Sol]

(Calculation)

Substituting ② into ①,

$$2x - 4 = x^2 - 2x$$
$$x^2 - 4x + 4 = 0$$
$$(x-2)^2 = 0$$

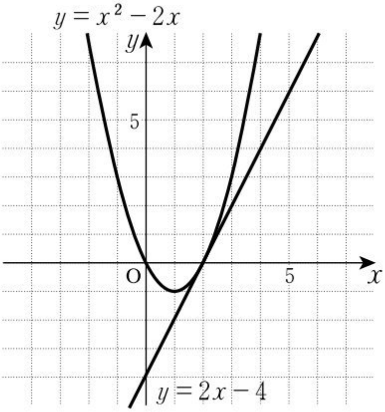
Therefore, $x = 2$

Substituting $x = 2$ into ②, $y = 0$.

Therefore, the point of intersection is **$(2,\ 0)$**.

(Graph)

From ①, $y = (x^2 - 2x + 1) - 1$
$= (x-1)^2 - 1$



(4) $\begin{cases} y = x^2 - 4x + 1 & \cdots① \\ y = -x^2 + 6x - 7 & \cdots② \end{cases}$

[Sol]

(Calculation)

Substituting ② into ①,

$$-x^2 + 6x - 7 = x^2 - 4x + 1$$
$$2x^2 - 10x + 8 = 0$$
$$x^2 - 5x + 4 = 0$$
$$(x-4)(x-1) = 0$$

Therefore, $x = 4,\ 1$

Substituting $x = 4$ into ①, $y = 1$.

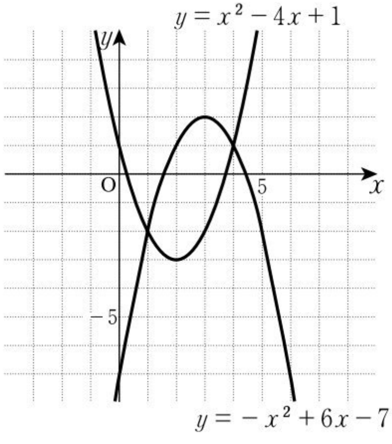
Substituting $x = 1$ into ①, $y = -2$.

Therefore, the points of intersection are **$(4,\ 1)$, $(1,\ -2)$**.

(Graph)

From ①, $y = (x^2 - 4x + 4) - 4 + 1$
$= (x-2)^2 - 3$

From ②, $y = -(x^2 - 6x) - 7$
$= -(x^2 - 6x + 9) + 9 - 7$
$= -(x-3)^2 + 2$



K 21 a KUMON

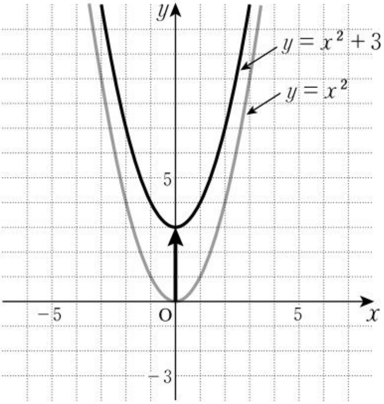
Quadratic Functions and Graphs

1. For each parabola, find the axis of symmetry and the vertex.

(1) $y = x^2 + 3$

Axis of symmetry: $x = \boxed{\mathbf{0}}$

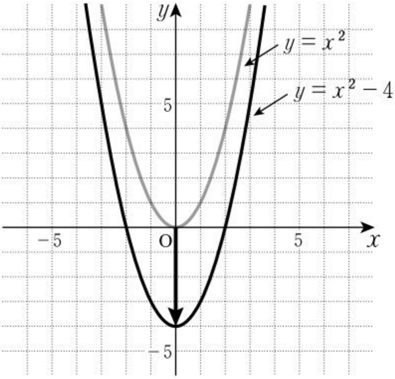
Vertex: (**0** , **3**)



(2) $y = x^2 - 4$

Axis of symmetry: $\boldsymbol{x = 0}$

Vertex: (**0** , **−4**)



Looking at the graphs, we see that:

- The graph of $y = x^2 + 3$ is a translation of $y = x^2$, 3 units along the y-axis.
- The graph of $y = x^2 - 4$ is a translation of $y = x^2$, -4 units along the y-axis.

The transformations shown on the graphs above are called ***translations***. Each point on the parabola is moved a fixed distance in the same direction.

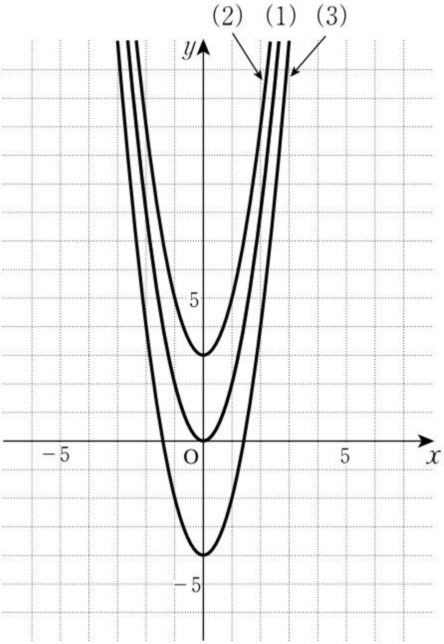
K21b

2. For each parabola, find the axis of symmetry and the vertex. Then draw all the graphs on the same coordinate plane below.

(1) $y = 2x^2$

Axis of symmetry: $\boldsymbol{x = 0}$

Vertex: (**0** , **0**)



(2) $y = 2x^2 + 3$

Axis of symmetry: $\boldsymbol{x = 0}$

Vertex: (**0** , **3**)

(3) $y = 2x^2 - 4$

Axis of symmetry: $\boldsymbol{x = 0}$

Vertex: (**0** , **−4**)

3. Complete the following, using the above graphs.

(1) The graph of $y = 2x^2 + 3$ is a translation of $y = 2x^2$,

$\boxed{\mathbf{3}}$ unit(s) along the y-axis.

(2) The graph of $y = 2x^2 - 4$ is a translation of $y = 2x^2$,

$\boxed{\mathbf{-4}}$ unit(s) along the y-axis.

K 22a KUMON

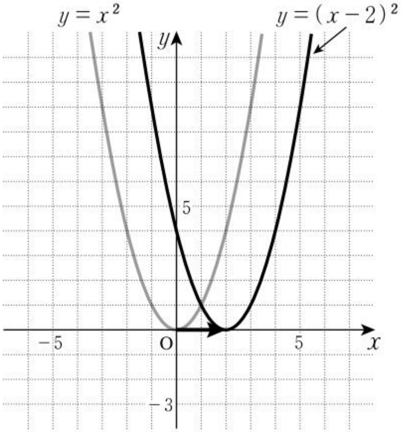
Quadratic Functions and Graphs

1. For each parabola, find the axis of symmetry and the vertex.

(1) $y=(x-2)^2$

Axis of symmetry: $\boldsymbol{x=2}$

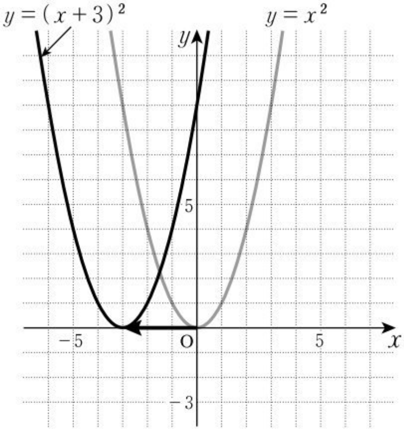
Vertex: **(2, 0)**



(2) $y=(x+3)^2$

Axis of symmetry: $\boldsymbol{x=-3}$

Vertex: **(−3, 0)**



Looking at the graphs, we see that:

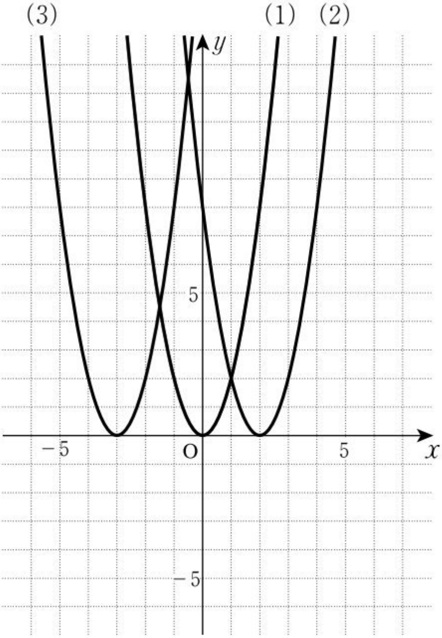
- The graph of $y=(x-2)^2$ is a translation of $y=x^2$, 2 units along the x-axis.
- The graph of $y=(x+3)^2$ is a translation of $y=x^2$, -3 units along the x-axis.

2. For each parabola, find the axis of symmetry and the vertex. Then draw all the graphs on the same coordinate plane below.

(1) $y=2x^2$

Axis of symmetry: $\boldsymbol{x=0}$

Vertex: **(0, 0)**



(2) $y=2(x-2)^2$

Axis of symmetry: $\boldsymbol{x=2}$

Vertex: **(2, 0)**

(3) $y=2(x+3)^2$

Axis of symmetry: $\boldsymbol{x=-3}$

Vertex: **(−3, 0)**

3. Complete the following, using the above graphs.

(1) The graph of $y=2(x-2)^2$ is a translation of $y=2x^2$,

$\boxed{\mathbf{2}}$ unit(s) along the x-axis.

(2) The graph of $y=2(x+3)^2$ is a translation of $y=2x^2$,

$\boxed{\mathbf{-3}}$ unit(s) along the x-axis.

K 23a KUMON

Quadratic Functions and Graphs

1. For each parabola, find the axis of symmetry and the vertex.

(1) $y=x^2+3$

Axis of symmetry: $\boldsymbol{x=0}$

Vertex: **(0, 3)**

(2) $y=(x-2)^2$

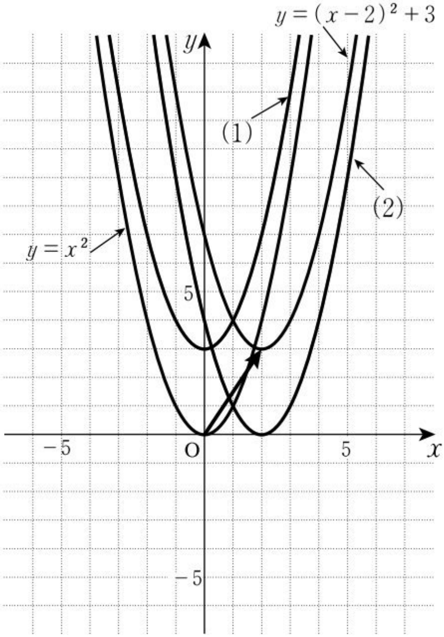
Axis of symmetry: $\boldsymbol{x=2}$

Vertex: **(2, 0)**

(3) $y=(x-2)^2+3$

Axis of symmetry: $\boldsymbol{x=2}$

Vertex: **(2, 3)**



2. Complete the following, using the above graphs.

(1) The graph of $y=x^2+3$ is a translation of $y=x^2$,

$\boxed{\mathbf{3}}$ unit(s) along the y-axis.

(2) The graph of $y=(x-2)^2$ is a translation of $y=x^2$,

$\boxed{\mathbf{2}}$ unit(s) along the x-axis.

The graph of $y=(x-2)^2+3$ is a translation of $y=x^2$,
2 units along the x-axis **and** 3 units along the y-axis.

K 23b

3. For each parabola, find the axis of symmetry and the vertex. Then draw all the graphs on the same coordinate plane below.

（1） $y=2x^2-5$

Axis of symmetry: $\boldsymbol{x=0}$

Vertex: **(0, −5)**

（2） $y=2(x+4)^2$

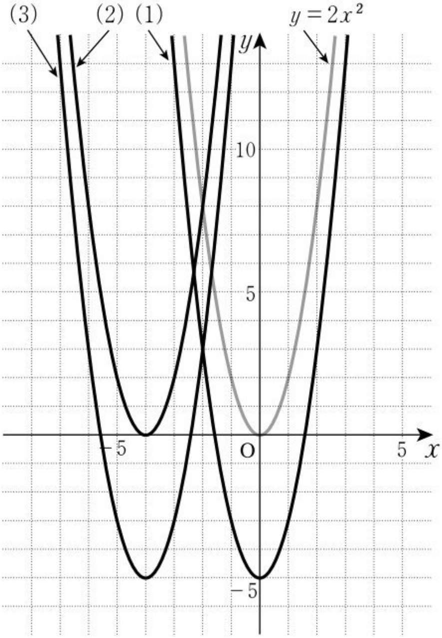
Axis of symmetry: $\boldsymbol{x=-4}$

Vertex: **(−4, 0)**

（3） $y=2(x+4)^2-5$

Axis of symmetry: $\boldsymbol{x=-4}$

Vertex: **(−4, −5)**



4. Complete the following, using the above graphs.

（1） The graph of $y=2x^2-5$ is a translation of $y=2x^2$,

$\boxed{\mathbf{-5}}$ unit(s) along the y-axis.

（2） The graph of $y=2(x+4)^2$ is a translation of $y=2x^2$,

$\boxed{\mathbf{-4}}$ unit(s) along the x-axis.

（3） The graph of $y=2(x+4)^2-5$ is a translation of $y=2x^2$,

$\boxed{\mathbf{-4}}$ unit(s) along the x-axis and $\boxed{\mathbf{-5}}$ unit(s) along the y-axis.

K 24a KUMON

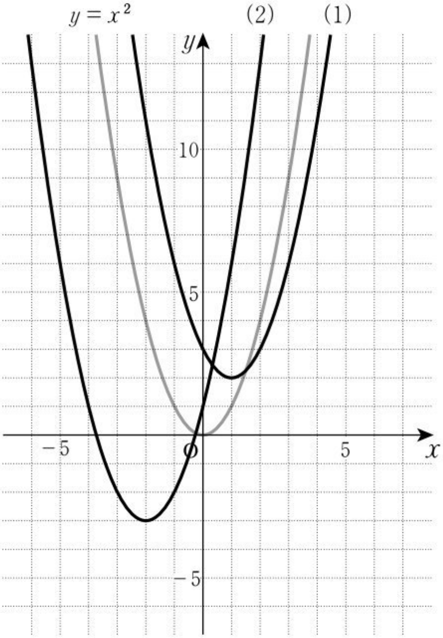
Quadratic Functions and Graphs

1. For each parabola, find the axis of symmetry and the vertex. Then draw all the graphs on the same coordinate plane below.

(1) $y=(x-1)^2+2$

Axis of symmetry: $\boldsymbol{x=1}$

Vertex: **(1, 2)**



(2) $y=(x+2)^2-3$

Axis of symmetry: $\boldsymbol{x=-2}$

Vertex: **(−2, −3)**

2. Complete the following, using the above graphs.

(1) The graph of $y=(x-1)^2+2$ is a translation of $y=x^2$,

$\boxed{\mathbf{1}}$ unit(s) along the x-axis and $\boxed{\mathbf{2}}$ unit(s) along the y-axis.

(2) The graph of $y=(x+2)^2-3$ is a translation of $y=x^2$,

$\boxed{\mathbf{-2}}$ unit(s) along the x-axis and $\boxed{\mathbf{-3}}$ unit(s) along the y-axis.

3. For each parabola, find the axis of symmetry and the vertex. Then draw all the graphs on the same coordinate plane below.

(1) $y=2(x-1)^2+2$

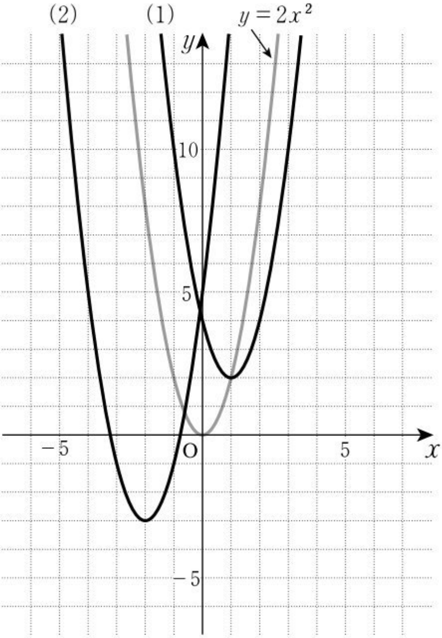
Axis of symmetry: $\boldsymbol{x=1}$

Vertex: **(1, 2)**

(2) $y=2(x+2)^2-3$

Axis of symmetry: $\boldsymbol{x=-2}$

Vertex: **(−2, −3)**



4. Complete the following, using the above graphs.

(1) The graph of $y=2(x-1)^2+2$ is a translation of $y=2x^2$,
$\boxed{\mathbf{1}}$ unit(s) along the x-axis and $\boxed{\mathbf{2}}$ unit(s) along the y-axis.

(2) The graph of $y=2(x+2)^2-3$ is a translation of $y=2x^2$,
$\boxed{\mathbf{-2}}$ unit(s) along the x-axis and $\boxed{\mathbf{-3}}$ unit(s) along the y-axis.

K 25a KUMON

Quadratic Functions and Graphs

1. For each parabola, find the axis of symmetry and the vertex. Then draw all the graphs on the same coordinate plane below.

(1) $y=-(x-1)^2+2$

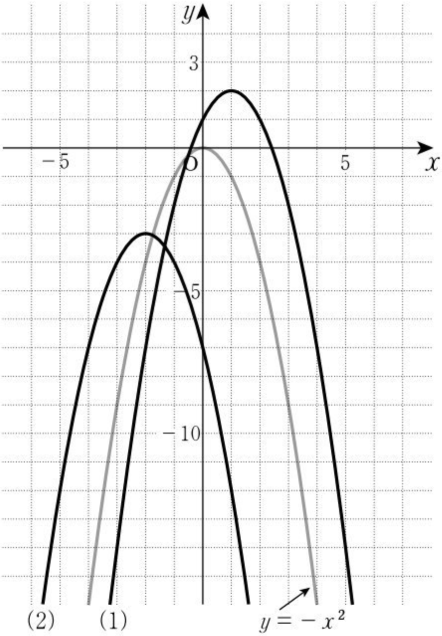
Axis of symmetry: $\boldsymbol{x=1}$

Vertex: **(1, 2)**

(2) $y=-(x+2)^2-3$

Axis of symmetry: $\boldsymbol{x=-2}$

Vertex: **(−2, −3)**



2. Complete the following, using the above graphs.

(1) The graph of $y=-(x-1)^2+2$ is a translation of $y=-x^2$, $\boxed{\mathbf{1}}$ unit(s) along the x-axis and $\boxed{\mathbf{2}}$ unit(s) along the y-axis.

(2) The graph of $y=-(x+2)^2-3$ is a translation of $y=-x^2$, $\boxed{\mathbf{-2}}$ unit(s) along the x-axis and $\boxed{\mathbf{-3}}$ unit(s) along the y-axis.

K 25b

3. For each parabola, find the axis of symmetry and the vertex. Then draw all the graphs on the same coordinate plane below.

(1) $y = -2(x-1)^2 + 2$

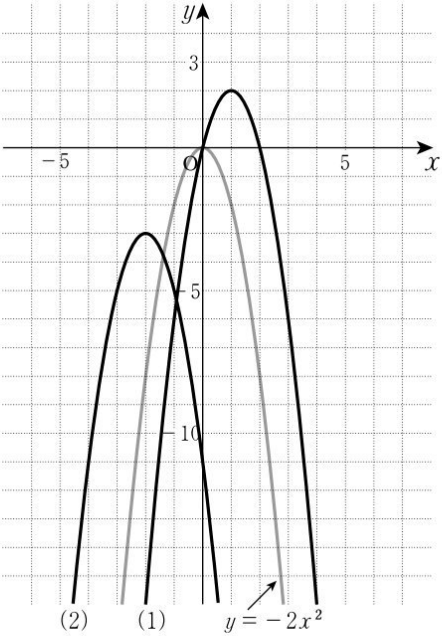
Axis of symmetry: $\boldsymbol{x = 1}$

Vertex: **(1, 2)**

(2) $y = -2(x+2)^2 - 3$

Axis of symmetry: $\boldsymbol{x = -2}$

Vertex: **(−2, −3)**



4. Complete the following, using the above graphs.

(1) The graph of $y = -2(x-1)^2 + 2$ is a translation of $y = -2x^2$,
$\boxed{\mathbf{1}}$ unit(s) along the x-axis and $\boxed{\mathbf{2}}$ unit(s) along the y-axis.

(2) The graph of $y = -2(x+2)^2 - 3$ is a translation of $y = -2x^2$,
$\boxed{\mathbf{-2}}$ unit(s) along the x-axis and $\boxed{\mathbf{-3}}$ unit(s) along the y-axis.

K 26a **KUMON**

Quadratic Functions and Graphs

In each question, state how the graph of each quadratic function has been translated from $y=2x^2$.

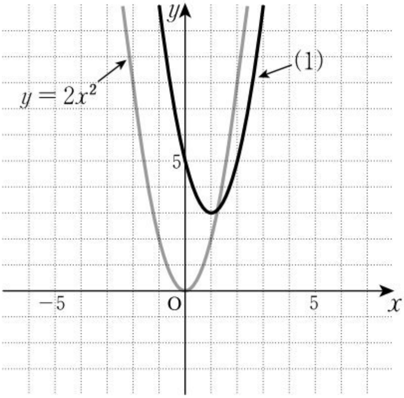
(1) $y=2x^2-4x+5$

[Sol] $y=2(x^2-2x)+5$

$=2[(x^2-2x+1)-1]+5$

$=2(x-1)^2-2+5$

$=2(x-\boxed{1})^2+\boxed{3}$



Therefore, the graph has been translated:

$\boxed{1}$ unit(s) along the x-axis and $\boxed{3}$ unit(s) along the y-axis.

(2) $y=2x^2+4x+5$

[Sol] $y=2(x^2+2x)+5$

$=2[(x^2+2x+1)-1]+5$

$=2(x+1)^2-2+5$

$=2(x+1)^2+3$

Therefore, the graph has been translated:

−1 unit along the x-axis and 3 units along the y-axis.

K 26b

(3) $y=2x^2-8x+9$

[Sol] $y=2(x^2-4x)+9$

$=2[(x^2-4x+4)-4]+9$

$=2(x-2)^2-8+9$

$=2(x-2)^2+1$

Therefore, the graph has been translated:
2 units along the x-axis and 1 unit along the y-axis.

(4) $y=2x^2-8x+7$

[Sol] $y=2(x^2-4x)+7$

$=2[(x^2-4x+4)-4]+7$

$=2(x-2)^2-8+7$

$=2(x-2)^2-1$

Therefore, the graph has been translated:
2 units along the x-axis and -1 unit along the y-axis.

(5) $y=2x^2+8x+7$

[Sol] $y=2(x^2+4x)+7$

$=2[(x^2+4x+4)-4]+7$

$=2(x+2)^2-8+7$

$=2(x+2)^2-1$

Therefore, the graph has been translated:
-2 units along the x-axis and -1 unit along the y-axis.

K 27a KUMON

Quadratic Functions and Graphs

1. For each function, find the vertex. Then state how the graph of the given quadratic function has been translated from $y=3x^2$.

(1) $y=3x^2+2$

The vertex: (**0** , **2**)

The graph has been translated $\boxed{\mathbf{2}}$ unit(s) along the y-axis.

(2) $y=3(x-1)^2$

The vertex: (**1** , **0**)

The graph has been translated $\boxed{\mathbf{1}}$ unit(s) along the x-axis.

(3) $y=3(x-1)^2+2$

The vertex: (**1** , **2**)

The graph has been translated $\boxed{\mathbf{1}}$ unit(s) along the x-axis and

$\boxed{\mathbf{2}}$ unit(s) along the y-axis.

(4) $y=3(x+1)^2-2$

The vertex: (**−1** , **−2**)

The graph has been translated $\boxed{\mathbf{-1}}$ unit(s) along the x-axis and

$\boxed{\mathbf{-2}}$ unit(s) along the y-axis.

K 27b

2. Find the equations of the functions whose graphs have been translated from $y=3x^2$ in the following ways.

（1） Translated 2 units along the y-axis.

[Sol] $y=3x^2+\boxed{\mathbf{2}}$

（2） Translated -2 units along the y-axis.

[Sol] $\boldsymbol{y=3x^2-2}$

（3） Translated 1 unit along the x-axis.

[Sol] $y=3(x-\boxed{\mathbf{1}})^2$

（4） Translated -1 unit along the x-axis.

[Sol] $\boldsymbol{y=3(x+1)^2}$

（5） Translated 1 unit along the x-axis and 2 units along the y-axis.

[Sol] $\boldsymbol{y=3(x-1)^2+2}$

（6） Translated -1 unit along the x-axis and -2 units along the y-axis.

[Sol] $\boldsymbol{y=3(x+1)^2-2}$

Note: The graph of $y=a(x-p)^2+q$ is a translation of $y=ax^2$, p units along the x-axis and q units along the y-axis. The vertex is (p, q).

K 28a KUMON

Quadratic Functions and Graphs

1. Find the equations of the functions whose graphs have been translated from $y=\frac{1}{2}x^2$ in the following ways.

(1) Translated 6 units along the x-axis and 4 units along the y-axis.

[Sol] $y=\frac{1}{2}(x-\boxed{\mathbf{6}})^2+\boxed{\mathbf{4}}$

(2) Translated 5 units along the x-axis.

[Sol] $\boldsymbol{y=\frac{1}{2}(x-5)^2}$

(3) Translated 2 units along the y-axis.

[Sol] $\boldsymbol{y=\frac{1}{2}x^2+2}$

(4) Translated 2 units along the x-axis and 3 units along the y-axis.

[Sol] $\boldsymbol{y=\frac{1}{2}(x-2)^2+3}$

(5) Translated -3 units along the x-axis and 3 units along the y-axis.

[Sol] $\boldsymbol{y=\frac{1}{2}(x+3)^2+3}$

(6) Translated -3 units along the x-axis and -5 units along the y-axis.

[Sol] $\boldsymbol{y=\frac{1}{2}(x+3)^2-5}$

K 28b

2. Find the equations of the functions whose graphs have been translated from $y = -3x^2$ in the following ways.

（1） Translated 6 units along the x-axis and 4 units along the y-axis.

[Sol] $\boldsymbol{y = -3(x-6)^2+4}$

（2） Translated 5 units along the x-axis.

[Sol] $\boldsymbol{y = -3(x-5)^2}$

（3） Translated 2 units along the y-axis.

[Sol] $\boldsymbol{y = -3x^2+2}$

（4） Translated 2 units along the x-axis and 3 units along the y-axis.

[Sol] $\boldsymbol{y = -3(x-2)^2+3}$

（5） Translated -3 units along the x-axis and 3 units along the y-axis.

[Sol] $\boldsymbol{y = -3(x+3)^2+3}$

（6） Translated -3 units along the x-axis and -5 units along the y-axis.

[Sol] $\boldsymbol{y = -3(x+3)^2-5}$

（7） Translated a units along the x-axis and b units along the y-axis.

[Sol] $\boldsymbol{y = -3(x-a)^2+b}$

K 29a

KUMON

Quadratic Functions and Graphs

Ex.

Find the equation of the resulting parabola when $y=2x^2-12x+19$ is translated 2 units along the x-axis and 1 unit along the y-axis. Then check your answer by drawing the graph.

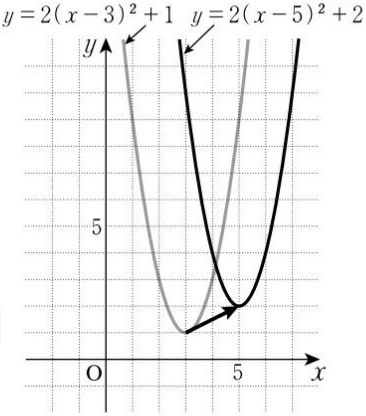
[Sol] $y=2(x^2-6x)+19$

$=2[(x^2-6x+9)-9]+19$

$=2(x-3)^2+1$

Therefore,
the vertex is $(3,\ 1)$.
Applying the translation,
the resulting parabola's
vertex is $(5,\ 2)$. ☞ $(3+2,\ 1+1)$
Therefore,

$y=2(x-5)^2+2$



1. Find the equation of the resulting parabola when $y=2x^2+4x$ is translated -3 units along the x-axis and 4 units along the y-axis. Then check your answer by drawing the graph.

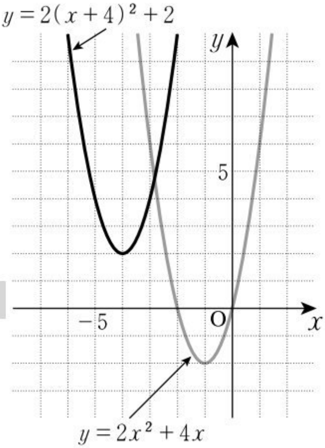
[Sol] $y=2(x^2+2x)$

$=2[(x^2+2x+1)-1]$

$=2(x+1)^2-2$

Therefore,
the vertex is $(-1,\ -2)$.
Applying the translation,
the resulting parabola's
vertex is $(-4,\ 2)$. ☞ $(-1-3,\ -2+4)$
Therefore,

$\boldsymbol{y=2(x+4)^2+2}$



K 29b

Ex.

Determine how the parabola $y=2x^2+8x+7$ has been translated from $y=2x^2-4x-1$. Then check your answer by drawing the graphs.

[Sol] $y=2x^2-4x-1$

$=2[(x^2-2x+1)-1]-1$

$=2(x-1)^2-3$

The vertex before translation is $(1,\ -3)$.

$y=2x^2+8x+7$

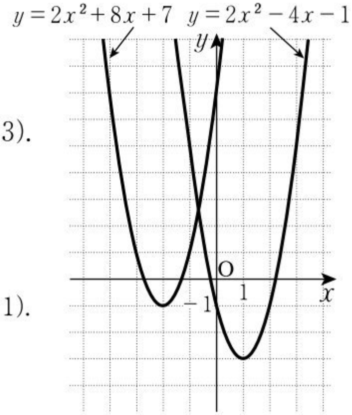
$=2[(x^2+4x+4)-4]+7$

$=2(x+2)^2-1$

The vertex after translation is $(-2,\ -1)$.

Thus, the parabola has been translated:

-3 units along the x-axis and 2 units along the y-axis.



☜ To find the translation, subtract the coordinates of the old vertex from the coordinates of the new vertex.

2. Determine how the parabola $y=-2x^2+12x-19$ has been translated from $y=-2x^2-4x+1$. Then check your answer by drawing the graphs.

[Sol] $y=-2x^2-4x+1$

$=-2[(x^2+2x+1)-1]+1$

$=-2(x+1)^2+3$

The vertex before translation is $(-1,\ 3)$.

$y=-2x^2+12x-19$

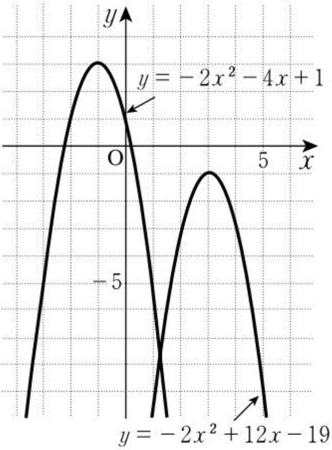
$=-2[(x^2-6x+9)-9]-19$

$=-2(x-3)^2-1$

The vertex after translation is $(3,\ -1)$.

Thus, the parabola has been translated:

4 units along the x-axis and -4 units along the y-axis.



K 30a KUMON

Quadratic Functions and Graphs

1. Find the equation of the resulting parabola when $y=-2x^2+4x+1$ is translated -3 units along the x-axis and 2 units along the y-axis.

[Sol] $y=-2[(x^2-2x+1)-1]+1$

$=-2(x-1)^2+3$

Therefore,

the vertex is $(1, 3)$.

Applying the translation, the resulting parabola's vertex is $(-2, 5)$.

Therefore,

$\boldsymbol{y=-2(x+2)^2+5}$

$\boldsymbol{(y=-2x^2-8x-3)}$

2. Determine how the parabola $y=2x^2-12x+20$ has been translated from $y=2x^2+4x+3$.

[Sol] $y=2x^2+4x+3$

$=2[(x^2+2x+1)-1]+3$

$=2(x+1)^2+1$

The vertex before translation is $(-1, 1)$.

$y=2x^2-12x+20$

$=2[(x^2-6x+9)-9]+20$

$=2(x-3)^2+2$

The vertex after the translation is $(3, 2)$.

Thus, the parabola has been translated:

4 units along the x-axis and 1 unit along the y-axis.

K 30b

3. A parabola $y = 2x^2$ is translated p units along the x-axis and q units along the y-axis. In each question, state the letter (A)〜(G) of the sketch below that satisfies the given conditions of p and q.

(1) $p>0,\ q>0$ … $\boxed{\textbf{(A)}}$

(2) $p>0,\ q<0$ … $\boxed{\textbf{(E)}}$

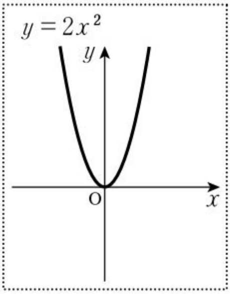
(3) $p<0,\ q>0$ … $\boxed{\textbf{(G)}}$

(4) $p<0,\ q<0$ … $\boxed{\textbf{(C)}}$

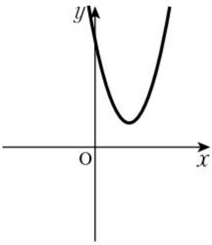
(5) $p>0,\ q=0$ … $\boxed{\textbf{(F)}}$

(6) $p<0,\ q=0$ … $\boxed{\textbf{(D)}}$

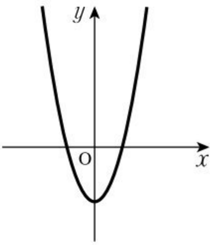
(7) $p=0,\ q<0$ … $\boxed{\textbf{(B)}}$



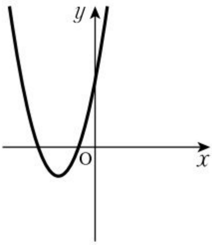
(A)



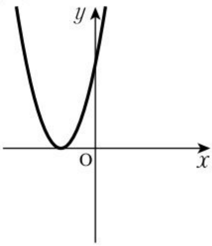
(B)



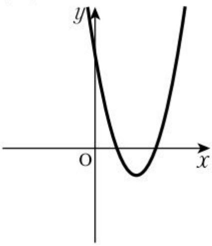
(C)



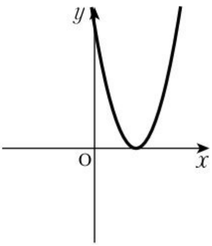
(D)



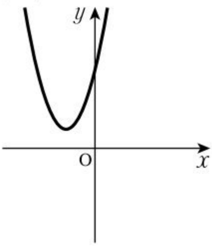
(E)



(F)



(G)



K31a KUMON

Determining Equations of Quadratic Functions

Draw the graph of each quadratic function by finding ① the vertex, ② the y-intercept and ③ the x-intercept(s).

Ex.

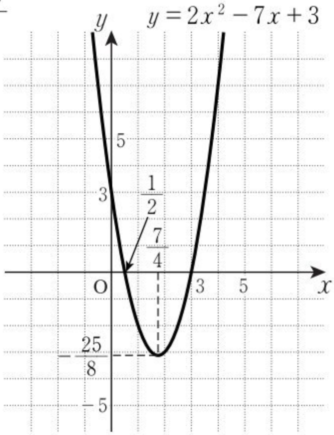
$y=2x^2-7x+3$

[Sol] $y=2\left(x^2-\frac{7}{2}x\right)+3=2\left(x-\frac{7}{4}\right)^2-\frac{25}{8}$

① The vertex is $\left(\frac{7}{4},\ -\frac{25}{8}\right)$.

② If $x=0$, then $y=3$.
Therefore, the y-intercept is $(0,\ 3)$.

③ If $y=0$, i.e. $2x^2-7x+3=0$,
then $x=\frac{1}{2},\ 3$.
Therefore, the x-intercepts are $\left(\frac{1}{2},\ 0\right)$ and $(3,\ 0)$.



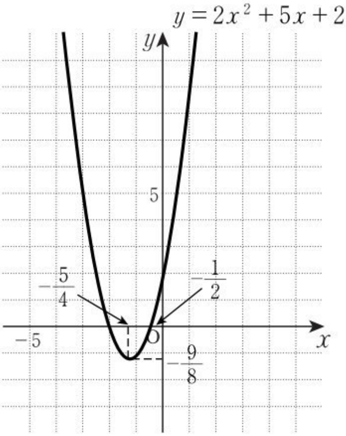
(1) $y=2x^2+5x+2$

[Sol] $y=2\left(x^2+\frac{5}{2}x\right)+2=2\left(x+\frac{5}{4}\right)^2-\frac{9}{8}$

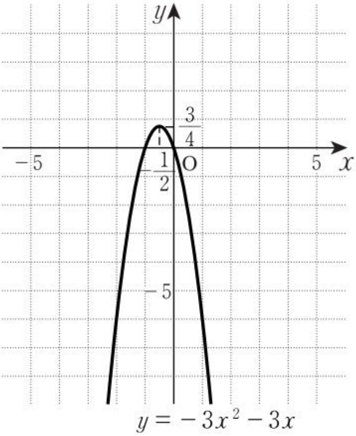
① The vertex is $\boldsymbol{\left(-\frac{5}{4},\ -\frac{9}{8}\right)}$.

② If $x=0$, then $y=2$.
Therefore, the y-intercept is $\boldsymbol{(0,\ 2)}$.

③ If $y=0$, i.e. $2x^2+5x+2=0$,
then $x=-\frac{1}{2},\ -2$.
Therefore, the x-intercepts are $\boldsymbol{\left(-\frac{1}{2},\ 0\right)}$ and $\boldsymbol{(-2,\ 0)}$.



(2) $y=-3x^2-3x$



[Sol] $y=-3(x^2+x)=-3\left(x+\frac{1}{2}\right)^2+\frac{3}{4}$

① The vertex is $\left(-\frac{1}{2},\ \frac{3}{4}\right)$.

② If $x=0$, then $y=0$.

Therefore, the y-intercept is $(0,\ 0)$.

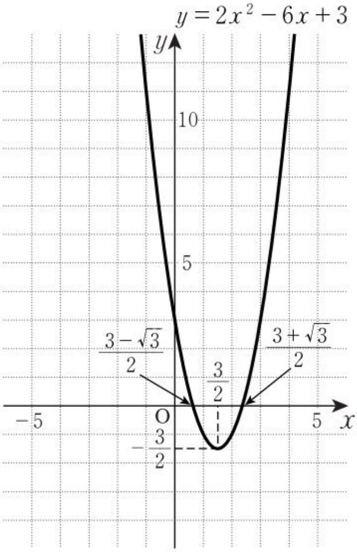
③ If $y=0$, i.e. $-3x^2-3x=0$,

then $x=0,\ -1$.

Therefore, the x-intercepts are

$(0,\ 0)$ and $(-1,\ 0)$.

(3) $y=2x^2-6x+3$ **Hint**



[Sol] $y=2(x^2-3x)+3=2\left(x-\frac{3}{2}\right)^2-\frac{3}{2}$

① The vertex is $\left(\frac{3}{2},\ -\frac{3}{2}\right)$.

② If $x=0$, then $y=3$.

Therefore, the y-intercept is $(0,\ 3)$.

③ If $y=0$, i.e. $2x^2-6x+3=0$,

then $x=\frac{3\pm\sqrt{3}}{2}$.

Therefore, the x-intercepts are

$\left(\frac{3+\sqrt{3}}{2},\ 0\right)$ and $\left(\frac{3-\sqrt{3}}{2},\ 0\right)$.

Hint

The x-coordinate of the x-intercept will be an irrational number.

Determining Equations of Quadratic Functions

Find the parabolas that satisfy the given conditions, and draw the graphs.

Ex.

The vertex is $(2, 1)$, and the parabola passes through point $(0, 5)$.

[Sol] Since the vertex is $(2, 1)$,
let $y=a(x-2)^2+1$.
Because the parabola passes through $(0, 5)$,

$$5=4a+1$$

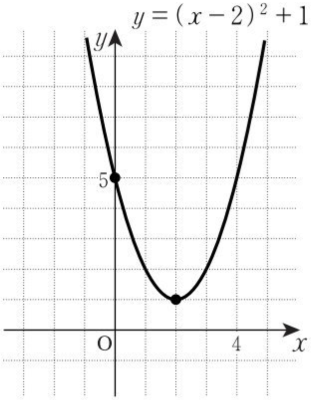
Substituting $x=0$ and $y=5$ into the equation above.

Solving this,

$$a=1$$

Therefore, the parabola is

$$y=(x-2)^2+1$$



(1) The vertex is $(2, 9)$, and the parabola passes through point $(4, -3)$.

[Sol] Since the vertex is $(2, 9)$,
let $y=a(x-2)^2+9$.
Because the parabola passes through $(4, -3)$,

$$-3=4a+9$$

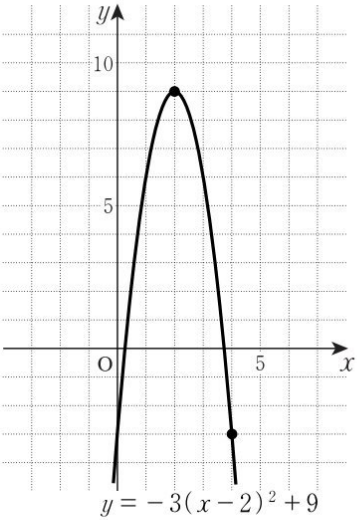
Substituting $x=4$ and $y=-3$ into the equation above.

Solving this,

$$a=-3$$

Therefore, the parabola is

$$\boldsymbol{y=-3(x-2)^2+9}$$



K 32b

(2) The vertex is (1, −2), and the parabola passes through point (2, −3).

[Sol] Since the vertex is (1, −2),

let $y=a(x-1)^2-2$.

Because the parabola passes through (2, −3),

$$-3=a-2$$ ☜

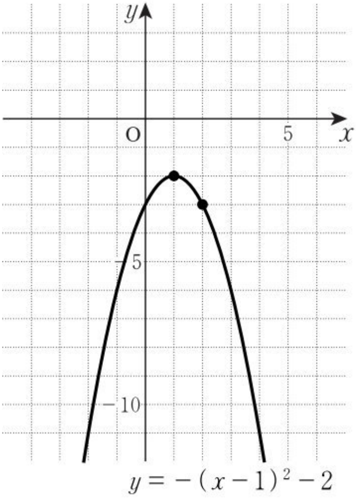
Substituting $x=2$ and $y=-3$ into the equation above.

Solving this,

$$a=-1$$

Therefore, the parabola is

$$\boldsymbol{y=-(x-1)^2-2}$$



(3) The vertex is (−1, −2), and the parabola intercepts the y-axis at point (0, 1).

[Sol] Since the vertex is (−1, −2),

let $y=a(x+1)^2-2$.

Because the parabola passes through (0, 1),

$$1=a-2$$ ☜

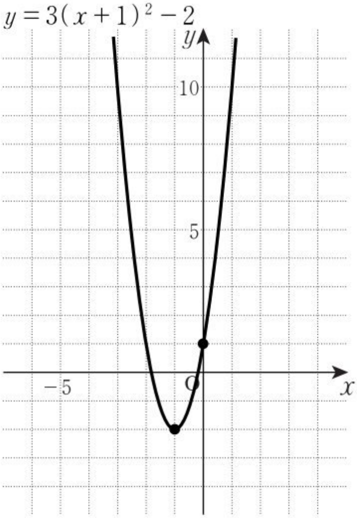
Substituting $x=0$ and $y=1$ into the equation above.

Solving this,

$$a=3$$

Therefore, the parabola is

$$\boldsymbol{y=3(x+1)^2-2}$$



Determining Equations of Quadratic Functions

Find the parabolas that satisfy the given conditions, and draw the graphs.

Ex.

The axis of symmetry is $x=-2$, and the parabola passes through points $(0,\ 1)$ and $(-3,\ 4)$.

[Sol] Since the axis of symmetry is $x=-2$,

let $y=a(x+2)^2+b$.

Since the parabola passes through $(0,\ 1)$ and $(-3,\ 4)$,

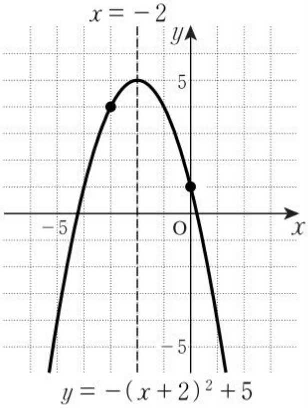
$$\begin{cases} 1=4a+b & \cdots① \\ 4=a+b & \cdots② \end{cases}$$

Substituting $x=0$, $y=1$.

Substituting $x=-3$, $y=4$.

From ① and ②, $a=-1$, $b=5$

Therefore, $y=-(x+2)^2+5$



(1) The axis of symmetry is $x=-1$, and the parabola passes through points $(1,\ 3)$ and $(-2,\ -3)$.

[Sol] Since the axis of symmetry is $x=-1$,

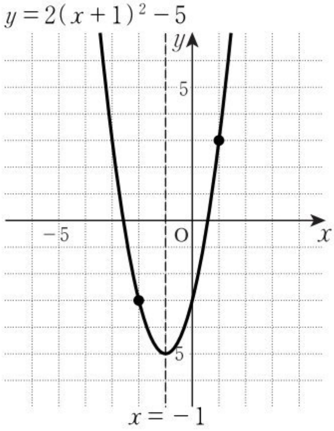
let $y=a(x+1)^2+b$.

Since the parabola passes through $(1,\ 3)$ and $(-2,\ -3)$,

$$\begin{cases} 3=4a+b & \cdots① \\ -3=a+b & \cdots② \end{cases}$$

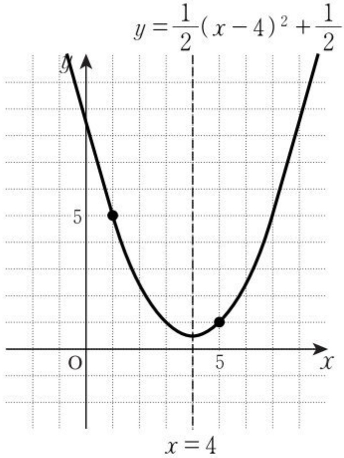
From ① and ②, $a=2$, $b=-5$

Therefore, $\boldsymbol{y=2(x+1)^2-5}$



K 33b

(2) The axis of symmetry is $x=4$, and the parabola passes through points $(1,\ 5)$ and $(5,\ 1)$.



[Sol] Since the axis of symmetry is $x=4$,

let $y=a(x-4)^2+b$.

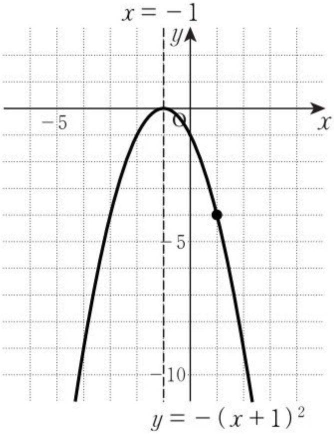
Since the parabola passes through $(1,\ 5)$ and $(5,\ 1)$,

$$\begin{cases} 5=9a+b & \cdots ① \\ 1=a+b & \cdots ② \end{cases}$$

From ① and ②, $a=\frac{1}{2}$, $b=\frac{1}{2}$

Therefore, $\boldsymbol{y=\frac{1}{2}(x-4)^2+\frac{1}{2}}$

(3)* The axis of symmetry is $x=-1$, the vertex is on the x-axis, and the parabola passes through point $(1,\ -4)$.



[Sol] Since the axis of symmetry is $x=-1$,

let $y=a(x+1)^2+b$.

Because the vertex is on the x-axis, $b=0$,

and the equation becomes $y=a(x+1)^2$.

Since the parabola passes through $(1,\ -4)$,

$$-4=4a$$ ☜ Substituting $x=1$, $y=-4$.

$$a=-1$$

Therefore, $\boldsymbol{y=-(x+1)^2}$

K 34a

KUMON

Determining Equations of Quadratic Functions

Find the parabolas that satisfy the given conditions, and draw the graphs.

Ex.

The parabola passes through three points $(-1,\ 0)$, $(2,\ 3)$ and $(3,\ -4)$.

[Sol] Let $y=ax^2+bx+c$.

Since the parabola passes through $(-1,\ 0)$, $(2,\ 3)$ and $(3,\ -4)$,

$$\begin{cases} a-b+c=0 & \cdots ① \\ 4a+2b+c=3 & \cdots ② \\ 9a+3b+c=-4 & \cdots ③ \end{cases}$$

From ②−①, $a+b=1$ $\cdots$④

From ③−②, $5a+b=-7$ $\cdots$⑤

From ⑤−④, $a=-2$

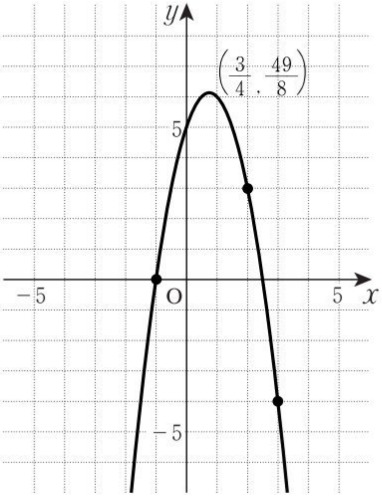
Therefore, $b=3,\ c=5$

Thus, $y=-2x^2+3x+5$

Rewriting,

$$y=-2\left(x^2-\frac{3}{2}x\right)+5$$

$$=-2\left(x-\boxed{\frac{3}{4}}\right)^2+\boxed{\frac{49}{8}}$$



Answers: in order $\frac{3}{4}$, $\frac{49}{8}$

K 34b

(1) The parabola passes through three points $(-1,\ 1)$, $(0,\ 4)$ and $(1,\ 5)$.

[Sol] Let $y = \boxed{\boldsymbol{ax^2+bx+c}}$.

Since the parabola passes through ($\mathbf{-1}$, $\mathbf{1}$), ($\mathbf{0}$, $\mathbf{4}$) and ($\mathbf{1}$, $\mathbf{5}$),

$$\begin{cases} a-b+c=1 & \cdots ① \\ c=4 & \cdots ② \\ a+b+c=5 & \cdots ③ \end{cases}$$

From ①−③,

$b=2$

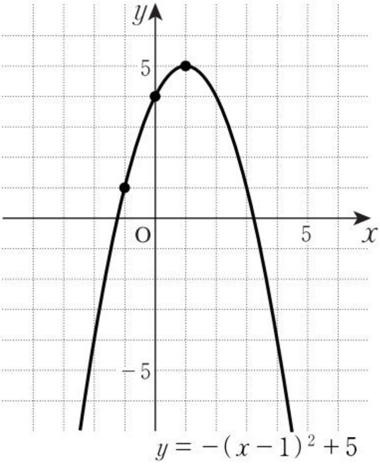
Substituting this value and ② into ①,

$a=-1$

Therefore, $a=-1$, $b=2$, $c=4$

Thus, $\boldsymbol{y=-x^2+2x+4}$

Rewriting, $y=-(x-1)^2+5$



(2) The parabola passes through the origin and the two points $(-1,\ -6)$ and $(2,\ 0)$.

[Sol] Let $y=ax^2+bx+c$.

Since the parabola passes through $(0,\ 0)$, $(-1,\ -6)$ and $(2,\ 0)$,

$$\begin{cases} c=0 & \cdots ① \\ a-b+c=-6 & \cdots ② \\ 4a+2b+c=0 & \cdots ③ \end{cases}$$

Substituting ① into ② and ③,

$a-b=-6 \quad \cdots ④$

$4a+2b=0 \quad \cdots ⑤$

From $2\times$④+⑤,

$a=-2$

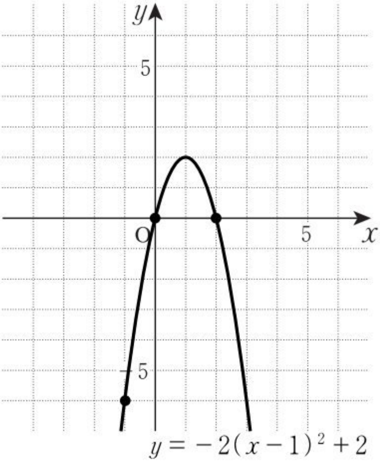
Substituting into ④,

$b=4$

Therefore, $a=-2$, $b=4$, $c=0$

Thus, $\boldsymbol{y=-2x^2+4x}$

Rewriting, $y=-2(x-1)^2+2$



K 35a KUMON

Determining Equations of Quadratic Functions

1. Find the parabola that satisfy the given conditions, and draw the graph.

Ex.

The parabola intersects the x-axis at points $(1,\ 0)$ and $(3,\ 0)$, and intersects the y-axis at point $(0,\ 6)$.

[Sol] Since the parabola intersects the x-axis at $(1,\ 0)$ and $(3,\ 0)$,

let $y=a(x-1)(x-3)$. ☜ When $x=1$, $y=0$ and when $x=3$, $y=0$.

Since the parabola intersects the y-axis at $(0,\ 6)$,

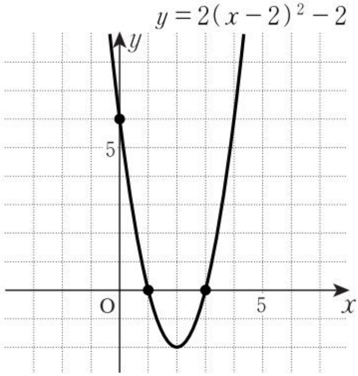
$6=3a$

$a=2$

Therefore, $y=2(x-1)(x-3)$

$[\,y=2x^2-8x+6\,]$

Rewriting, $y=2(x-2)^2-2$



(1) The parabola intersects the x-axis at points $(1,\ 0)$ and $(4,\ 0)$, and intersects the y-axis at point $(0,\ -4)$.

[Sol] Since the parabola intersects the x-axis at $(1,\ 0)$ and $(4,\ 0)$,

let $y=a(x-1)(x-4)$. ☜ When $x=1$, $y=0$ and when $x=4$, $y=0$.

Since the parabola intersects the y-axis at $(0,\ -4)$,

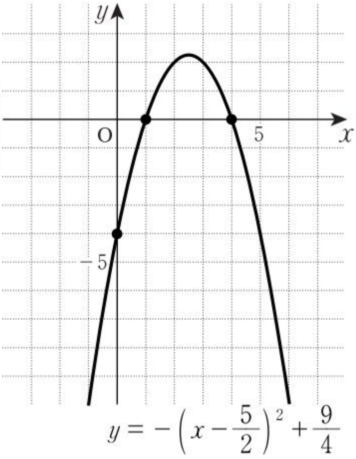
$-4=4a$

$a=-1$

Therefore, $\boldsymbol{y=-(x-1)(x-4)}$

$[\,\boldsymbol{y=-x^2+5x-4}\,]$

Rewriting, $y=-\left(x-\frac{5}{2}\right)^2+\frac{9}{4}$



2. Find the parabola that intersects the x-axis at points $(-1,\ 0)$ and $(3,\ 0)$, and passes through point $(2,\ 6)$, using the following two methods.

(1) Let $y = a(x-\alpha)(x-\beta)$

[Sol] Since the parabola intersects the x-axis at $(-1,\ 0)$ and $(3,\ 0)$,

$$y = a(x+1)(x-3)$$

Since the parabola passes through $(2,\ 6)$,

$$6 = -3a$$

$$a = -2$$

Therefore, $\boldsymbol{y = -2(x+1)(x-3)}$

$\boldsymbol{[\,y = -2x^2+4x+6\,]}$

(2) Let $y = ax^2+bx+c$

[Sol] Since the parabola passes through $(-1,\ 0)$, $(3,\ 0)$ and $(2,\ 6)$,

$$\begin{cases} a-b+c=0 & \cdots ① \\ 9a+3b+c=0 & \cdots ② \\ 4a+2b+c=6 & \cdots ③ \end{cases}$$

From 3×①+②,

$$3a+c=0 \quad \cdots ④$$

From 2×①+③,

$$2a+c=2 \quad \cdots ⑤$$

From ④−⑤,

$$a=-2$$

Substituting into ④,

$$c=6$$

Substituting $a=-2$ and $c=6$ into ①,

$$b=4$$

Therefore, $a=-2$, $b=4$, $c=6$

Thus, $\boldsymbol{y = -2x^2+4x+6}$

Note: If the graph of a parabola crosses the x-axis at points $(\alpha,\ 0)$ and $(\beta,\ 0)$, the function can be written as $y = a(x-\alpha)(x-\beta)$.

K 36a KUMON

Determining Equations of Quadratic Functions

Find the parabolas that satisfy the given conditions, and draw the graphs.

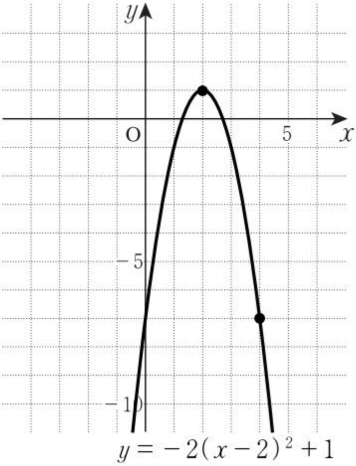
(1) The vertex is $(2,\ 1)$, and the parabola passes through point $(4,\ -7)$.

[Sol] Since the vertex is $(2,\ 1)$,
let $y=a(x-2)^2+1$.
Since the parabola passes
through $(4,\ -7)$,

$$-7=4a+1$$

$$a=-2$$

Therefore, $\boldsymbol{y=-2(x-2)^2+1}$



(2) The axis of symmetry is $x=1$, and the parabola passes through points $(-1,\ 10)$ and $(2,\ 1)$.

[Sol] Since the axis of symmetry is $x=1$,
let $y=a(x-1)^2+b$.
Since the parabola passes through
$(-1,\ 10)$ and $(2,\ 1)$,

$$\begin{cases} 4a+b=10 & \cdots ① \\ a+b=1 & \cdots ② \end{cases}$$

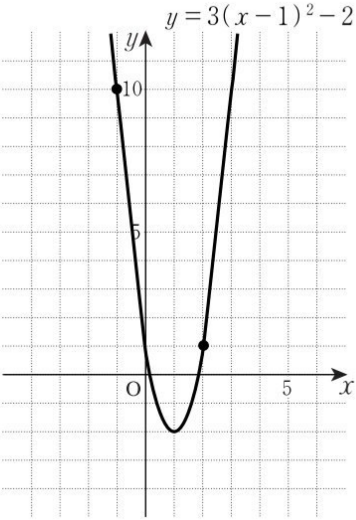
From ①−②,

$$a=3$$

Substituting into ②,

$$b=-2$$

Therefore, $\boldsymbol{y=3(x-1)^2-2}$



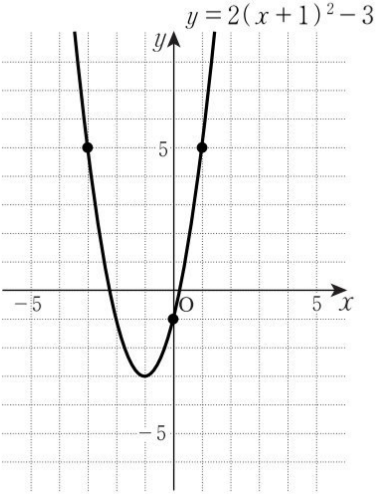
K 36b

(3) The parabola passes through points $(-3,\ 5)$, $(0,\ -1)$ and $(1,\ 5)$.

[Sol] Let $y = ax^2 + bx + c$.

Since the parabola passes through $(-3,\ 5)$, $(0,\ -1)$ and $(1,\ 5)$,

$$\begin{cases} 9a - 3b + c = 5 & \cdots ① \\ c = -1 & \cdots ② \\ a + b + c = 5 & \cdots ③ \end{cases}$$



Substituting ② into ① and ③,

$$9a - 3b = 6 \qquad \cdots ④$$

$$a + b = 6 \qquad \cdots ⑤$$

From ④ $+ 3 \times$ ⑤,

$$a = 2$$

Substituting into ⑤,

$$b = 4$$

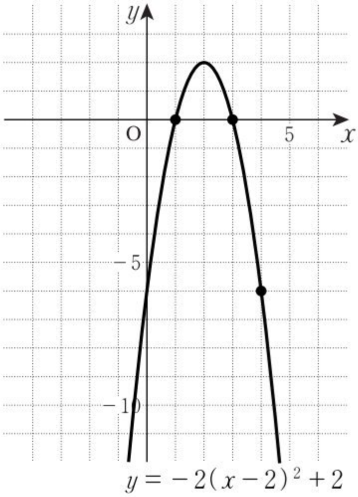
Therefore, $a = 2$, $b = 4$, $c = -1$

Thus, $\boldsymbol{y = 2x^2 + 4x - 1}$

Rewriting, $y = 2(x+1)^2 - 3$

(4) The parabola intersects the x-axis at $(1,\ 0)$ and $(3,\ 0)$, and passes through point $(4,\ -6)$.

[Sol] Since the parabola intersects the x-axis at $(1,\ 0)$ and $(3,\ 0)$, let $y = a(x-1)(x-3)$.



Since the parabola passes through $(4,\ -6)$,

$$-6 = 3a$$

$$a = -2$$

Therefore, $\boldsymbol{y = -2(x-1)(x-3)}$

$$\left[\ \boldsymbol{y = -2x^2 + 8x - 6}\ \right]$$

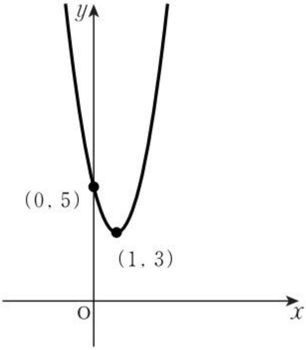
Rewriting, $y = -2(x-2)^2 + 2$

K 37a KUMON

Determining Equations of Quadratic Functions

Find the equations of the parabolas shown on the graphs below.

(1)



[Sol] Since the vertex is $(1, 3)$,

let $y=a(x-1)^2+3$.

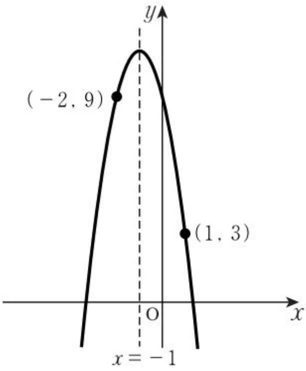
Since the parabola passes through $(0, 5)$,

$5=a+3$

$a=2$

Therefore, $\boldsymbol{y=2(x-1)^2+3}$

(2)



[Sol] Since the axis of symmetry is $x=-1$, let $y=a(x+1)^2+b$.

Since the parabola passes through $(-2, 9)$ and $(1, 3)$,

$$\begin{cases} a+b=9 & \cdots① \\ 4a+b=3 & \cdots② \end{cases}$$

From ②−①,

$a=-2$

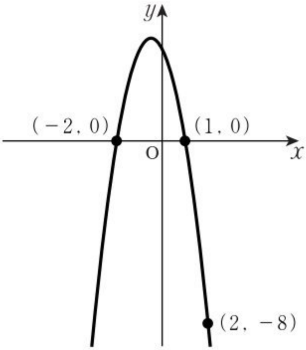
Substituting into ①,

$b=11$

Therefore, $\boldsymbol{y=-2(x+1)^2+11}$

K 37b

(3)



[Sol] Since the parabola intersects the x-axis at $(-2,\ 0)$ and $(1,\ 0)$, let $y=a(x+2)(x-1)$.

Since the parabola passes through $(2,\ -8)$,

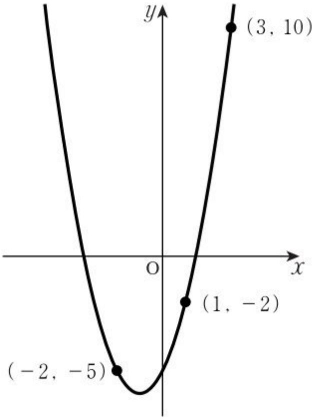
$$-8=4a$$

$$a=-2$$

Therefore, $\boldsymbol{y=-2(x+2)(x-1)}$

$\left[\ \boldsymbol{y=-2x^2-2x+4}\ \right]$

(4)



[Sol] Let $y=ax^2+bx+c$.

Since the parabola passes through $(-2,\ -5)$, $(1,\ -2)$ and $(3,\ 10)$,

$$\begin{cases} 4a-2b+c=-5 & \cdots① \\ a+b+c=-2 & \cdots② \\ 9a+3b+c=10 & \cdots③ \end{cases}$$

From ①−②,

$$a-b=-1 \qquad \cdots④$$

From ③−②,

$$4a+b=6 \qquad \cdots⑤$$

From ④+⑤,

$$a=1$$

Substituting into ④,

$$b=2$$

Substituting $a=1$ and $b=2$ into ②,

$$c=-5$$

Therefore, $\boldsymbol{y=x^2+2x-5}$

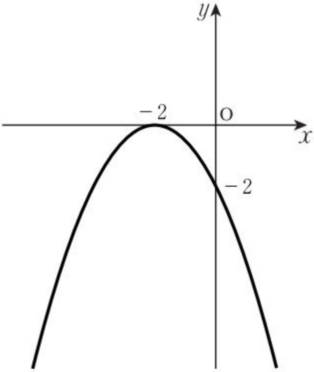
K 38a

KUMON

Determining Equations of Quadratic Functions

Find the equations of the parabolas shown on the graphs below.

(1)



[Sol] Since the vertex is $(-2,\ 0)$,

let $y=a(x+2)^2$.

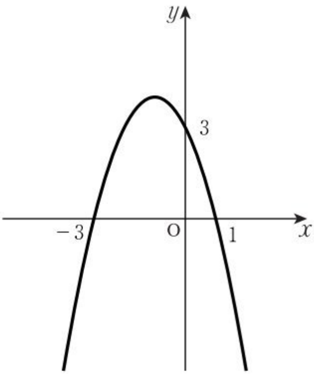
Since the parabola passes through $(0,\ -2)$,

$$-2=4a$$

$$a=-\frac{1}{2}$$

Therefore, $\boldsymbol{y=-\frac{1}{2}(x+2)^2}$

(2)



[Sol] Since the parabola intersects the x-axis at $(-3,\ 0)$ and $(1,\ 0)$,

let $y=a(x+3)(x-1)$.

Since the parabola passes through $(0,\ 3)$,

$$3=-3a$$

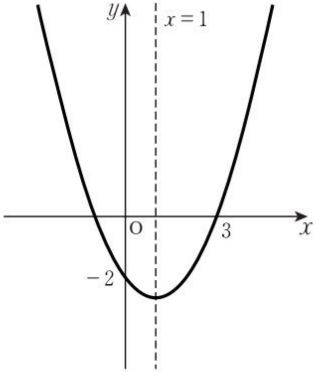
$$a=-1$$

Therefore, $\boldsymbol{y=-(x+3)(x-1)}$

〔$\boldsymbol{y=-x^2-2x+3}$〕

K 38b

(3)



[Sol] Since the axis of symmetry is $x=1$,

let $y=a(x-1)^2+b$.

Since the parabola passes through (3, 0) and (0, -2),

$$\begin{cases} 4a+b=0 & \cdots① \\ a+b=-2 & \cdots② \end{cases}$$

From ①$-$②,

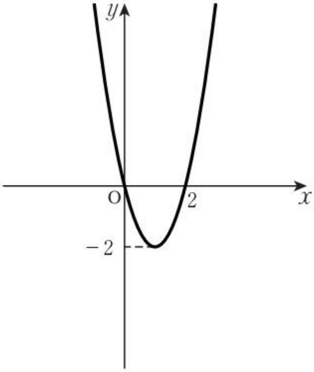
$$a=\frac{2}{3}$$

Substituting into ②,

$$b=-\frac{8}{3}$$

Therefore, $\boldsymbol{y=\frac{2}{3}(x-1)^2-\frac{8}{3}}$

(4)*



[Sol] Since the parabola intersects the x-axis at (0, 0) and (2, 0),

let $y=ax(x-2)$.

Rearranging and completing the square,

$$\begin{aligned} y&=a(x^2-2x) \\ &=a(x^2-2x+1)-a \\ &=a(x-1)^2-a \end{aligned}$$

Since the y-coordinate of the vertex is -2,

$$-a=-2$$

$$a=2$$

Therefore, $\boldsymbol{y=2x(x-2)}$

〔 $\boldsymbol{y=2x^2-4x}$ 〕

K 39a KUMON

Determining Equations of Quadratic Functions

1. Find the equation of the parabola that is a translation of $y=x^2-3x+4$, and passes through points $(-3,\ 3)$ and $(2,\ 8)$. Then draw the graph.

Hint

[Sol] Since the parabola is a translation of $y=x^2-3x+4$, let $y=x^2+bx+c$.

Since the parabola passes through $(-3,\ 3)$ and $(2,\ 8)$,

$$\begin{cases} 3=9-3b+c \\ 8=4+2b+c \end{cases}$$

Rearranging,

$$\begin{cases} -3b+c=-6 & \cdots ① \\ 2b+c=4 & \cdots ② \end{cases}$$

From ②−①,

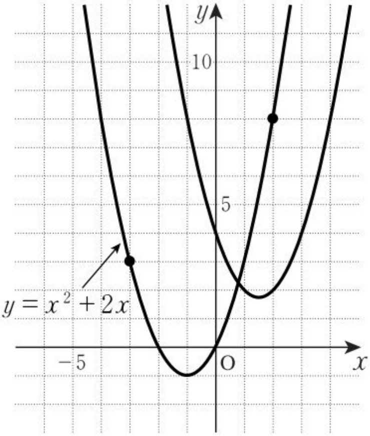
$b=2$

Substituting into ②,

$c=0$

Therefore, $\boldsymbol{y=x^2+2x}$

Rewriting, $y=(x+1)^2-1$



Hint

The parabola $y=x^2+bx+c$ is a translation of $y=x^2$.

K 39b

2. Find the equation of the parabola that is a translation of $y=2x^2-3x+4$, passes through point $(2, -1)$, and has $x=1$ as its axis of symmetry. Then draw the graph.

[Sol] Since $x=1$ is the axis of symmetry,

let $y=2(x-1)^2+b$.

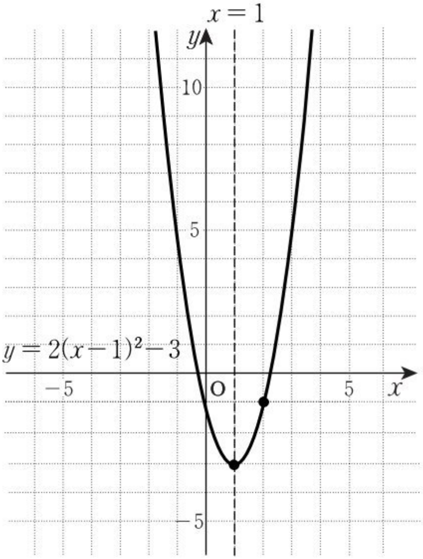
Since the parabola passes through

$(2, -1)$,

$$-1=2+b$$

$$b=-3$$

Therefore, $\boldsymbol{y=2(x-1)^2-3}$



K 40a KUMON

Determining Equations of Quadratic Functions

In each question, indicate the letter(s), (A)～(J), of the equation below that satisfies the given conditions.

(1) The vertex is (1, 2). … **(A)**

(2) The parabola intersects the y-axis at point (0, 3). … **(I)**

(3) The axis of symmetry is $x=2$. … **(B)**

(4) The parabola is a translation of $y=x^2-3x+1$. … **(E)**, **(H)**

(5) The parabola intersects the x-axis at (−3, 0) and (2, 0). … **(C)**

(6) The vertex is on the x-axis. … **(G)**

(7) The parabola passes through the origin. … **(D)**, **(F)**

(8) The vertex is on the y-axis. … **(J)**

(A) $y=a(x-1)^2+2 \quad (a\neq 0)$

(B) $y=a(x-2)^2+b \quad (a\neq 0)$

(C) $y=a(x+3)(x-2) \quad (a\neq 0)$

(D) $y=ax(x-b) \quad (a\neq 0)$

(E) $y=(x-b)^2+c$

(F) $y=ax^2+bx \quad (a\neq 0)$

(G) $y=a(x-b)^2 \quad (a\neq 0)$

(H) $y=x^2+bx+c$

(I) $y=ax^2+bx+3 \quad (a\neq 0)$

(J) $y=ax^2+c \quad (a\neq 0)$

K 40b

Let's try this!

Find the equation of the parabola that passes through the points $(1,\ 0)$, $(3,\ 0)$ and $(0,\ -6)$, using the following methods.

(i) Let $y=a(x-p)^2+q$

[Sol] Since the parabola passes through $(1,\ 0)$, $(3,\ 0)$ and $(0,\ -6)$,

$$\begin{cases} a(1-p)^2+q=0 & \cdots ① \\ a(3-p)^2+q=0 & \cdots ② \\ ap^2+q=-6 & \cdots ③ \end{cases}$$

From ①, ② and ③,

$a=-2$, $p=2$ and $q=2$.

Therefore, $y=-2(x-2)^2+2$ 〔$y=-2x^2+8x-6$〕

(ii) Let $y=a(x-\boxed{\mathbf{1}})(x-\boxed{\mathbf{3}})$

[Sol] Since the parabola passes through $(0,\ -6)$,

$$-6=3a$$

$$a=-2$$

Therefore, $y=-2(x-1)(x-3)$ 〔$y=-2x^2+8x-6$〕

(iii) Let $y=ax^2+bx+c$

[Sol] Since the parabola passes through $(1,\ 0)$, $(3,\ 0)$ and $(0,\ -6)$,

$$\begin{cases} a+b+c=0 & \cdots ① \\ 9a+3b+c=0 & \cdots ② \\ c=-6 & \cdots ③ \end{cases}$$

Substituting ③ into ① and ②,

$$a+b=6 \qquad \cdots ④$$

$$9a+3b=6 \qquad \cdots ⑤$$

From ⑤$-3\times$④,

$$a=-2$$

Substituting into ④, $b=8$

Therefore, $y=-2x^2+8x-6$

Answers: $\boxed{1}$, $\boxed{3}$

Did you get the same answer for all three methods (i), (ii) and (iii)?
From this, we can see that when we rearrange the equation *before substituting*, the calculations are done more easily.

K41a KUMON

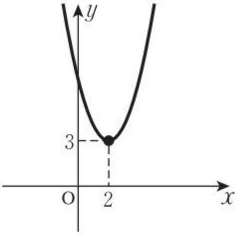
Maxima and Minima of Quadratic Functions I

1. Draw the graph of the quadratic function, and locate the vertex. Then, find the value of x at which y is smallest, and state that value of y (the ***minimum value***).

Ex.

$y=x^2-4x+7$

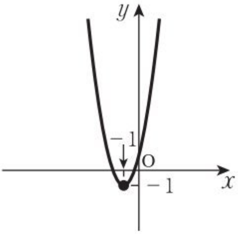
[Sol] $y=(x-2)^2+3$



From the graph,
at $x=2$ there is a
minimum value of 3.

(1) $y=2x^2+4x+1$

[Sol] $y=\mathbf{2(x+1)^2-1}$



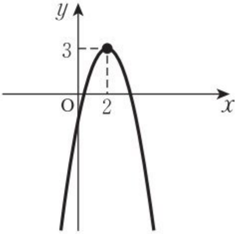
From the graph,
at $x=\boxed{\mathbf{-1}}$ there is a minimum
value of $\boxed{\mathbf{-1}}$.

2. Draw the graph of the quadratic function, and locate the vertex. Then, find the value of x at which y is largest, and state that value of y (the ***maximum value***).

Ex.

$y=-x^2+4x-1$

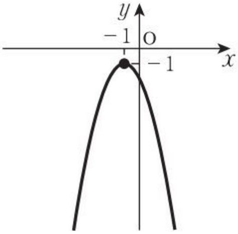
[Sol] $y=-(x-2)^2+3$



From the graph,
at $x=2$ there is a
maximum value of 3.

(1) $y=-2x^2-4x-3$

[Sol] $y=-2(x+1)^2-1$

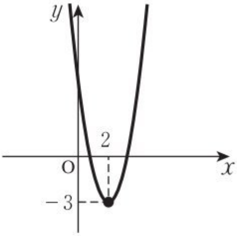


From the graph,
at $x=\boxed{\mathbf{-1}}$ there is a maximum
value of $\boxed{\mathbf{-1}}$.

3. Draw the graph of each quadratic function, and locate the vertex. If there is a maximum or minimum value, find the value, and indicate the corresponding value of x.

(1) $y=2x^2-8x+5$

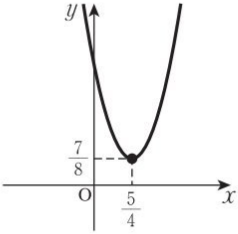
[Sol] $y=2(x-2)^2-3$



From the graph,
there is a minimum value of
$\boxed{\mathbf{-3}}$, at $x=\boxed{\mathbf{2}}$.

(2) $y=2x^2-5x+4$

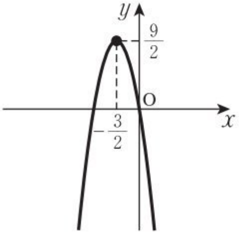
[Sol] $y=2\left(x-\dfrac{5}{4}\right)^2+\dfrac{7}{8}$



From the graph,
there is a minimum value of $\mathbf{\dfrac{7}{8}}$,
at $\boldsymbol{x=\dfrac{5}{4}}$**.**

(3) $y=-2x^2-6x$

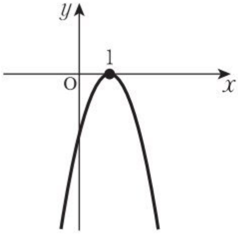
[Sol] $y=-2\left(x+\dfrac{3}{2}\right)^2+\dfrac{9}{2}$



From the graph,
there is a maximum value of $\mathbf{\dfrac{9}{2}}$,
at $\boldsymbol{x=-\dfrac{3}{2}}$**.**

(4) $y=-x^2+2x-1$

[Sol] $y=-(x-1)^2$



From the graph,
there is a maximum value of **0,**
at $\boldsymbol{x=1}$**.**

Note: Given a quadratic function of the form $y=ax^2+bx+c$:

- When $a>0$, the graph of the parabola opens upward, and it has a minimum value at the vertex. (It does not have a maximum value.)
- When $a<0$, the graph of the parabola opens downward, and it has a maximum value at the vertex. (It does not have a minimum value.)

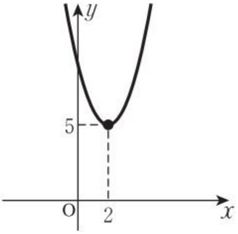
K 42a KUMON

Maxima and Minima of Quadratic Functions I

Draw the graph of each quadratic function, and locate the vertex. If there is a maximum or minimum value, find the value, and indicate the corresponding value of x.

(1) $y=(x-2)^2+5$

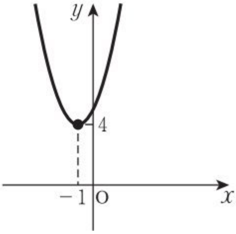
[Sol]



From the graph,
there is a minimum value of **5,**
at $x=2$.

(2) $y=x^2+2x+5$

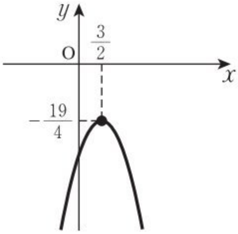
[Sol] $y=(x+1)^2+4$



From the graph,
there is a minimum value of **4,**
at $x=-1$.

(3) $y=-x^2+3x-7$

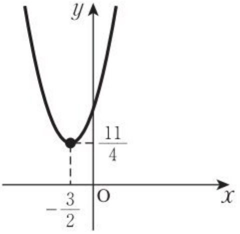
[Sol] $y=-\left(x-\frac{3}{2}\right)^2-\frac{19}{4}$



From the graph,
there is a maximum value of
$\boldsymbol{-\frac{19}{4}}$**, at $x=\frac{3}{2}$.**

(4) $y=x^2+3x+5$

[Sol] $y=\left(x+\frac{3}{2}\right)^2+\frac{11}{4}$

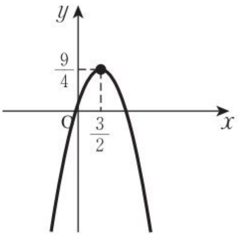


From the graph,
there is a minimum value of
$\boldsymbol{\frac{11}{4}}$**, at $x=-\frac{3}{2}$.**

K 42b

(5)　$y=-x^2+3x$

[Sol]　$y=-\left(x-\dfrac{3}{2}\right)^2+\dfrac{9}{4}$



From the graph,

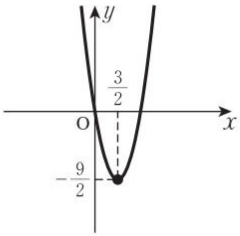
there is a maximum value of

$\boldsymbol{\dfrac{9}{4}}$**, at** $\boldsymbol{x=\dfrac{3}{2}}$**.**

(6)　$y=2x(x-3)$

[Sol]　$y=2(x^2-3x)$

$$=2\left(x-\frac{3}{2}\right)^2-\frac{9}{2}$$



From the graph,

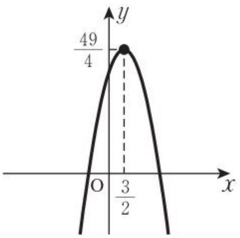
there is a minimum value of

$\boldsymbol{-\dfrac{9}{2}}$**, at** $\boldsymbol{x=\dfrac{3}{2}}$**.**

(7)　$y=(x+2)(5-x)$

[Sol]　$y=-x^2+3x+10$

$$=-\left(x-\frac{3}{2}\right)^2+\frac{49}{4}$$



From the graph,

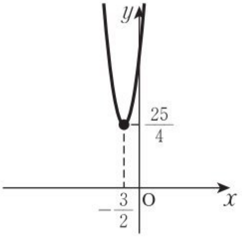
there is a maximum value of

$\boldsymbol{\dfrac{49}{4}}$**, at** $\boldsymbol{x=\dfrac{3}{2}}$**.**

(8)　$y=3(x+1)(x+2)+7$

[Sol]　$y=3x^2+9x+13$

$$=3\left(x+\frac{3}{2}\right)^2+\frac{25}{4}$$



From the graph,

there is a minimum value of

$\boldsymbol{\dfrac{25}{4}}$**, at** $\boldsymbol{x=-\dfrac{3}{2}}$**.**

Maxima and Minima of Quadratic Functions I

1. Given $y=x^2-6x+4$, draw the graph corresponding to each given condition. Then find the corresponding set of y values (the ***range***).

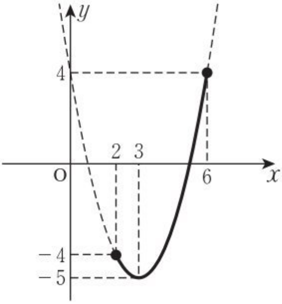
Ex.

When $2 \leqslant x \leqslant 6$.

[Sol] $y=x^2-6x+4$

$\quad\;\, =(x-3)^2-5$

Since $2 \leqslant x \leqslant 6$, the graph is the solid curve shown on the figure below.



From the graph:

- There is a maximum value of 4, at $x=6$.
- There is a minimum value of -5, at $x=3$.

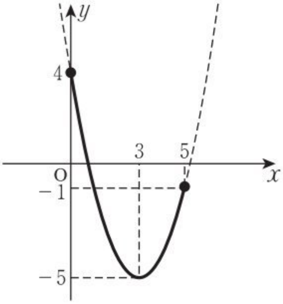
Therefore, the range is $-5 \leqslant y \leqslant 4$.

(1) When $0 \leqslant x \leqslant 5$.

[Sol] $y=x^2-6x+4$

$\quad = \boxed{(x-3)^2-5}$

Since $\boxed{0} \leqslant x \leqslant \boxed{5}$, draw the graph on the figure below.



From the graph:

- There is a maximum value of 4, at $x=0$.
- There is a minimum value of -5, at $x=3$.

Therefore, the range is $\boldsymbol{-5 \leqslant y \leqslant 4}$.

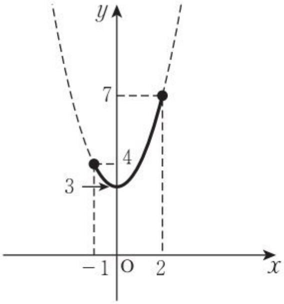
- The set of x values is called the ***domain***.
  The set of y values is called the ***range***.
- The domain is normally written in brackets next to the function, e.g. $y=x^2-6x+4 \;(2 \leqslant x \leqslant 6)$

K 43b

2. Draw the graph of each quadratic function, find the maximum and minimum values, and state the range (the set of y values) for the given domain.

(1) $y = x^2 + 3 \quad (-1 \leqslant x \leqslant 2)$

[Sol]



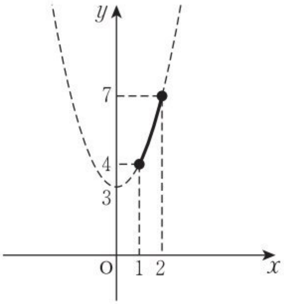
Maximum value: $\boxed{7}$, at $x = \boxed{2}$

Minimum value: $\boxed{3}$, at $x = \boxed{0}$

Range: $\boxed{3} \leqslant y \leqslant \boxed{7}$

(2) $y = x^2 + 3 \quad (1 \leqslant x \leqslant 2)$

[Sol]



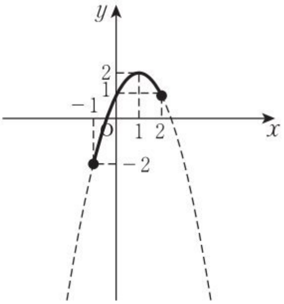
Maximum value: 7, at $x = 2$

Minimum value: 4, at $x = 1$

Range: $4 \leqslant y \leqslant 7$

(3) $y = -x^2 + 2x + 1 \quad (-1 \leqslant x \leqslant 2)$

[Sol] $y = -(x-1)^2 + 2$



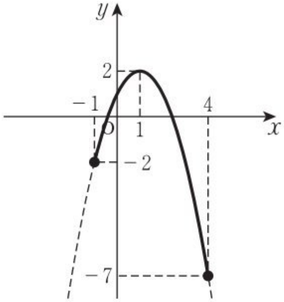
Maximum value: 2, at $x = 1$

Minimum value: -2, at $x = -1$

Range: $-2 \leqslant y \leqslant 2$

(4) $y = -x^2 + 2x + 1 \quad (-1 \leqslant x \leqslant 4)$

[Sol] $y = -(x-1)^2 + 2$



Maximum value: 2, at $x = 1$

Minimum value: -7, at $x = 4$

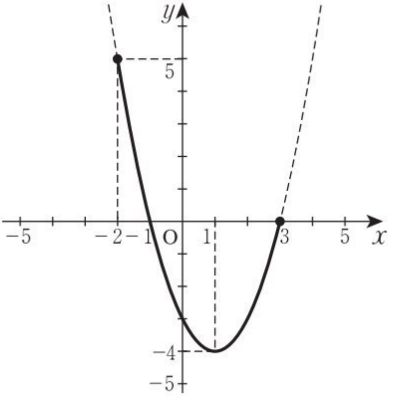
Range: $-7 \leqslant y \leqslant 2$

K 44a KUMON

Maxima and Minima of Quadratic Functions I

Draw the graph of each quadratic function. Then, find the maximum and minimum values and the range.

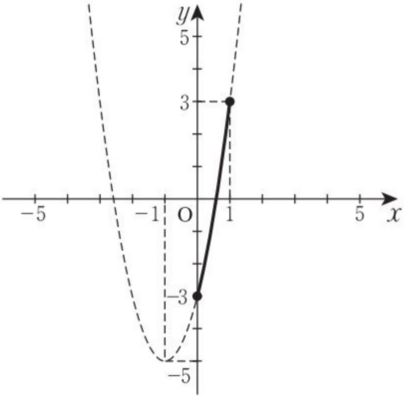
(1) $y=x^2-2x-3 \quad (-2\leqslant x\leqslant 3)$



$y=(x-1)^2-4$

Maximum value: 5, at $x=-2$
Minimum value: -4, at $x=1$
Range: $-4\leqslant y\leqslant 5$

(2) $y=2x^2+4x-3 \quad (0\leqslant x\leqslant 1)$

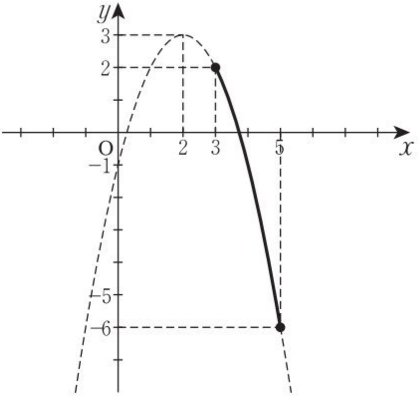


$y=2(x+1)^2-5$

Maximum value: 3, at $x=1$
Minimum value: -3, at $x=0$
Range: $-3\leqslant y\leqslant 3$

K 44b

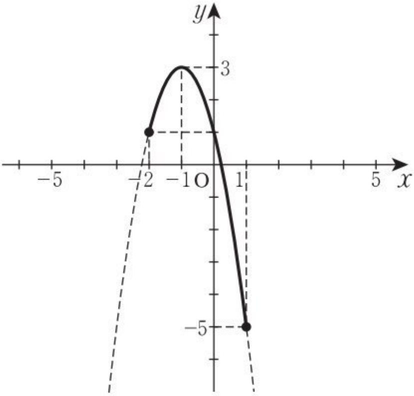
(3)　$y=-x^2+4x-1$　$(3 \leqq x \leqq 5)$



$y=-(x-2)^2+3$

Maximum value: 2, at $x=3$
Minimum value: -6, at $x=5$
Range: $-6 \leqq y \leqq 2$

(4)　$y=-2x^2-4x+1$　$(-2 \leqq x \leqq 1)$



$y=-2(x+1)^2+3$

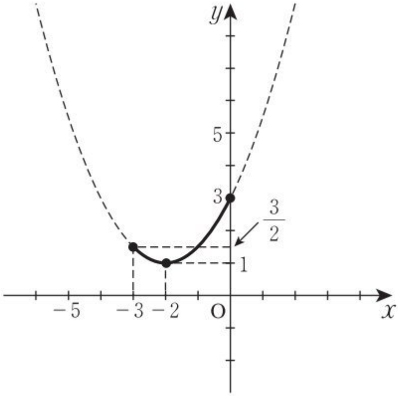
Maximum value: 3, at $x=-1$
Minimum value: -5, at $x=1$
Range: $-5 \leqq y \leqq 3$

K 45a KUMON

Maxima and Minima of Quadratic Functions I

Draw the graph of each quadratic function. Then find the maximum and minimum values and the range.

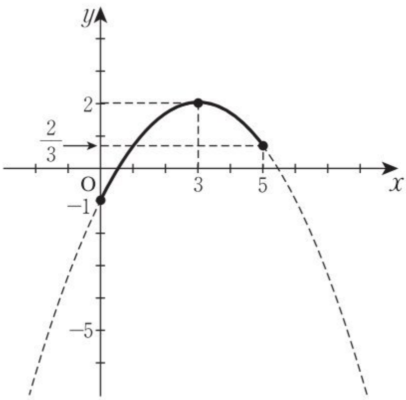
(1) $y=\frac{1}{2}x^2+2x+3 \quad (-3 \leqslant x \leqslant 0)$



$y=\frac{1}{2}(x+2)^2+1$

Maximum value: 3, at $x=0$
Minimum value: 1, at $x=-2$
Range: $1 \leqslant y \leqslant 3$

(2) $y=-\frac{1}{3}x^2+2x-1 \quad (0 \leqslant x \leqslant 5)$

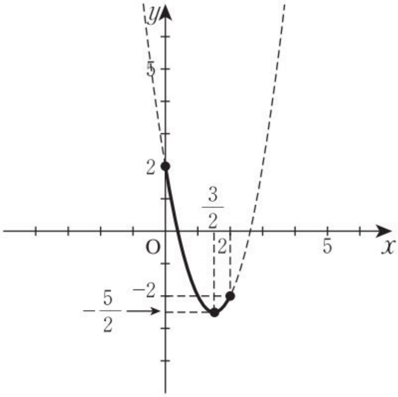


$y=-\frac{1}{3}(x-3)^2+2$

Maximum value: 2, at $x=3$
Minimum value: -1, at $x=0$
Range: $-1 \leqslant y \leqslant 2$

K 45b

(3) $y=2x^2-6x+2 \quad (0 \leqq x \leqq 2)$



$$y=2\left(x-\frac{3}{2}\right)^2-\frac{5}{2}$$

Maximum value: 2, at $x=0$

Minimum value: $-\frac{5}{2}$, at $x=\frac{3}{2}$

Range: $-\frac{5}{2} \leqq y \leqq 2$

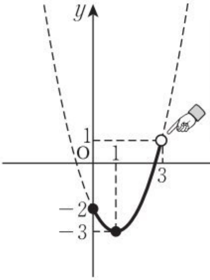
Let’s try this!

Given a domain that contains the inequality sign, “less than” (<), find the maximum value or the minimum value, and state the range.

Case 1

$y=x^2-2x-2 \quad (0 \leqq x < 3)$

〔 $y=x^2-2x-2=(x-1)^2-3$ 〕



This point is marked with an open circle (○) to indicate that the graph reaches, but does not include, this point.

Maximum value: no value

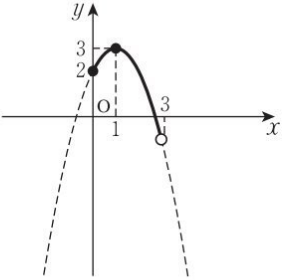
Minimum value: -3, at $x=1$

Range: $-3 \leqq y < 1$

Case 2

$y=-x^2+2x+2 \quad (0 \leqq x < 3)$

〔 $y=-x^2+2x+2=-(x-1)^2+3$ 〕



Maximum value: 3, at $x=1$

Minimum value: $\boxed{\textbf{no value}}$

Range: $\boxed{\mathbf{-1}} < y \leqq 3$

Answers: in order no value, −1

K 46a KUMON

Maxima and Minima of Quadratic Functions I

1. Find the quadratic function that satisfies the following conditions.

Ex.

The minimum value is 2, at $x=1$, and when $x=3$, $y=6$.

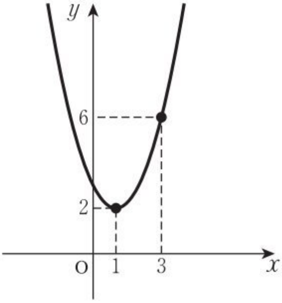
[Sol] Since the minimum value is 2, at $x=1$,

let $y=a(x-1)^2+2$ (where $a>0$).

Substituting $x=3$ and $y=6$,

$$6=4a+2$$
$$a=1$$

Therefore, $y=(x-1)^2+2$



(1) The maximum value is 3, at $x=-1$, and when $x=1$, $y=-5$.

[Sol] Since the maximum value is 3, at $x=-1$,

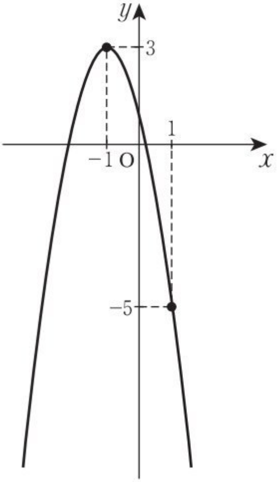
let $y=a(x+1)^2+3$ $(a<0)$.

Substituting $x=1$ and $y=-5$,

$$-5=4a+3$$
$$a=-2$$

Therefore, $\boldsymbol{y=-2(x+1)^2+3}$

〔 $\boldsymbol{y=-2x^2-4x+1}$ 〕



2. Find the quadratic function whose minimum value is -4 at $x=2$, and when $x=0$, $y=4$.

[Sol] Since the minimum value is -4, at $x=2$,

let $y=a(x-2)^2-4 \quad (a>0)$.

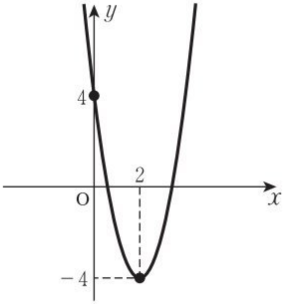
Substituting $x=0$ and $y=4$,

$4=4a-4$

$a=2$

Therefore, $\boldsymbol{y=2(x-2)^2-4}$

〔 $\boldsymbol{y=2x^2-8x+4}$ 〕



3. Find the parabola with maximum value 8 at $x=-1$, and passing through point $(-3,\ 5)$.

[Sol] Since the maximum value is 8, at $x=-1$,

let $y=a(x+1)^2+8 \quad (a<0)$.

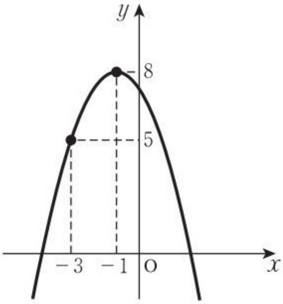
Substituting $x=-3$ and $y=5$,

$5=4a+8$

$a=-\dfrac{3}{4}$

Therefore, $\boldsymbol{y=-\dfrac{3}{4}(x+1)^2+8}$

$$\left[\ \boldsymbol{y=-\frac{3}{4}x^2-\frac{3}{2}x+\frac{29}{4}}\ \right]$$



K 47a

KUMON

Maxima and Minima of Quadratic Functions I

Ex.

Given the quadratic function $y=4x^2+bx+c$, find the values of the constants b and c so that 3 becomes the minimum value, at $x=-1$.

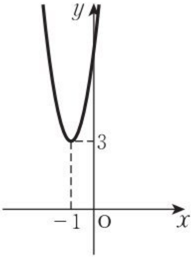
[Sol] Since the minimum value is 3, at $x=-1$,

$$y=4(x+1)^2+3$$
$$=4x^2+8x+7$$

So $4x^2+bx+c=4x^2+8x+7$

Comparing coefficients,

$b=8,\ c=7$



1. Given the quadratic function $y=x^2+bx+c$, find the values of the constants b and c so that 1 becomes the minimum value, at $x=2$. Use the method shown above.

[Sol] Since the minimum value is 1, at $x=2$,

$$y=(x-2)^2+1$$
$$=x^2-4x+5$$

So $x^2+bx+c=x^2-4x+5$

Comparing coefficients,

$\boldsymbol{b=-4,\ c=5}$

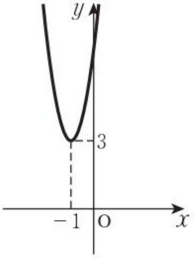
K 47b

Ex.

Given the quadratic function $y=4x^2+bx+c$, find the values of the constants b and c so that 3 becomes the minimum value, at $x=-1$.

[Sol] $y=4x^2+bx+c$

$$=4\left(x+\frac{b}{8}\right)^2-\frac{b^2}{16}+c$$



Since the minimum value is 3, at $x=-1$,

$$\begin{cases} -\dfrac{b}{8}=-1 & \cdots ① \\ -\dfrac{b^2}{16}+c=3 & \cdots ② \end{cases}$$

From ① and ②, $b=8$, $c=7$

2. Given the quadratic function $y=x^2+bx+c$, find the values of the constants b and c so that 1 becomes the minimum value, at $x=2$. Use the method shown above.

[Sol] $y=x^2+bx+c$

$$=\left(x+\frac{b}{2}\right)^2-\frac{b^2}{4}+c$$

Since the minimum value is 1, at $x=2$,

$$\begin{cases} -\dfrac{b}{2}=2 & \cdots ① \\ -\dfrac{b^2}{4}+c=1 & \cdots ② \end{cases}$$

From ① and ②, $\boldsymbol{b=-4,\ c=5}$

K 48a KUMON

Maxima and Minima of Quadratic Functions I

1. Given the quadratic function $y=2x^2+bx+c$, find the values of the constants b and c so that -3 becomes the minimum value, at $x=2$. (Use either the method shown on K 47a or that shown on 47b.)

[Sol] Since the minimum value is -3, at $x=2$,

$$y=2(x-2)^2-3$$
$$=2x^2-8x+5$$

So $2x^2+bx+c=2x^2-8x+5$

Comparing coefficients,

$\boldsymbol{b=-8,\ c=5}$

Alternative Solution

$$y=2x^2+bx+c=2\left(x+\frac{b}{4}\right)^2-\frac{b^2}{8}+c$$

Since the minimum value is -3, at $x=2$,

$$\begin{cases} -\dfrac{b}{4}=2 & \cdots① \\ -\dfrac{b^2}{8}+c=-3 & \cdots② \end{cases}$$

From ① and ②, $\boldsymbol{b=-8,\ c=5}$

2. Given the quadratic function $y=-2x^2+bx+c$, find the values of the constants b and c so that 9 becomes the maximum value, at $x=-3$.

[Sol] Since the maximum value is 9, at $x=-3$,

$$y=-2(x+3)^2+9$$
$$=-2x^2-12x-9$$

So $-2x^2+bx+c=-2x^2-12x-9$

Comparing coefficients,

$\boldsymbol{b=-12,\ c=-9}$

Alternative Solution

$$y=-2x^2+bx+c=-2\left(x-\frac{b}{4}\right)^2+\frac{b^2}{8}+c$$

Since the maximum value is 9, at $x=-3$,

$$\begin{cases} \dfrac{b}{4}=-3 & \cdots① \\ \dfrac{b^2}{8}+c=9 & \cdots② \end{cases}$$

From ① and ②, $\boldsymbol{b=-12,\ c=-9}$

3. Given the quadratic function $y=ax^2+2x+c$, find the values of the constants a and c so that 2 becomes the maximum value, at $x=1$.

[Sol] Since the maximum value is 2, at $x=1$,

$$\begin{aligned} y &= a(x-1)^2+2 \quad (a<0) \\ &= ax^2-2ax+a+2 \end{aligned}$$

So $ax^2+2x+c=ax^2-2ax+a+2$

Comparing coefficients,

$$\begin{cases} 2=-2a & \cdots ① \\ c=a+2 & \cdots ② \end{cases}$$

From ① and ②,

$\boldsymbol{a=-1,\ c=1}$

Alternative Solution

$y=ax^2+2x+c=a\left(x+\dfrac{1}{a}\right)^2-\dfrac{1}{a}+c$

Since the maximum value is 2, at $x=1$,

$$\begin{cases} -\dfrac{1}{a}=1 & \cdots ① \\ -\dfrac{1}{a}+c=2 & \cdots ② \end{cases}$$

From ① and ②, $\boldsymbol{a=-1,\ c=1}$

4. Given the quadratic function $y=ax^2+bx+5$, find the values of the constants a and b so that 3 becomes the minimum value, at $x=-1$.

[Sol] Since the minimum value is 3, at $x=-1$,

$$\begin{aligned} y &= a(x+1)^2+3 \quad (a>0) \\ &= ax^2+2ax+a+3 \end{aligned}$$

So $ax^2+bx+5=ax^2+2ax+a+3$

Comparing coefficients,

$$\begin{cases} b=2a & \cdots ① \\ 5=a+3 & \cdots ② \end{cases}$$

From ① and ②,

$\boldsymbol{a=2,\ b=4}$

Alternative Solution

$y=ax^2+bx+5=a\left(x+\dfrac{b}{2a}\right)^2-\dfrac{b^2}{4a}+5$

Since the minimum value is 3, at $x=-1$,

$$\begin{cases} -\dfrac{b}{2a}=-1 & \cdots ① \\ -\dfrac{b^2}{4a}+5=3 & \cdots ② \end{cases}$$

From ① and ②, $\boldsymbol{a=2,\ b=4}$

K 49a KUMON

Maxima and Minima of Quadratic Functions I

1. Given the quadratic function $y=ax^2+bx+c$, the maximum value is a^2+4 at $x=1$, and the graph passes through point $(3,\ 1)$. Find the values of the constants a, b and c.

[Sol] Since the maximum value is a^2+4, at $x=1$,

$$\begin{aligned} y &= a(x-1)^2+a^2+4\ (a<0) \\ &= ax^2-2ax+a^2+a+4 \end{aligned}$$

So $ax^2+bx+c=ax^2-2ax+a^2+a+4$

Comparing coefficients,

$$\begin{cases} b=-2a & \cdots ① \\ c=a^2+a+4 & \cdots ② \end{cases}$$

Since the graph of $y=ax^2+bx+c$ passes through $(3,\ 1)$,

$$1=9a+3b+c\ \ \cdots ③$$

From ①, ② and ③,

$$\begin{cases} \boldsymbol{a=-1} \\ \boldsymbol{b=2} \\ \boldsymbol{c=4} \end{cases} \qquad \begin{cases} \boldsymbol{a=-3} \\ \boldsymbol{b=6} \\ \boldsymbol{c=10} \end{cases}$$

Alternative Solution

$$y=ax^2+bx+c=a\left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a}+c$$

Since the maximum value is a^2+4, at $x=1$,

$$\begin{cases} -\dfrac{b}{2a}=1 & \cdots ① \\ -\dfrac{b^2}{4a}+c=a^2+4 & \cdots ② \end{cases}$$

Since the graph of $y=ax^2+bx+c$ passes through $(3,\ 1)$,

$$1=9a+3b+c \qquad \cdots ③$$

From ①, ② and ③,

$$\begin{cases} \boldsymbol{a=-1} \\ \boldsymbol{b=2} \\ \boldsymbol{c=4} \end{cases} \qquad \begin{cases} \boldsymbol{a=-3} \\ \boldsymbol{b=6} \\ \boldsymbol{c=10} \end{cases}$$

K 49b

2. Given the quadratic function $y=x^2+ax+b$, the minimum value is -3, and the graph passes through point $(1,\ 1)$. Find the values of the constants a and b.

Hint Refer to K 47b

[Sol] $y=x^2+ax+b=\left(x+\dfrac{a}{2}\right)^2-\dfrac{a^2}{4}+b$

Since the minimum value is -3,

$$-\frac{a^2}{4}+b=-3 \quad \cdots①$$

Since the graph of $y=x^2+ax+b$ passes through $(1,\ 1)$,

$$1=1+a+b$$

$$b=-a \quad \cdots②$$

From ① and ②,

$$a^2+4a-12=0$$

$$(a+6)(a-2)=0$$

$$a=-6,\ 2$$

Therefore, $\begin{cases}\boldsymbol{a=-6}\\ \boldsymbol{b=6}\end{cases}$ $\begin{cases}\boldsymbol{a=2}\\ \boldsymbol{b=-2}\end{cases}$

Let's try this!

You can solve the question above using the following method:

Let $y=(x-k)^2-3$

$y=x^2-2kx+k^2-3$

Therefore, $a=-2k,\ b=k^2-3$

Since the graph passes through $(1,\ 1)$,

$1=1-2k+k^2-3$

$k^2-2k-3=0$

$(k-3)(k+1)=0$

Therefore, $k=3,\ -1$

When $k=3$,

$\begin{cases}a=\boxed{\mathbf{-6}}\\ b=\boxed{\mathbf{6}}\end{cases}$

When $k=-1$,

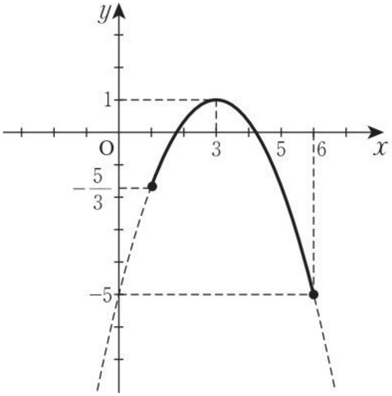
$\begin{cases}a=\boxed{\mathbf{2}}\\ b=\boxed{\mathbf{-2}}\end{cases}$

Answers: in order $\begin{cases}a=-6\\ b=6\end{cases}$, $\begin{cases}a=2\\ b=-2\end{cases}$

K 50a KUMON

Maxima and Minima of Quadratic Functions I

1. Given the quadratic function $y=-\frac{2}{3}x^2+4x-5\ (1\leqslant x\leqslant 6)$, find the maximum and minimum values.



[Sol]

$$y=-\frac{2}{3}(x-3)^2+1$$

Maximum value: 1, at $x=3$

Minimum value: -5, at $x=6$

2. Given the quadratic function $y=2x^2+bx+c$, the minimum value is -3, and the graph intersects the y-axis at point $(0,\ -1)$. Find the values of the constants b and c.

[Sol] $y=2x^2+bx+c$

$$=2\left(x+\frac{b}{4}\right)^2-\frac{b^2}{8}+c$$

Since the minimum value is -3,

$$-\frac{b^2}{8}+c=-3 \quad \cdots①$$

Since the graph passes through $(0,\ -1)$,

$$c=-1 \quad \cdots②$$

From ① and ②,

$$b^2=16$$

$$b=\pm 4$$

Therefore, $\begin{cases}\boldsymbol{b=4}\\ \boldsymbol{c=-1}\end{cases}$ $\begin{cases}\boldsymbol{b=-4}\\ \boldsymbol{c=-1}\end{cases}$

Alternative Solution

Let $y=2(x-k)^2-3$

$y=2x^2-4kx+2k^2-3$

Therefore, $b=-4k$, $c=2k^2-3$

Since the graph passes through $(0,\ -1)$,

$-1=2k^2-3$

$k^2=1$

$k=\pm 1$

When $k=1$, $\begin{cases}\boldsymbol{b=-4}\\ \boldsymbol{c=-1}\end{cases}$

When $k=-1$, $\begin{cases}\boldsymbol{b=4}\\ \boldsymbol{c=-1}\end{cases}$

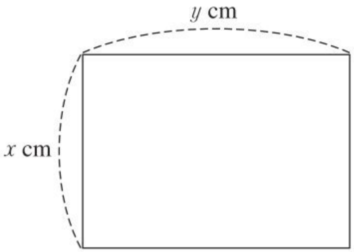
K 50b

Let's think about this!

Given a rectangle with perimeter 20 cm, what would the height and width have to be in order to maximize the rectangle's area, S?

[Sol]

As shown in the figure on the right, let the length be x cm, and let the width be y cm.



$0 < x < \boxed{10}$ …①

$0 < y < \boxed{10}$ …②

$x + y = 10$ …③

Lastly the area of the rectangle is

$S = xy$ …④

From ③ and ④, express S in terms of x,

$S = xy = x(10 - x) = -x^2 + 10x = -(x - 5)^2 + 25$ …⑤

Therefore, the maximum value of S is 25 when $x = 5$. This value of x satisfies ①. From ③, $y = 5$.

Accordingly, you can see that the height and width would be $\boxed{5}$ cm.

Let's try this!

We can also solve the above exercise using the ***arithmetic*** and ***geometric means***. (Refer to J 197.)

[Sol] $\dfrac{x+y}{2} \geqslant \sqrt{xy}$ (This becomes an equation when $x = y$.) …⑥

$x + y = 10$

Therefore, $x = y = \boxed{5}$

Answers: in order 10, 10, 5, 5

Further explanation

- The LHS of ⑥ has a fixed value here, as $x + y = 10$.
- The RHS of ⑥ is the square root of the area $S = xy$.

Substituting, we find $\dfrac{10}{2} \geqslant \sqrt{S}$, i.e. $S \leqslant 25$.

Also, this inequality becomes an equation when $x = y$.
Therefore, S has a maximum value of 25 when $x = y = 5$.

K 51 a **KUMON**

Maxima and Minima of Quadratic Functions II

1. For each value of a, find the minimum value of the function $f(x)=(x-5)^2+3$ when $a\leqslant x\leqslant a+4$.

Ex.

When $a=0$:

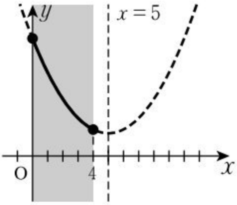
[Sol]

The domain becomes $0\leqslant x\leqslant 4$. ☜ Substituting $a=0$ into $a\leqslant x\leqslant a+4$.

From the graph, the minimum value is at $x=4$.

Therefore,

minimum value: $f(4)=(4-5)^2+3=4$



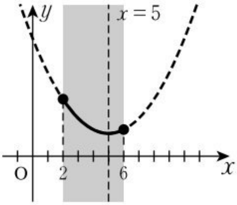
(1) When $a=2$:

[Sol] The domain becomes $2\leqslant x\leqslant 6$.

From the graph, the minimum value is at $x=5$.

Therefore,

minimum value: $\boldsymbol{f(5)}=(5-5)^2+3\boldsymbol{=3}$



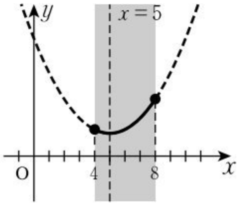
(2) When $a=4$:

[Sol] The domain becomes $4\leqslant x\leqslant 8$.

From the graph, the minimum value is at $x=5$.

Therefore,

minimum value: $\boldsymbol{f(5)}=(5-5)^2+3\boldsymbol{=3}$



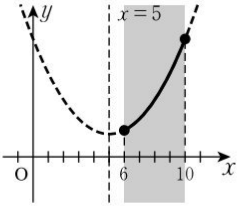
(3) When $a=6$:

[Sol] The domain becomes $6\leqslant x\leqslant 10$.

From the graph, the minimum value is at $x=6$.

Therefore,

minimum value: $\boldsymbol{f(6)}=(6-5)^2+3\boldsymbol{=4}$

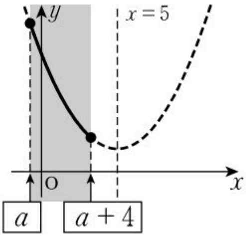


Note:
- y can be expressed as a function of x, as in $y=f(x)$ or $y=g(x)$. When $x=a$, the value of y is $f(a)$ or $g(a)$.
- For example, $y=(x-5)^2+3$ can be written as a function $f(x)=(x-5)^2+3$. When $x=2$, the value of y is $f(2)=(2-5)^2+3=12$.

K51b

2. The minimum value of the quadratic function $f(x)=(x-5)^2+3\ (a \leqslant x \leqslant a+4)$ varies, depending on the constant a. For each graph, find the value of x at which $f(x)$ has a minimum value, and then find the minimum value.

Ex.

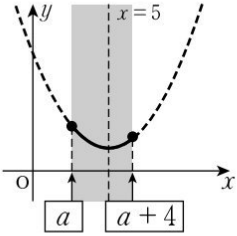


[Sol] At $x=a+4$, ☜ *The right end of the domain*

minimum value:

$$f(a+4)=(a+4-5)^2+3$$
$$=a^2-2a+4$$

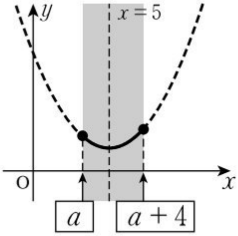
(1)



[Sol] At $x=5$, ☜ *The vertex*

minimum value: $f(\ \mathbf{5}\ )=\mathbf{3}$

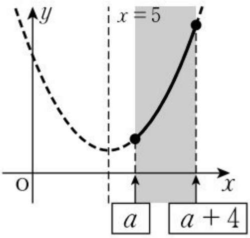
(2)



[Sol] At $x=\boxed{\mathbf{5}}$,

minimum value: $f(\ \mathbf{5}\ )=\mathbf{3}$

(3)



[Sol] At $x=a$, ☜ *The left end of the domain*

minimum value: $f(\ \boldsymbol{a}\ )=(a-5)^2+3$

$$=\boldsymbol{a^2-10a+28}$$

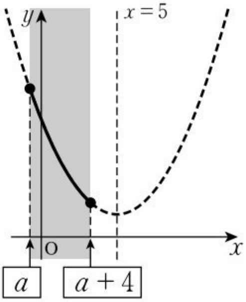
Note: Looking at the graphs above, we can see that the minimum value of a quadratic function is given either at the left end of the domain, or at the vertex, or at the right end of the domain.

K 52a KUMON

Maxima and Minima of Quadratic Functions II

1. Given the quadratic function $f(x)=(x-5)^2+3$ $(a \leqslant x \leqslant a+4)$, for each different graph (1)∼(5), find the corresponding range of values of a. Then find the minimum value.

(1)



[Sol] $a+4<5$ ☞ The right end of the domain < the axis of symmetry

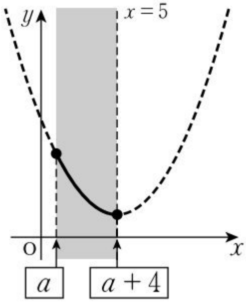
Therefore, $a<\boxed{\mathbf{1}}$

At $x=a+4$, ☞ The right end of the domain gives the minimum value.

minimum value: $f(a+4)=(a+4-5)^2+3$

$=\boldsymbol{a^2-2a+4}$

(2)



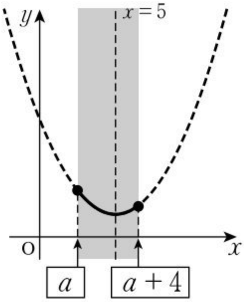
[Sol] $a+4=5$ ☞ The right end of the domain = the axis of symmetry

Therefore, $a=\boxed{\mathbf{1}}$

At $x=5$, ☞ The vertex gives the minimum value.

minimum value: $f(\ \mathbf{5}\ )=\mathbf{3}$

(3)



[Sol] $a<5<a+4$ ☞ The left end of the domain < the axis of symmetry < the right end of the domain

The inequality may be split into 2 parts:

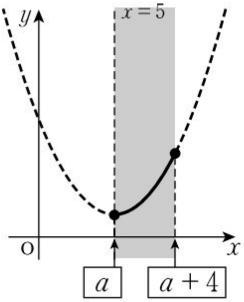
$$\begin{cases} a<5 \\ 5<a+4 \end{cases}$$

Therefore, $\boxed{\mathbf{1}}<a<\boxed{\mathbf{5}}$

At $x=\boxed{\mathbf{5}}$, ☞ The vertex gives the minimum value.

minimum value: $f(\ \mathbf{5}\ )=\mathbf{3}$

(4)

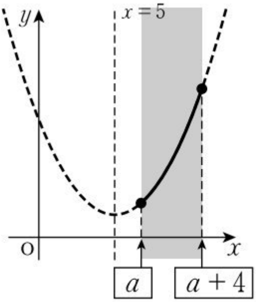


[Sol] $a = \boxed{\mathbf{5}}$ ☜ The axis of symmetry = the left end of the domain

At $x = \boxed{\mathbf{5}}$, ☜ The vertex gives the minimum value.

minimum value: $f(\ \mathbf{5}\ ) = \mathbf{3}$

(5)



[Sol] $a > \boxed{\mathbf{5}}$ ☜ The axis of symmetry < the left end of the domain

At $x = \boxed{\boldsymbol{a}}$, ☜ The left end of the domain gives the minimum value.

minimum value: $f(\ \boldsymbol{a}\ ) = (a-5)^2 + 3$

$$= \boldsymbol{a^2 - 10a + 28}$$

2. Fill in the blank boxes.

Looking at the results of (2), (3) and (4), these three situations give the same minimum value (the value at the vertex). Therefore, the range of a can be divided into three:

(i) $a < \boxed{\mathbf{1}}$ … The right end of the domain gives the minimum value.

(ii) $\boxed{\mathbf{1}} \leqslant a \leqslant \boxed{\mathbf{5}}$ … The vertex gives the minimum value. ☜ As in questions (2), (3) and (4)

(iii) $a > \boxed{\mathbf{5}}$ … The left end of the domain gives the minimum value.

K 53a KUMON

Maxima and Minima of Quadratic Functions II

Ex.

Given the quadratic function $f(x)=(x-5)^2+3\ \ (a\leqslant x\leqslant a+4)$, find the minimum value.

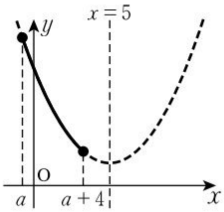
[Sol] The axis of symmetry is $x=5$.

(i) $a+4<\boxed{5}$ The right end of the domain < the axis of symmetry

Therefore, when $a<1$,

minimum value:

$$f(a+4)=(a+4-5)^2+3$$
$$=a^2-2a+4$$

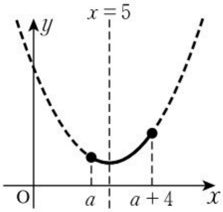


(ii) $a\leqslant\boxed{5}\leqslant a+4$ The left end of the domain ⩽ the axis of symmetry ⩽ the right end of the domain

This means $\begin{cases} a\leqslant\boxed{5} \\ \boxed{5}\leqslant a+4 \end{cases}$

Therefore, when $1\leqslant a\leqslant\boxed{5}$,

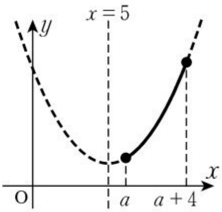
minimum value: $f(5)=3$



(iii) When $a>\boxed{5}$, The axis of symmetry < the left end of the domain

minimum value:

$$f(a)=(a-5)^2+3$$
$$=a^2-10a+28$$



Answers: All the answers are the same, 5.

K 53b

1. Given the quadratic function $f(x)=(x-5)^2+3\ (a\leqslant x\leqslant a+2)$, find the minimum value.

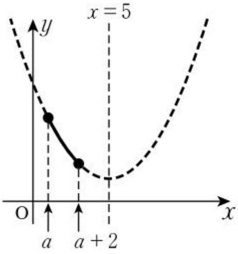
[Sol] The axis of symmetry is $\boxed{\boldsymbol{x=5}}$.

(i) $a+2<\boxed{\mathbf{5}}$

Therefore, when $\boxed{\boldsymbol{a<3}}$,

minimum value:

$$f(\ \boldsymbol{a+2}\ )=(a+2-5)^2+3$$
$$=\boldsymbol{a^2-6a+12}$$

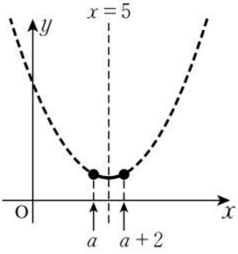


(ii) $a\leqslant 5\leqslant a+2$

This means $\begin{cases} a\leqslant 5 \\ 5\leqslant a+2 \end{cases}$

Therefore, **when $\boldsymbol{3\leqslant a\leqslant 5}$,**

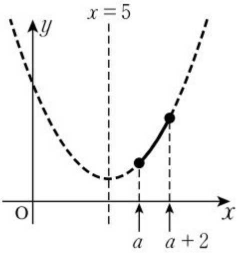
minimum value: $\boldsymbol{f(5)=3}$



(iii) **When $\boldsymbol{a>5}$,**

minimum value: $\boldsymbol{f(a)}=(a-5)^2+3$

$$\boldsymbol{=a^2-10a+28}$$



K 54a KUMON

Maxima and Minima of Quadratic Functions II

1. Given the quadratic function $f(x)=x^2-4x+3\ (a\leqslant x\leqslant a+1)$, find the minimum value.

[Sol] $f(x)=(x-2)^2-1$

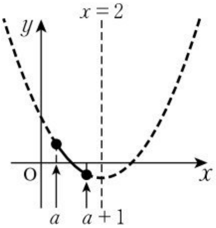
The axis of symmetry is $x=2$.

(i) $a+1<2$

Therefore, **when $\boldsymbol{a<1}$**,

minimum value: $\boldsymbol{f(a+1)}=(a+1-2)^2-1$

$\boldsymbol{=a^2-2a}$

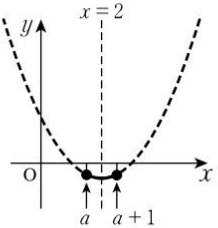


(ii) $a\leqslant 2\leqslant a+1$

This means $\begin{cases}a\leqslant 2\\2\leqslant a+1\end{cases}$

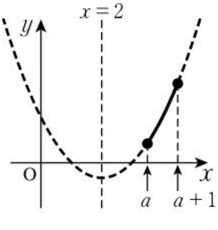
Therefore, **when $\boldsymbol{1\leqslant a\leqslant 2}$**,

minimum value: $\boldsymbol{f(2)=-1}$



(iii) **When $\boldsymbol{a>2}$**,

minimum value: $\boldsymbol{f(a)=a^2-4a+3}$



K 54b

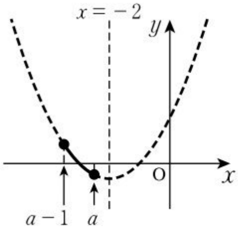
2. Given the quadratic function $f(x)=x^2+4x+3\ \ (a-1\leqq x\leqq a)$, find the minimum value.

[Sol] $f(x)=(x+2)^2-1$

The axis of symmetry is $x=-2$.

(i) **When $a<-2$,**

minimum value: $\boldsymbol{f(a)=a^2+4a+3}$

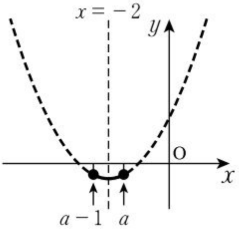


(ii) $a-1\leqq -2\leqq a$

This means $\begin{cases} a-1\leqq -2 \\ -2\leqq a \end{cases}$

Therefore, **when $-2\leqq a\leqq -1$,**

minimum value: $\boldsymbol{f(-2)=-1}$

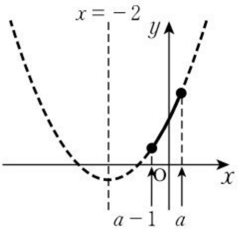


(iii) $-2<a-1$

Therefore, **when $a>-1$,**

minimum value: $\boldsymbol{f(a-1)}=(a-1+2)^2-1$

$$=\boldsymbol{a^2+2a}$$

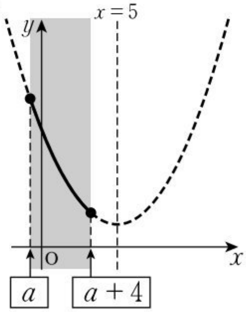


K 55a KUMON

Maxima and Minima of Quadratic Functions II

1. Given the quadratic function $f(x)=(x-5)^2+3\ (a \leqslant x \leqslant a+4)$, for each different graph (1)～(5), find the corresponding range of values of a. Then find the maximum value.

(1)



[Sol] $a+4<5$ ☜ The right end of the domain < the axis of symmetry

Therefore, $a< \boxed{\mathbf{1}}$

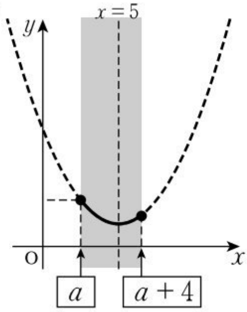
At $x=a$, ☜ The left end of the domain gives the maximum value.

maximum value:

$$f(a)=(a-5)^2+3$$

$$=\boldsymbol{a^2-10a+28}$$

(2)



[Sol] From the graph, $1 \leqslant a<3$ ☜ You can see this from (3).

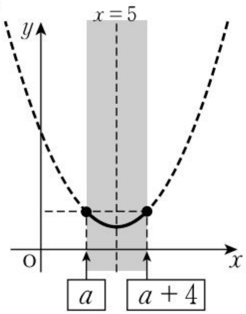
Thus, at $x=\boxed{\boldsymbol{a}}$, ☜ The left end of the domain gives the maximum value.

maximum value:

$$f(\ \boldsymbol{a}\ )=(a-5)^2+3$$

$$=\boldsymbol{a^2-10a+28}$$

(3)



[Sol] From $f(a)=f(a+4)$, ☜ Both ends of the domain give the maximum value.

$$a^2-10a+28=a^2-2a+4$$

Solving this, $a=3$

Also $a+4=\boxed{\mathbf{7}}$

Therefore,

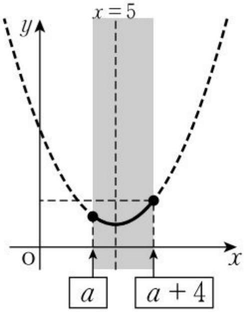
at $x=\boxed{\mathbf{3}}, \boxed{\mathbf{7}}$, ☜ Both ends of the domain give the maximum value.

maximum value:

$$f(\ \mathbf{3}\ )=f(\ \mathbf{7}\ )=(3-5)^2+3=\mathbf{7}$$

K 55b

(4)



[Sol] From the graph,

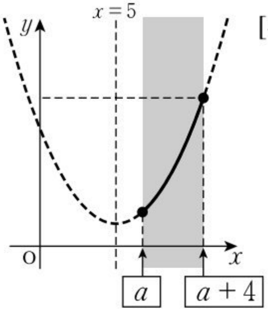
$\boxed{3} < a \leqq \boxed{5}$ ☜ You can see this from (3).

Thus, at $x = \boxed{a+4}$, ☜ The right end of the domain gives the maximum value.

maximum value:

$$f(\ \boldsymbol{a+4}\ ) = (a+4-5)^2+3$$
$$= \boldsymbol{a^2-2a+4}$$

(5)



[Sol] From the graph,

$a > \boxed{5}$

Thus, at $x = \boxed{a+4}$, ☜ The right end of the domain gives the maximum value.

maximum value:

$$f(\ \boldsymbol{a+4}\ ) = (a+4-5)^2+3$$
$$= \boldsymbol{a^2-2a+4}$$

2. Fill in the blank boxes.

The key is finding the particular value of a in (3). Summarizing (1)～(5), the range of a can be divided into three:

(i) $a < \boxed{3}$ … The left end of the domain gives the maximum value. ☜ As in questions (1) and (2)

(ii) $a = \boxed{3}$ … Both ends of the domain give the maximum value. ☜ As in question (3)

(iii) $a > \boxed{3}$ … The right end of the domain gives the maximum value. ☜ As in questions (4) and (5)

K 56a KUMON

Maxima and Minima of Quadratic Functions II

Ex.

Given the quadratic function $f(x)=(x-5)^2+3\ \ (a \leqslant x \leqslant a+4)$, find the maximum value.

[Sol] $f(a)=(a-5)^2+3=a^2-10a+28$

$f(a+4)=(a+4-5)^2+3=a^2-2a+4$

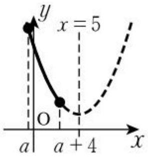
Setting $f(a)=f(a+4)$ to find the value of a,

$$a^2-10a+28=a^2-2a+4$$

$$a=3$$

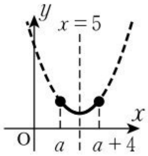
(i) When $a<\boxed{3}$, The left end of the domain gives the maximum value.

maximum value: $f(a)=a^2-10a+28$



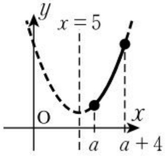
(ii) When $a=\boxed{3}$, Both ends of the domain give the maximum value.

maximum value: $f(3)=f(7)=7$



(iii) When $a>\boxed{3}$, The right end of the domain gives the maximum value.

maximum value: $f(a+4)=a^2-2a+4$



Answers: All the answers are the same, 3.

K 56b

1. Given the quadratic function $f(x)=(x-5)^2+3\ (a \leqq x \leqq a+2)$, find the maximum value.

[Sol] $f(a)=(a-5)^2+3=\boldsymbol{a^2-10a+28}$

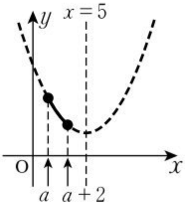
$f(\ \boldsymbol{a+2}\ )=(a+2-5)^2+3=\boldsymbol{a^2-6a+12}$

Setting $f(a)=f(\ \boldsymbol{a+2}\ )$ to find the value of a,

$$a^2-10a+28=a^2-6a+12$$
$$4a=16$$
$$a=4$$

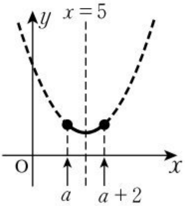
(i) When $a<\boxed{4}$,

maximum value: $f(\ \boldsymbol{a}\ )=\boldsymbol{a^2-10a+28}$



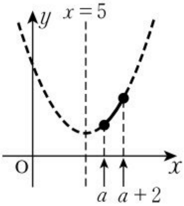
(ii) **When $\boldsymbol{a=4}$,**

maximum value: $\boldsymbol{f(4)=f(6)}=(4-5)^2+3\boldsymbol{=4}$



(iii) **When $\boldsymbol{a>4}$,**

maximum value: $\boldsymbol{f(a+2)=a^2-6a+12}$



K 57a KUMON

Maxima and Minima of Quadratic Functions II

1. Given the quadratic function $f(x)=x^2-4x+3$ $(a \leqslant x \leqslant a+1)$, find the maximum value.

[Sol] $f(x)=(x-2)^2-1$

$f(a)=a^2-4a+3$

$f(a+1)=(a+1-2)^2-1=a^2-2a$

Setting $f(a)=f(a+1)$ to find the value of a,

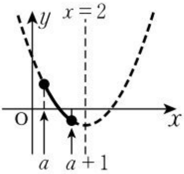
$$a^2-4a+3=a^2-2a$$

$$2a=3$$

$$a=\frac{3}{2}$$

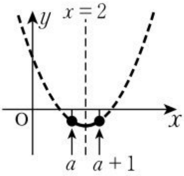
(i) **When $\boldsymbol{a<\frac{3}{2}}$,**

maximum value: $\boldsymbol{f(a)=a^2-4a+3}$



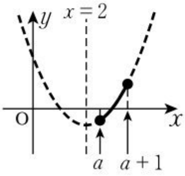
(ii) **When $\boldsymbol{a=\frac{3}{2}}$,**

maximum value: $\boldsymbol{f\left(\frac{3}{2}\right)=f\left(\frac{5}{2}\right)=-\frac{3}{4}}$



(iii) **When $\boldsymbol{a>\frac{3}{2}}$,**

maximum value: $\boldsymbol{f(a+1)=a^2-2a}$



K 57b

2. Given the quadratic function $f(x)=x^2+4x+3\ \ (a-1 \leqslant x \leqslant a)$, find the maximum value.

[Sol] $f(x)=(x+2)^2-1$

$f(a-1)=(a-1+2)^2-1=a^2+2a$

$f(a)=a^2+4a+3$

Setting $f(a-1)=f(a)$ to find the value of a,

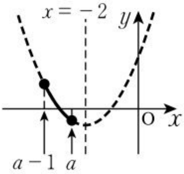
$$a^2+2a=a^2+4a+3$$

$$2a=-3$$

$$a=-\frac{3}{2}$$

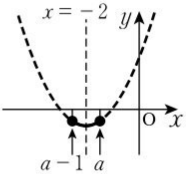
(i) **When $\boldsymbol{a<-\frac{3}{2}}$,**

maximum value: $\boldsymbol{f(a-1)=a^2+2a}$



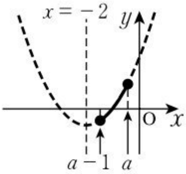
(ii) **When $\boldsymbol{a=-\frac{3}{2}}$,**

maximum value: $\boldsymbol{f\left(-\frac{5}{2}\right)=f\left(-\frac{3}{2}\right)=-\frac{3}{4}}$



(iii) **When $\boldsymbol{a>-\frac{3}{2}}$,**

maximum value: $\boldsymbol{f(a)=a^2+4a+3}$



K 58a KUMON

Maxima and Minima of Quadratic Functions II

1. Given the quadratic function $f(x) = -x^2 + 10x - 28$ $(a \leqslant x \leqslant a+4)$, find the maximum value.

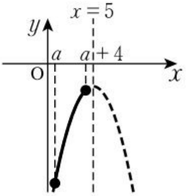
[Sol] $f(x) = \boldsymbol{-(x-5)^2 - 3}$ *Hint*

Therefore, the axis of symmetry is $\boxed{\boldsymbol{x=5}}$.

(i) When $a+4<5$, i.e. **when $a<1$,**

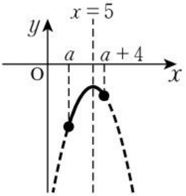
maximum value: $\boldsymbol{f(a+4)} = -(a+4-5)^2-3$

$= \boldsymbol{-a^2+2a-4}$



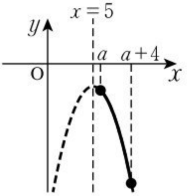
(ii) **When $1 \leqslant a \leqslant 5$,**

maximum value: $\boldsymbol{f(5) = -3}$



(iii) **When $a>5$,**

maximum value: $\boldsymbol{f(a) = -a^2+10a-28}$



Hint

As on K 53, consider the three conditions "the left end of the domain gives the maximum value", "the vertex gives the maximum value" and "the right end of the domain gives the maximum value".

K 58b

2. Given the quadratic function $f(x)=-x^2+10x-28\ (a\leqslant x\leqslant a+4)$, find the minimum value.

[Sol] $f(x)=-(x-5)^2-3$

$f(a)=-a^2+10a-28$

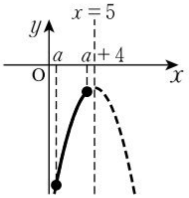
$f(a+4)=-(a+4-5)^2-3=-a^2+2a-4$

Setting $f(a)=f(a+4)$ to find the value of a,

$$-a^2+10a-28=-a^2+2a-4$$
$$8a=24$$
$$a=3$$

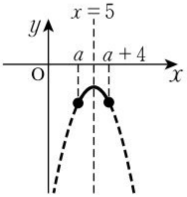
(i) **When $a<3$,**

minimum value: $\boldsymbol{f(a)=-a^2+10a-28}$



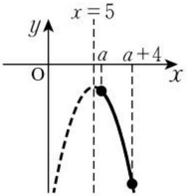
(ii) **When $a=3$,**

minimum value: $\boldsymbol{f(3)=f(7)=-7}$



(iii) **When $a>3$,**

minimum value: $\boldsymbol{f(a+4)=-a^2+2a-4}$



K 59a KUMON

Maxima and Minima of Quadratic Functions II

1. Given the quadratic function $f(x)=(x-5)^2+3\ (a\leqslant x\leqslant a+4)$, find the maximum and minimum values.

[Sol] The axis of symmetry is $x=\boxed{5}$.

$$f(a)=(a-5)^2+3=a^2-10a+28$$
$$f(a+4)=(a+4-5)^2+3=a^2-2a+4$$

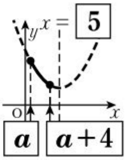
Setting $f(a)=f(a+4)$ to find the values of a and $a+4$,

from $a^2-10a+28=a^2-2a+4$, we find $a=\boxed{3}$ and $a+4=\boxed{7}$.

(i) When $a+4<\boxed{5}$, i.e. when $a<\boxed{1}$,

maximum value: $f(a)=a^2-10a+28$

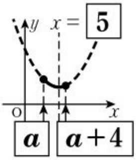
minimum value: $f(a+4)=a^2-2a+4$



(ii) When $\boxed{1}\leqslant a<\boxed{3}$,

maximum value: $f(a)=a^2-10a+28$

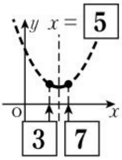
minimum value: $f(\boxed{5})=3$



(iii) When $a=\boxed{3}$,

maximum value: $f(\boxed{3})=f(\boxed{7})=7$

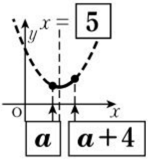
minimum value: $f(\boxed{5})=3$



(iv) When $\boxed{3}<a\leqslant\boxed{5}$,

maximum value: $f(a+4)=a^2-2a+4$

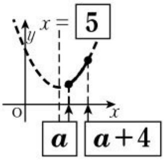
minimum value: $f(\boxed{5})=3$



(v) When $a>\boxed{5}$,

maximum value: $f(a+4)=a^2-2a+4$

minimum value: $f(a)=a^2-10a+28$



2. Given the quadratic function $f(x)=-(x-3)^2-1\ (a\leqq x\leqq a+2)$, find the maximum and minimum values.

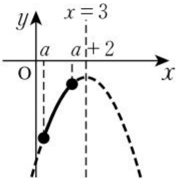
[Sol] The axis of symmetry is $x=\boxed{\mathbf{3}}$.

$$f(a)=-(a-3)^2-1=-a^2+6a-10$$

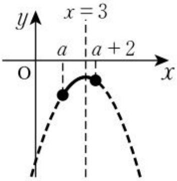
$$f(a+2)=-(a+2-3)^2-1=-a^2+2a-2$$

Setting $f(a)=f(a+2)$ to find the values of a and $a+2$,
from $-a^2+6a-10=-a^2+2a-2$,
we find $a=2$ and $a+2=4$.

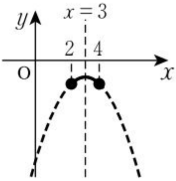
(i) When $\boxed{\boldsymbol{a+2<3}}$, i.e. when $\boxed{\boldsymbol{a<1}}$,
maximum value: $f(\ \boldsymbol{a+2}\ )=\boldsymbol{-a^2+2a-2}$
minimum value: $f(\ \boldsymbol{a}\ )=\boldsymbol{-a^2+6a-10}$



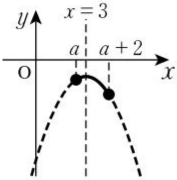
(ii) **When $\boldsymbol{1\leqq a<2}$,**
maximum value: $\boldsymbol{f(3)=-1}$
minimum value: $\boldsymbol{f(a)=-a^2+6a-10}$



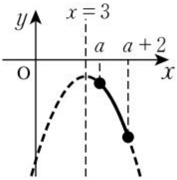
(iii) **When $\boldsymbol{a=2}$,**
maximum value: $\boldsymbol{f(3)=-1}$
minimum value: $\boldsymbol{f(2)=f(4)=-2}$



(iv) **When $\boldsymbol{2<a\leqq 3}$,**
maximum value: $\boldsymbol{f(3)=-1}$
minimum value: $\boldsymbol{f(a+2)=-a^2+2a-2}$



(v) **When $\boldsymbol{a>3}$,**
maximum value: $\boldsymbol{f(a)=-a^2+6a-10}$
minimum value: $\boldsymbol{f(a+2)=-a^2+2a-2}$



K 60a KUMON

Maxima and Minima of Quadratic Functions II

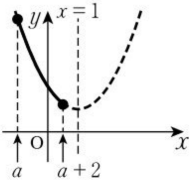
1. Given the quadratic function $f(x)=(x-1)^2+3\ (a\leqslant x\leqslant a+2)$, find the maximum and minimum values.

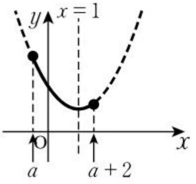
[Sol] The axis of symmetry is $x=1$.

$$f(a)=(a-1)^2+3=a^2-2a+4$$
$$f(a+2)=(a+2-1)^2+3=a^2+2a+4$$

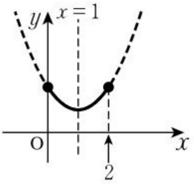
Setting $f(a)=f(a+2)$ to find the values of a and $a+2$,
from $a^2-2a+4=a^2+2a+4$,
we find $a=0$ and $a+2=2$.

(i) When $a+2<1$, i.e. **when $a<-1$,**
maximum value: $f(a)=a^2-2a+4$
minimum value: $f(a+2)=a^2+2a+4$

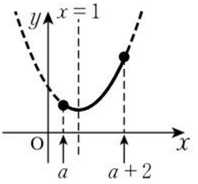




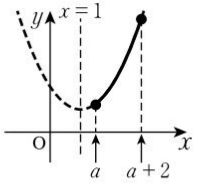
(ii) **When $-1\leqslant a<0$,**
maximum value: $f(a)=a^2-2a+4$
minimum value: $f(1)=3$



(iii) **When $a=0$,**
maximum value: $f(0)=f(2)=4$
minimum value: $f(1)=3$



(iv) **When $0<a\leqslant 1$,**
maximum value: $f(a+2)=a^2+2a+4$
minimum value: $f(1)=3$



(v) **When $a>1$,**
maximum value: $f(a+2)=a^2+2a+4$
minimum value: $f(a)=a^2-2a+4$

K 60b

Let's try this!

A parabola is symmetric with respect to the axis of symmetry.
Given the quadratic function $y=(x-5)^2+3\ (a\leqslant x\leqslant a+4)$, find the maximum value.

[Sol] First, find the centre of the domain.

$$x=\frac{a+(a+4)}{2}=a+2 \quad ☞ \quad \frac{\text{left end } + \text{ right end}}{2}$$

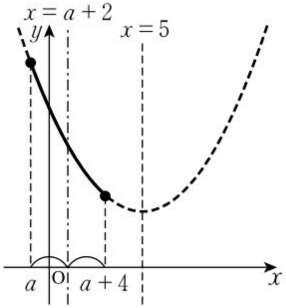
The axis of symmetry is $x=5$.

(i) **When the centre of the domain < the axis of symmetry,**

$\boxed{a+2}<5$, i.e. when $a<3$,

the maximum value is:

$$f(a)=(a-5)^2+3$$
$$=a^2-10a+28$$

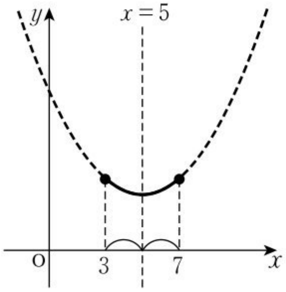


(ii) **When the centre of the domain = the axis of symmetry,**

$\boxed{a+2}=5$, i.e. when $a=3$,

the maximum value is:

$$f(3)=f(7)=7$$

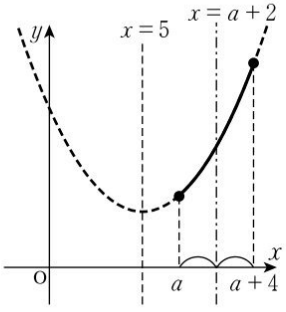


(iii) **When the centre of the domain > the axis of symmetry,**

$\boxed{a+2}>5$, i.e. when $a>3$,

the maximum value is:

$$f(a+4)=(a+4-5)^2+3$$
$$=a^2-2a+4$$



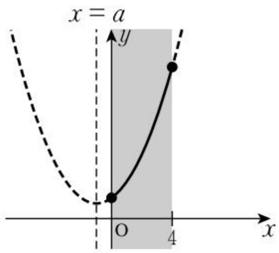
Compare these results with the example on K 56a.
Note that all the answers are the same!

Answers: All the answers are the same, $a+2$.

Maxima and Minima of Quadratic Functions III

1. Given the quadratic function $f(x)=(x-a)^2+3\ (0\leqslant x\leqslant 4)$, for each different graph (1)~(5), find the corresponding range of a. Then find the minimum value.

(1)

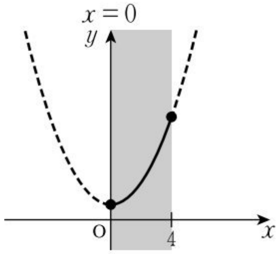


$a<\boxed{\mathbf{0}}$ ☜ The axis of symmetry $<$ the left end of the domain

At $x=\boxed{\mathbf{0}}$, ☜ The left end of the domain gives the minimum value.

minimum value: $f(\ \mathbf{0}\ )=(0-a)^2+3=\boldsymbol{a^2+3}$

(2)

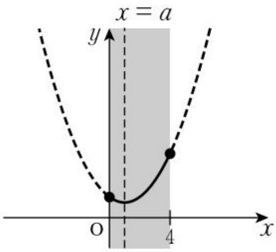


$a=\boxed{\mathbf{0}}$ ☜ The axis of symmetry $=$ the left end of the domain

At $x=\boxed{\mathbf{0}}$, ☜ The vertex gives the minimum value.

minimum value: $f(\ \mathbf{0}\ )=(0-0)^2+3=\mathbf{3}$

(3)



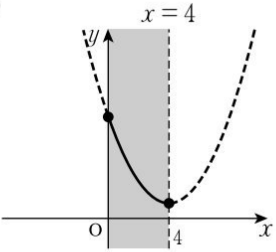
$\boxed{\mathbf{0}}<a<\boxed{\mathbf{4}}$ ☜ The left end of the domain $<$ the axis of symmetry $<$ the right end of the domain

At $x=\boxed{\boldsymbol{a}}$, ☜ The vertex gives the minimum value.

minimum value: $f(\ \boldsymbol{a}\ )=(a-a)^2+3=\mathbf{3}$

K61b

(4)

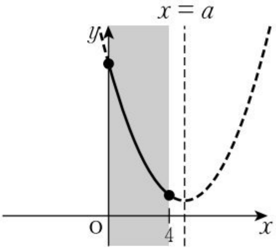


$a = \boxed{\mathbf{4}}$ ☞ The axis of symmetry = the right end of the domain

At $x = \boxed{\mathbf{4}}$, ☞ The vertex gives the minimum value.

minimum value: $f(\ \mathbf{4}\ ) = (4-4)^2 + 3 = \mathbf{3}$

(5)



$a > \boxed{\mathbf{4}}$ ☞ The axis of symmetry > the right end of the domain

At $x = \boxed{\mathbf{4}}$, ☞ The right end of the domain gives the minimum value.

minimum value: $f(\ \mathbf{4}\ ) = (4-a)^2 + 3$

$$= \boldsymbol{a}^2 - \mathbf{8}\boldsymbol{a} + \mathbf{19}$$

2. Fill in the blank boxes.

Looking at (2), (3) and (4), these three situations give the same minimum value (the value at the vertex). Therefore, the range of a can be divided into three:

(i) $a < \boxed{\mathbf{0}}$ … The left end of the domain gives the minimum value.

(ii) $\boxed{\mathbf{0}} \leqslant a \leqslant \boxed{\mathbf{4}}$ … The vertex gives the minimum value. ☞ As in questions (2), (3) and (4)

(iii) $a > \boxed{\mathbf{4}}$ … The right end of the domain gives the minimum value.

K 62a

KUMON

Maxima and Minima of Quadratic Functions III

Ex.

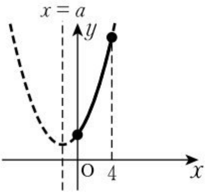
Given the quadratic function $f(x)=(x-a)^2+3\ (0 \leqslant x \leqslant 4)$, find the minimum value.

[Sol] The axis of symmetry is $x=a$.

(i) When $a<\boxed{0}$, ☞ The axis of symmetry < the left end of the domain

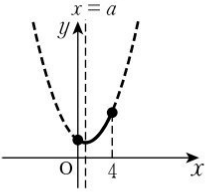
the minimum value is $f(0)=(0-a)^2+3$

$$=a^2+3$$



(ii) When $\boxed{0} \leqslant a \leqslant \boxed{4}$, ☞ The left end of the domain ⩽ the axis of symmetry ⩽ the right end of the domain

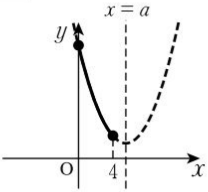
the minimum value is $f(a)=3$



(iii) When $a>\boxed{4}$, ☞ The axis of symmetry > the right end of the domain

the minimum value is $f(4)=(4-a)^2+3$

$$=a^2-8a+19$$



Answers: (i) 0 (ii) 0, 4 (iii) 4

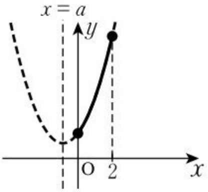
K 62b

1. Given the quadratic function $f(x)=(x-a)^2+1\ (0\leqslant x\leqslant 2)$, find the minimum value.

[Sol] The axis of symmetry is $\boxed{\boldsymbol{x=a}}$.

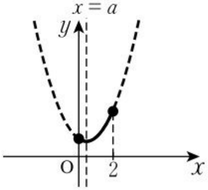
(i) When $a<\boxed{\mathbf{0}}$,

the minimum value is $f(\ \mathbf{0}\ )=\boldsymbol{a}^2+\mathbf{1}$



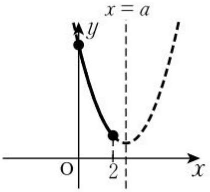
(ii) **When $\mathbf{0}\leqslant \boldsymbol{a}\leqslant \mathbf{2}$,**

the minimum value is $\boldsymbol{f(a)=1}$



(iii) **When $\boldsymbol{a}>\mathbf{2}$,**

the minimum value is $\boldsymbol{f(2)=a^2-4a+5}$



K 63a KUMON

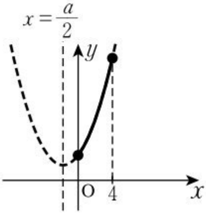
Maxima and Minima of Quadratic Functions III

1. Given the quadratic function $f(x)=\left(x-\frac{a}{2}\right)^2+3\ (0\leqslant x\leqslant 4)$, find the minimum value.

[Sol] The axis of symmetry is $x=\frac{a}{2}$.

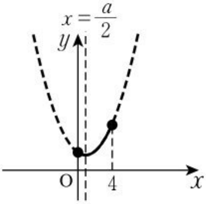
(i) When $\frac{a}{2}<0$, i.e. when $a<0$,

the minimum value is $f(\ \mathbf{0}\ )=\boldsymbol{\frac{a^2}{4}+3}$



(ii) When $0\leqslant\frac{a}{2}\leqslant 4$, i.e. **when** $\boldsymbol{0\leqslant a\leqslant 8}$,

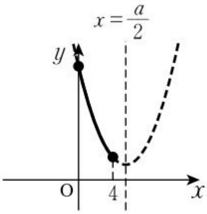
the minimum value is $\boldsymbol{f\left(\frac{a}{2}\right)=3}$



(iii) When $\frac{a}{2}>4$, i.e. **when** $\boldsymbol{a>8}$,

the minimum value is $\boldsymbol{f(4)}=\left(4-\frac{a}{2}\right)^2+3$

$$=\boldsymbol{\frac{a^2}{4}-4a+19}$$



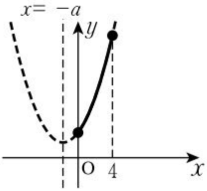
K 63b

2. Given the quadratic function $f(x)=(x+a)^2+3\ (0 \leqslant x \leqslant 4)$, find the minimum value.

[Sol] The axis of symmetry is $x=-a$.

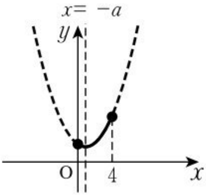
(i) When $-a<0$, i.e. when $a>\boxed{0}$,

the minimum value is $\boldsymbol{f(0)=a^2+3}$



(ii) When $0 \leqslant -a \leqslant 4$, i.e. **when** $\boldsymbol{-4 \leqslant a \leqslant 0}$,

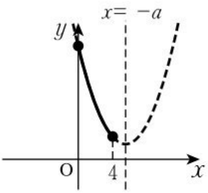
the minimum value is $\boldsymbol{f(-a)=3}$



(iii) When $-a>4$, i.e. **when** $\boldsymbol{a<-4}$,

the minimum value is $\boldsymbol{f(4)}=(4+a)^2+3$

$\boldsymbol{=a^2+8a+19}$



K 64a

KUMON

Maxima and Minima of Quadratic Functions III

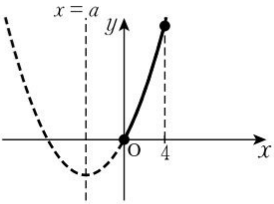
1. Given the quadratic function $f(x)=x^2-2ax\ (0\leqslant x\leqslant 4)$, find the minimum value.

[Sol] $f(x)=x^2-2ax=\boxed{(x-a)^2-a^2}$

Therefore, the axis of symmetry is $x=\boxed{a}$.

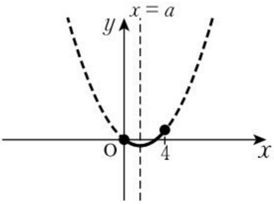
(i) **When $a<0$,**

the minimum value is $f(0)=0$



(ii) **When $0\leqslant a\leqslant 4$,**

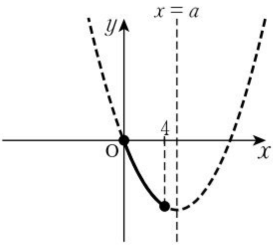
the minimum value is $f(a)=-a^2$



(iii) **When $a>4$,**

the minimum value is $f(4)=4^2-2a\times 4$

$= -8a+16$



K 64b

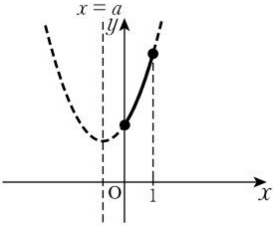
2. Given the quadratic function $f(x)=x^2-2ax+2\ (0\leqslant x\leqslant 1)$, find the minimum value.

[Sol] $f(x)=(x-a)^2-a^2+2$

Therefore, the axis of symmetry is $x=a$.

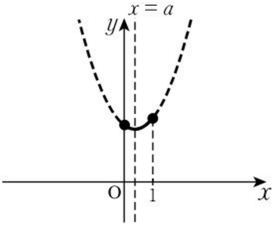
(i) **When $\boldsymbol{a<0}$,**

the minimum value is $\boldsymbol{f(0)=2}$



(ii) **When $\boldsymbol{0\leqslant a\leqslant 1}$,**

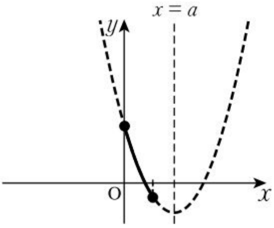
the minimum value is $\boldsymbol{f(a)=-a^2+2}$



(iii) **When $\boldsymbol{a>1}$,**

the minimum value is $\boldsymbol{f(1)}=1^2-2a\times 1+2$

$$\boldsymbol{=-2a+3}$$



K 65a KUMON

Maxima and Minima of Quadratic Functions III

Ex.

Given the quadratic function $f(x)=(x-a)^2+3\ (0 \leqslant x \leqslant 4)$, find the maximum value.

[Sol] $f(0)=(0-a)^2+3=a^2+3$ ☜ The value at the left end of the domain

$f(4)=(4-a)^2+3=a^2-8a+19$ ☜ The value at the right end of the domain

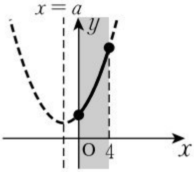
Setting $f(0)=f(4)$ to find the value of a,

$a^2+3=a^2-8a+19$

$a=2$

(i) When $a<\boxed{2}$, ☜ The right end of the domain gives the maximum value.

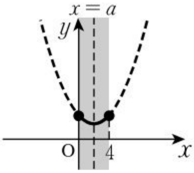
the maximum value is $f(4)=a^2-8a+19$



(ii) When $a=\boxed{2}$, ☜ Both ends of the domain give the maximum value.

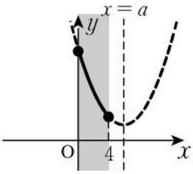
the maximum value is

$f(0)=f(4)=(0-2)^2+3=7$



(iii) When $a>\boxed{2}$, ☜ The left end of the domain gives the maximum value.

the maximum value is $f(0)=a^2+3$



Answers: All the answers are the same, 2.

K 65b

1. Given the quadratic function $f(x)=(x-a)^2+3\ (0\leqslant x\leqslant 2)$, find the maximum value.

[Sol] $f(0)=(0-a)^2+3=a^2+3$

$f(\ \mathbf{2}\ )=(2-a)^2+3=\boldsymbol{a^2-4a+7}$

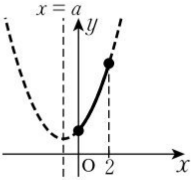
Setting $f(0)=f(\ \mathbf{2}\ )$ to find the value of a,

$a^2+3=\boldsymbol{a^2-4a+7}$

$\boldsymbol{a=1}$

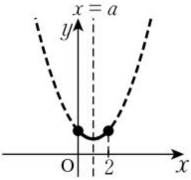
(i) When $a<\boxed{1}$,

the maximum value is $f(\ \mathbf{2}\ )=\boldsymbol{a^2-4a+7}$



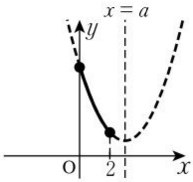
(ii) **When $\boldsymbol{a=1}$,**

the maximum value is $\boldsymbol{f(0)=f(2)}=(0-1)^2+3\ \mathbf{=4}$



(iii) **When $\boldsymbol{a>1}$,**

the maximum value is $\boldsymbol{f(0)=a^2+3}$



Maxima and Minima of Quadratic Functions III

1. Given the quadratic function $f(x)=\left(x-\frac{a}{2}\right)^2+3\ (0\leqslant x\leqslant 4)$, find the maximum value.

[Sol] $f(\ \mathbf{0}\ )=\left(0-\frac{a}{2}\right)^2+3=\frac{\boldsymbol{a}^2}{\mathbf{4}}+\mathbf{3}$

$f(\ \mathbf{4}\ )=\left(4-\frac{a}{2}\right)^2+3=\frac{\boldsymbol{a}^2}{\mathbf{4}}-\mathbf{4}\boldsymbol{a}+\mathbf{19}$

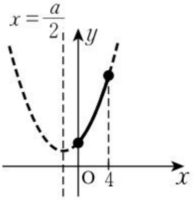
Setting $f(0)=f(4)$ to find the value of a,

$$\frac{a^2}{4}+3=\frac{a^2}{4}-4a+19$$

$$a=4$$

(i) **When $\boldsymbol{a}<4$,**

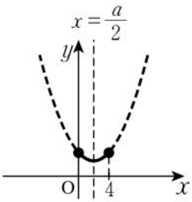
the maximum value is $\boldsymbol{f}(\mathbf{4})=\frac{\boldsymbol{a}^2}{\mathbf{4}}-\mathbf{4}\boldsymbol{a}+\mathbf{19}$



(ii) **When $\boldsymbol{a}=4$,**

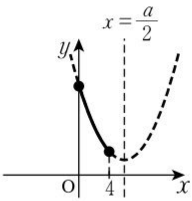
the maximum value is

$\boldsymbol{f}(\mathbf{0})=\boldsymbol{f}(\mathbf{4})=\left(0-\frac{4}{2}\right)^2+3=\mathbf{7}$



(iii) **When $\boldsymbol{a}>4$,**

the maximum value is $\boldsymbol{f}(\mathbf{0})=\frac{\boldsymbol{a}^2}{\mathbf{4}}+\mathbf{3}$



K 66b

2. Given the quadratic function $f(x)=(x+a)^2+3\ (0 \leqslant x \leqslant 4)$, find the maximum value.

[Sol] $f(0)=(0+a)^2+3=a^2+3$

$f(4)=(4+a)^2+3=a^2+8a+19$

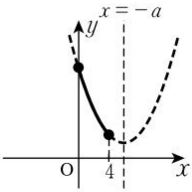
Setting $f(0)=f(4)$ to find the value of a,

$$a^2+3=a^2+8a+19$$

$$a=-2$$

(i) When $a<\boxed{\mathbf{-2}}$,

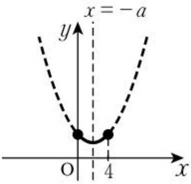
the maximum value is $\boldsymbol{f(0)=a^2+3}$



(ii) **When $\boldsymbol{a=-2}$,**

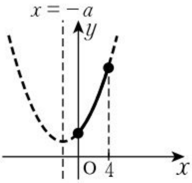
the maximum value is

$\boldsymbol{f(0)=f(4)}=(0-2)^2+3\boldsymbol{=7}$



(iii) **When $\boldsymbol{a>-2}$,**

the maximum value is $\boldsymbol{f(4)=a^2+8a+19}$



K 67a KUMON

Maxima and Minima of Quadratic Functions III

1. Given the quadratic function $f(x)=x^2-2ax\ (0 \leqslant x \leqslant 4)$, find the maximum value.

[Sol] $f(x)=(x-a)^2-a^2$

$f(0)=0$

$f(4)=4^2-2a\times 4=-8a+16$

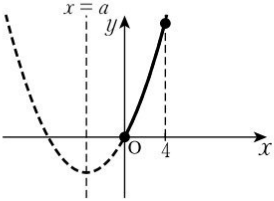
Setting $f(0)=f(4)$ to find the value of a,

$0=-8a+16$

$a=2$

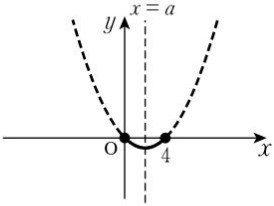
(i) **When $a<2$,**

the maximum value is $\boldsymbol{f(4)=-8a+16}$



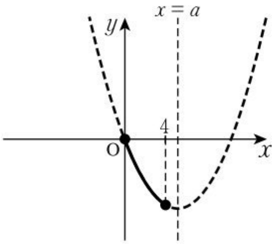
(ii) **When $a=2$,**

the maximum value is $\boldsymbol{f(0)=f(4)=0}$



(iii) **When $a>2$,**

the maximum value is $\boldsymbol{f(0)=0}$



K 67b

2. Given the quadratic function $f(x) = x^2 - 2ax + 2\ (0 \leqslant x \leqslant 1)$, find the maximum value.

[Sol] $f(x) = (x-a)^2 - a^2 + 2$

$$f(0) = 2$$

$$f(1) = 1^2 - 2a \times 1 + 2 = -2a + 3$$

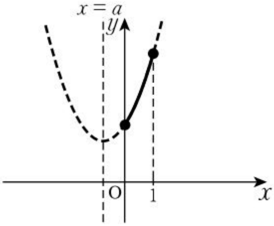
Setting $f(0) = f(1)$ to find the value of a,

$$2 = -2a + 3$$

$$a = \frac{1}{2}$$

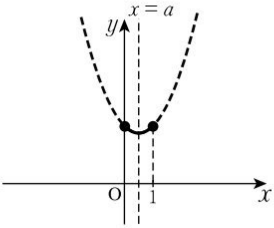
(i) **When $\boldsymbol{a < \frac{1}{2}}$,**

the maximum value is $\boldsymbol{f(1) = -2a + 3}$



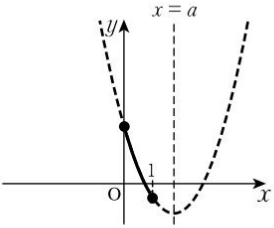
(ii) **When $\boldsymbol{a = \frac{1}{2}}$,**

the maximum value is $\boldsymbol{f(0) = f(1) = 2}$



(iii) **When $\boldsymbol{a > \frac{1}{2}}$,**

the maximum value is $\boldsymbol{f(0) = 2}$



K 68a KUMON

Maxima and Minima of Quadratic Functions III

1. Given the quadratic function $f(x) = -x^2 + 2ax - 2\ (0 \leqslant x \leqslant 1)$, find the maximum value.

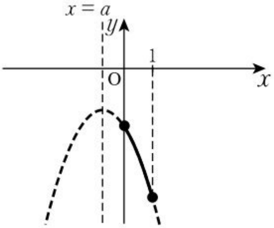
[Sol] $f(x) = -(x^2 - 2ax) - 2$

$= -(x-a)^2 + a^2 - 2$

The axis of symmetry is $x = a$.

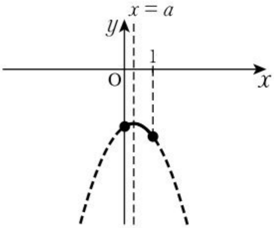
(i) **When $a < 0$,**

the maximum value is $\boldsymbol{f(0) = -2}$



(ii) **When $0 \leqslant a \leqslant 1$,**

the maximum value is $\boldsymbol{f(a) = a^2 - 2}$

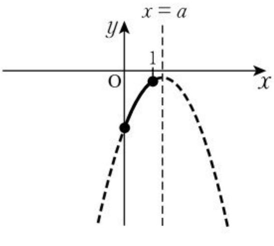


(iii) **When $a > 1$,**

the maximum value is

$\boldsymbol{f(1)} = -1^2 + 2a \times 1 - 2$

$= \boldsymbol{2a - 3}$



K 68b

2. Given the quadratic function $f(x) = -x^2 + 2ax - 2$ $(0 \leqslant x \leqslant 1)$, find the minimum value.

[Sol] $f(x) = -(x-a)^2 + a^2 - 2$

$$f(0) = -2$$

$$f(1) = -1^2 + 2a \times 1 - 2 = 2a - 3$$

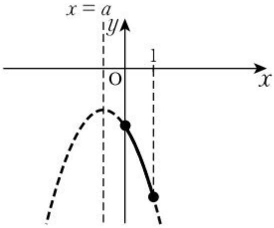
Setting $f(0) = f(1)$ to find the value of a,

$$-2 = 2a - 3$$

$$a = \frac{1}{2}$$

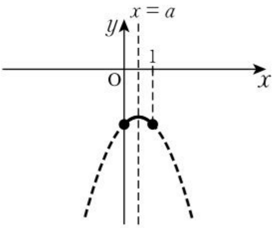
(i) **When $a < \frac{1}{2}$,**

the minimum value is $\boldsymbol{f(1) = 2a - 3}$



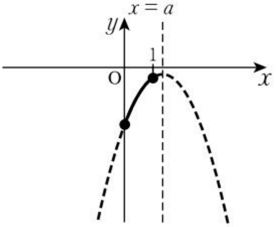
(ii) **When $a = \frac{1}{2}$,**

the minimum value is $\boldsymbol{f(0) = f(1) = -2}$



(iii) **When $a > \frac{1}{2}$,**

the minimum value is $\boldsymbol{f(0) = -2}$



K 69a KUMON

Maxima and Minima of Quadratic Functions III

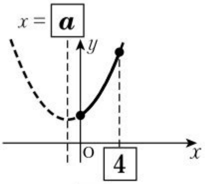
1. Given the quadratic function $f(x) = (x-a)^2 + 3\ (0 \leqslant x \leqslant 4)$, find the maximum and minimum values.

[Sol] The axis of symmetry is $x = \boxed{a}$.

$$f(0) = (0-a)^2 + 3 = a^2 + 3$$
$$f(4) = (4-a)^2 + 3 = a^2 - 8a + 19$$

Setting $f(0) = f(4)$ to find the value of a,

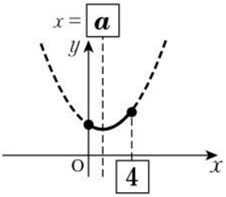
$$a^2 + 3 = a^2 - 8a + 19$$
$$a = \boxed{2}$$



(i) When $a < \boxed{0}$,

the maximum value is $f(4) = a^2 - 8a + 19$

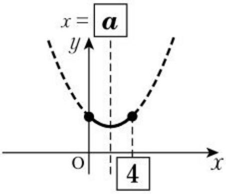
the minimum value is $f(0) = a^2 + 3$



(ii) When $\boxed{0} \leqslant a < \boxed{2}$,

the maximum value is $f(4) = a^2 - 8a + 19$

the minimum value is $f(\boxed{a}) = 3$

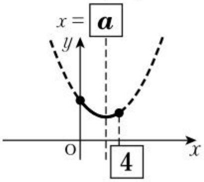


(iii) When $a = \boxed{2}$,

the maximum value is $f(\boxed{0}) = f(\boxed{4}) = 7$

the minimum value is $f(\boxed{a}) = 3$

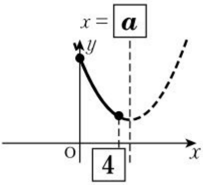
〔$f(\boxed{2}) = 3$〕



(iv) When $\boxed{2} < a \leqslant \boxed{4}$,

the maximum value is $f(0) = a^2 + 3$

the minimum value is $f(\boxed{a}) = 3$



(v) When $a > \boxed{4}$,

the maximum value is $f(0) = a^2 + 3$

the minimum value is $f(4) = a^2 - 8a + 19$

K 69b

2. Given the quadratic function $f(x) = -(x-a)^2 - 1\ (0 \leqslant x \leqslant 2)$, find the maximum and minimum values.

[Sol] The axis of symmetry is $x = a$.

$f(0) = -(0-a)^2 - 1 = -a^2 - 1$

$f(2) = -(2-a)^2 - 1 = -a^2 + 4a - 5$

Setting $f(0) = f(2)$ to find the value of a,

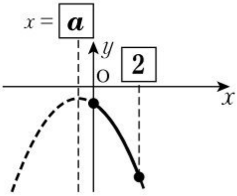
$-a^2 - 1 = -a^2 + 4a - 5$

$a = 1$

(i) When $\boxed{a < 0}$,

the maximum value is $f(\ \mathbf{0}\ ) = \boldsymbol{-a^2 - 1}$

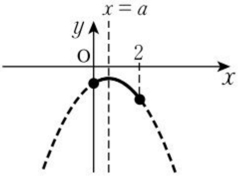
the minimum value is $f(\ \mathbf{2}\ ) = \boldsymbol{-a^2 + 4a - 5}$



(ii) **When $0 \leqslant a < 1$,**

the maximum value is $f(a) = -1$

the minimum value is $f(2) = -a^2 + 4a - 5$

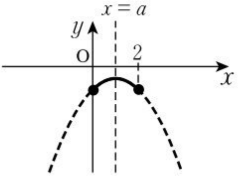


(iii) **When $a = 1$,**

the maximum value is $f(a) = -1$ 〔$f(1) = -1$〕

the minimum value is $f(0) = f(2) = -1^2 - 1$

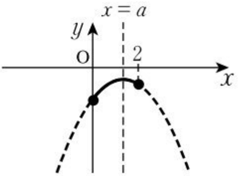
$= -2$



(iv) **When $1 < a \leqslant 2$,**

the maximum value is $f(a) = -1$

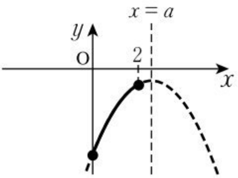
the minimum value is $f(0) = -a^2 - 1$



(v) **When $a > 2$,**

the maximum value is $f(2) = -a^2 + 4a - 5$

the minimum value is $f(0) = -a^2 - 1$



K 70a KUMON

Maxima and Minima of Quadratic Functions III

1. Given the quadratic function $f(x)=(x-a)^2+2\ (0 \leqslant x \leqslant 1)$, find the maximum and minimum values.

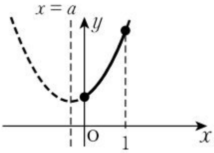
[Sol] The axis of symmetry is $x=a$.

$$f(0)=(0-a)^2+2=a^2+2$$
$$f(1)=(1-a)^2+2=a^2-2a+3$$

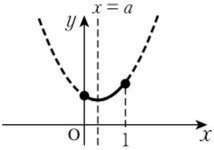
Setting $f(0)=f(1)$ to find the value of a,

$$a^2+2=a^2-2a+3$$
$$a=\frac{1}{2}$$

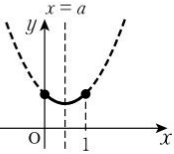
(i) **When $a<0$,**
the maximum value is $f(1)=a^2-2a+3$
the minimum value is $f(0)=a^2+2$



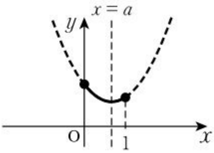
(ii) **When $0 \leqslant a < \frac{1}{2}$,**
the maximum value is $f(1)=a^2-2a+3$
the minimum value is $f(a)=2$



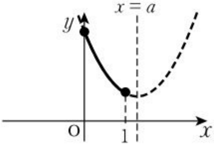
(iii) **When $a=\frac{1}{2}$,**
the maximum value is $f(0)=f(1)=\left(\frac{1}{2}\right)^2+2=\frac{9}{4}$
the minimum value is $f(a)=2$ $\left[f\left(\frac{1}{2}\right)=2\right]$



(iv) **When $\frac{1}{2}<a \leqslant 1$,**
the maximum value is $f(0)=a^2+2$
the minimum value is $f(a)=2$



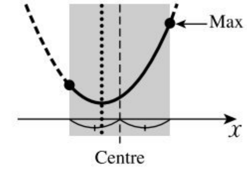
(v) **When $a>1$,**
the maximum value is $f(0)=a^2+2$
the minimum value is $f(1)=a^2-2a+3$



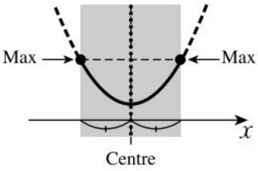
K 70b

Consider this!

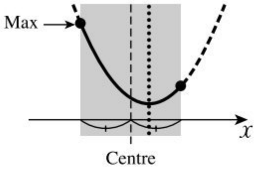
Since a parabola is symmetric with respect to the axis of symmetry:



When the axis of symmetry is to the left of the centre of the domain, the maximum value is given at the right end of the domain.



When the axis of symmetry coincides with the centre of the domain, the maximum value is given at both ends of the domain.



When the axis of symmetry is to the right of the centre of the domain, the maximum value is given at the left end of the domain.

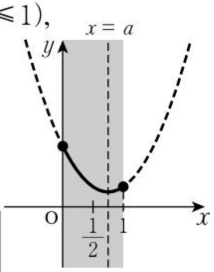
Let's try this!

Using the above hints, try the following.

Given the quadratic function $f(x)=x^2-2ax+2\ (0 \leqslant x \leqslant 1)$, find the maximum value. Assume $\frac{1}{2}<a<1$.

[Sol] $f(x)=(x-a)^2-a^2+2$

Therefore, the axis of symmetry is $x=a$.



The centre of the domain is $x=\frac{\boxed{\mathbf{0}}+\boxed{\mathbf{1}}}{2}=\boxed{\mathbf{\frac{1}{2}}}$

Since the axis of symmetry is to the right of the centre of the domain, the maximum value is given at the left end of the domain.

Therefore, the maximum value is $\boxed{\mathbf{2}}$, at $x=0$.

Answers: in order 0, 1, $\frac{1}{2}$, 2

K71a KUMON

Quadratic Functions and Equations

For each quadratic function, find ① the vertex, ② the y-intercept, ③ the graph of the function, ④ the number of ***common points*** with the x-axis and ⑤ the sign of the discriminant, *D*, when $y=0$.

Ex.

$y=x^2-2x-3$

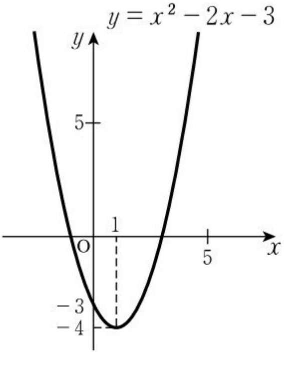
[Sol]

① $y=(x-1)^2-4$
Vertex: $(1, -4)$

② The y-intercept is $(0, -3)$.

Substituting $x=0$.

③



④ From the graph, there are 2 common points with the x-axis.

⑤ $D=(-2)^2-4\times1\times(-3)$
$=16>0$

(1) $y=x^2-2x$

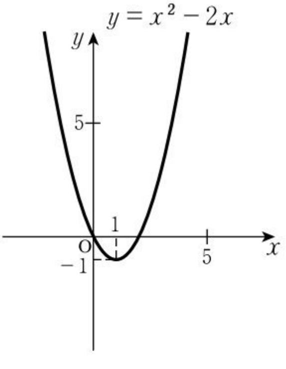
[Sol]

① $y=(x-1)^2-1$
Vertex: (**1** , **−1**)

② The y-intercept is (**0** , **0**).

Substituting $x=0$.

③



④ From the graph, there is/are $\boxed{\mathbf{2}}$ common point(s) with the x-axis.

⑤ $D=(-2)^2-4\times1\times0$
$=\mathbf{4>0}$

- A point where a graph touches another graph (or an axis) is called a ***point of contact***.
- A point where a graph intersects or has a point of contact with another graph (or an axis) is called a ***common point***.

(2) $y = x^2 - 2x + 1$

[Sol]

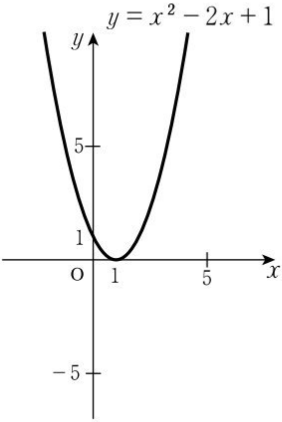
① $y = (x-1)^2$

Vertex: (**1** , **0**)

② The y-intercept ☜ is (**0** , **1**).

Substituting $x = 0$.

③



④ From the graph, there is/are $\boxed{\mathbf{1}}$ common point(s) with the x-axis.

⑤ $D = (-2)^2 - 4 \times 1 \times 1 = \mathbf{0}$

(3) $y = -x^2 + 2x + 3$

[Sol]

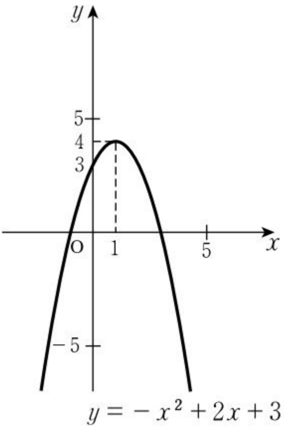
① $y = -(x-1)^2 + 4$

Vertex: (**1** , **4**)

② The y-intercept ☜ is (**0** , **3**).

Substituting $x = 0$.

③



④ From the graph, there is/are $\boxed{\mathbf{2}}$ common point(s) with the x-axis.

⑤ $D = 2^2 - 4 \times (-1) \times 3 = \mathbf{16 > 0}$

When the graph has 2 common points with the x-axis, $D > 0$.
When it has 1 common point, $D = 0$.

K 72a KUMON

Quadratic Functions and Equations

For each quadratic function, find ① the vertex, ② the y-intercept, ③ the graph of the function, ④ the number of common points with the x-axis and ⑤ the sign of the discriminant, D, when $y=0$.

Ex.

$y=x^2-2x+3$

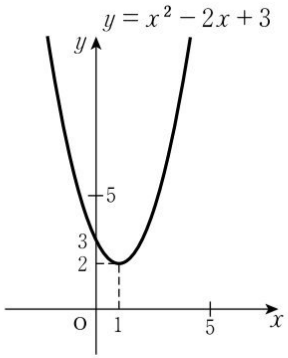
[Sol]

① $y=(x-1)^2+2$

Vertex: $(1, 2)$

② The y-intercept is $(0, 3)$. ☞ Substituting $x=0$.

③



④ From the graph, there are 0 common points with the x-axis.

⑤ $D=(-2)^2-4\times1\times3=-8<0$

(1) $y=x^2+4x+5$

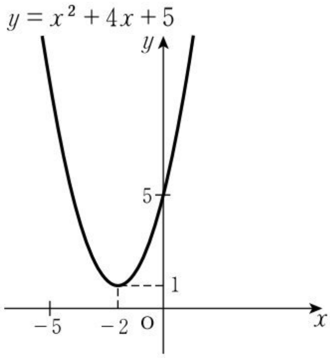
[Sol]

① $y=(x+2)^2+1$

Vertex: ($\mathbf{-2}$, **1**)

② The y-intercept is (**0** , **5**).

③



④ From the graph, there is/are $\boxed{\mathbf{0}}$ common point(s) with the x-axis.

⑤ $D=4^2-4\times1\times5=\mathbf{-4<0}$

When there are 0 common points with the x-axis, $D<0$.

K72b

(2)　$y=-2x^2+4x-2$

[Sol]

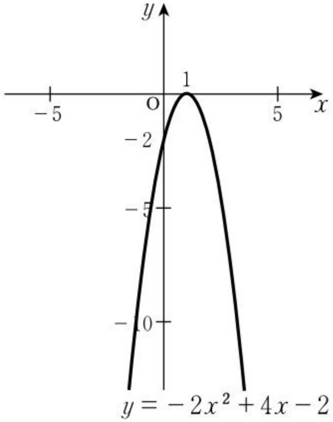
① $y=-2(x-1)^2$

Vertex: (**1** , **0**)

② The y-intercept

is (**0** , **−2**).

③



④ From the graph, there is/are $\boxed{\mathbf{1}}$ common point(s) with the x-axis.

⑤ $D=4^2-4\times(-2)\times(-2)=\mathbf{0}$

(3)　$y=-2x^2-8x-9$

[Sol]

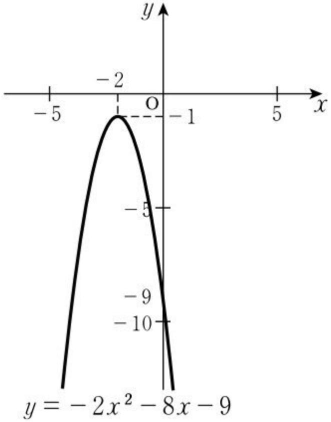
① $y=-2(x+2)^2-1$

Vertex: (**−2** , **−1**)

② The y-intercept

is (**0** , **−9**).

③



④ From the graph, there is/are $\boxed{\mathbf{0}}$ common point(s) with the x-axis.

⑤ $D=(-8)^2-4\times(-2)\times(-9)$

$=\mathbf{-8<0}$

Note Summary

Given a quadratic function $y=ax^2+bx+c$, when the number of common points with the x-axis is:

2, then $D=b^2-4ac>0$
1, then $D=b^2-4ac=0$
0, then $D=b^2-4ac<0$

K 73a KUMON

Quadratic Functions and Equations

1. For each quadratic function, find the number of common points that the graph has with the x-axis.

Ex.

$y=2x^2-3x-1$

[Sol] $D=(-3)^2-4\times2\times(-1)=17>0$ There are 2 common points.

(1) $y=x^2+x-2$

[Sol] $D=1^2-4\times1\times(-2)=9>0$ There is/are $\boxed{2}$ common point(s).

(2) $y=2x^2-4x$

[Sol] $D=(-4)^2-4\times2\times0=16>0$ There is/are $\boxed{2}$ common point(s).

(3) $y=2x^2-4x+2$

[Sol] $D=(-4)^2-4\times2\times2=0$ There is/are $\boxed{1}$ common point(s).

(4) $y=2x^2-4x+3$

[Sol] $D=(-4)^2-4\times2\times3=-8<0$ There is/are $\boxed{0}$ common point(s).

(5) $y=-3x^2+6x-3$

[Sol] $D=6^2-4\times(-3)\times(-3)=0$ There is/are $\boxed{1}$ common point(s).

(6) $y=-3x^2+6x-2$

[Sol] $D=6^2-4\times(-3)\times(-2)=12>0$ There is/are $\boxed{2}$ common point(s).

(7) $y=-3x^2+6x-4$

[Sol] $D=6^2-4\times(-3)\times(-4)=-12<0$ There is/are $\boxed{0}$ common point(s).

If the coefficient of the x-term is even, we can calculate the discriminant as $\frac{D}{4}$.

i.e. if $ax^2+2b'x+c=0$, then $\frac{D}{4}=b'^2-ac$.

K 73b

Ex.

Given the quadratic function $y = 2x^2 + 3x + k$, find the range of k for which the graph has 2 common points with the x-axis.

[Sol] $D = 3^2 - 4 \times 2 \times k = 9 - 8k > 0$ ☜ The discriminant must be positive.

$$8k < 9$$

Therefore, $k < \dfrac{9}{8}$

2. Given the quadratic function $y = 2x^2 - x + k$, find the range of k for which:

(1) The graph has 2 common points with the x-axis.

[Sol] $D = (-1)^2 - 4 \times 2 \times k = \boldsymbol{1 - 8k > 0}$

$$8k < 1$$

Therefore, $\boldsymbol{k < \dfrac{1}{8}}$

(2) The graph has 1 common point with the x-axis. ☜ The discriminant equals 0.

[Sol] $D = 1 - 8k = 0$

$$8k = 1$$

Therefore, $\boldsymbol{k = \dfrac{1}{8}}$

(3) The graph does not have any common points with the x-axis. ☜ The discriminant must be negative.

[Sol] $D = 1 - 8k < 0$

$$8k > 1$$

Therefore, $\boldsymbol{k > \dfrac{1}{8}}$

Quadratic Functions and Equations

Ex.

Given the quadratic function $y=2x^2-x+k$, find the number of common points the graph has with the x-axis depending on the value of k.

[Sol] $D=(-1)^2-4\times2\times k=1-8k$

Therefore,

When $1-8k>0$, i.e. when $k<\frac{1}{8}$, ☞ $D>0$

there are 2 common points.

When $1-8k=0$, i.e. when $k=\frac{1}{8}$, ☞ $D=0$

there is 1 common point.

When $1-8k<0$, i.e. when $k>\frac{1}{8}$, ☞ $D<0$

there are 0 common points.

1. Given the quadratic function $y=x^2+3x+k$, find the number of common points the graph has with the x-axis.

[Sol] $D=3^2-4\times1\times k=9-4k$

When $9-4k>0$, i.e. **when $k<\frac{9}{4}$, there are 2 common points**.

When $9-4k=0$, i.e. **when $k=\frac{9}{4}$, there is 1 common point**.

When $9-4k<0$, i.e. **when $k>\frac{9}{4}$, there are 0 common points**.

K 74b

2. Given the quadratic function $y=-2x^2+6x+k$, find the number of common points the graph has with the x-axis.

   (Here, it is easier to calculate the discriminant as $\frac{D}{4}$.)

[Sol] $\frac{D}{4}=3^2-(-2)\times k=9+2k$

When $9+2k>0$, i.e. **when $k>-\frac{9}{2}$, there are 2 common points**.

When $9+2k=0$, i.e. **when $k=-\frac{9}{2}$, there is 1 common point**.

When $9+2k<0$, i.e. **when $k<-\frac{9}{2}$, there are 0 common points**.

3. Given the quadratic function $y=2x^2-6x+(k+1)$, find the number of common points the graph has with the x-axis.

[Sol] $\frac{D}{4}=(-3)^2-2\times(k+1)=7-2k$

When $7-2k>0$, i.e. **when $k<\frac{7}{2}$, there are 2 common points**.

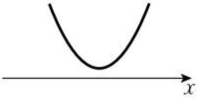
When $7-2k=0$, i.e. **when $k=\frac{7}{2}$, there is 1 common point**.

When $7-2k<0$, i.e. **when $k>\frac{7}{2}$, there are 0 common points**.

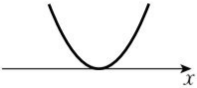
Quadratic Functions and Equations

The relationship between the graph of a quadratic function and the x-axis can be classified as follows.

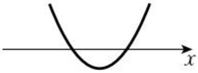
(A) Above the x-axis but not touching:



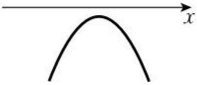
(B) Above the x-axis and touching it:



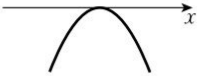
(C) Crosses the x-axis with open top (vertex below):



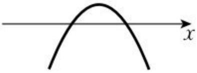
(D) Below the x-axis but not touching:



(E) Below the x-axis and touching it:



(F) Crosses the x-axis with open bottom (vertex above):



1. Classify the following graphs of quadratic functions as A, B, C, D, E or F by considering the coefficient of x^2 and the sign of the discriminant.

Ex.

$y=x^2+3x-1$ (**C**)

[Sol] The coefficient of x^2 is positive.
The discriminant, $D=13>0$.

(1) $y=x^2+x+1$ (**A**)

[Sol] The coefficient of x^2 is positive.
The discriminant, $D=-3<0$.

(2) $y=-3x^2+4x+5$ (**F**)

[Sol] The coefficient of x^2 is negative.
The discriminant, $\frac{D}{4}=19>0$.

(3) $y=2x^2-12x+18$ (**B**)

[Sol] The coefficient of x^2 is positive.
The discriminant, $\frac{D}{4}=0$.

(4) $y=-4x^2+4x-1$ (**E**)

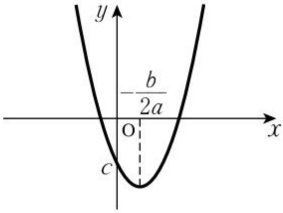
[Sol] The coefficient of x^2 is negative.
The discriminant, $\frac{D}{4}=0$.

(5) $y=-x^2+x-1$ (**D**)

[Sol] The coefficient of x^2 is negative.
The discriminant, $D=-3<0$.

2. Given the quadratic function $y = ax^2 + bx + c$, fill in the signs $(+, -, 0)$ of a, b, c and the discriminant, D, for each of the graphs below.

Ex.



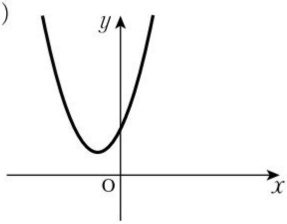
[Sol] a ⋯ $\boxed{+}$ ☜ The parabola opens upward.

b ⋯ $\boxed{-}$ ☜ From $a > 0$ and $-\dfrac{b}{2a} > 0$

c ⋯ $\boxed{-}$ ☜ From the y-intercept.

D ⋯ $\boxed{+}$ ☜ The parabola crosses the x-axis at 2 points.

(1)



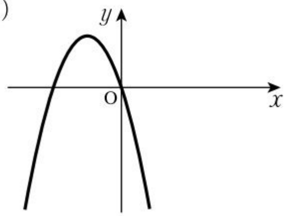
[Sol] a ⋯ $\boxed{+}$

b ⋯ $\boxed{+}$

c ⋯ $\boxed{+}$

D ⋯ $\boxed{-}$

(2)



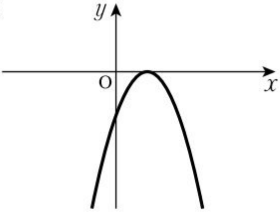
[Sol] a ⋯ $\boxed{-}$

b ⋯ $\boxed{-}$

c ⋯ $\boxed{0}$

D ⋯ $\boxed{+}$

(3)



[Sol] a ⋯ $\boxed{-}$

b ⋯ $\boxed{+}$

c ⋯ $\boxed{-}$

D ⋯ $\boxed{0}$

Note:
To determine the sign of a, consider the shape of the parabola:

- If it opens upwards, a is positive.
- If it opens downwards, a is negative.

To determine the sign of b, consider the position of the vertex:

- If it is to the right of the y-axis, i.e. $-\dfrac{b}{2a}$ is positive, then a and b have opposite signs.
- If it is to the left of the y-axis, i.e. $-\dfrac{b}{2a}$ is negative, then a and b have the same sign.
- If it is on the y-axis, then $b = 0$.

To determine the sign of c, consider the y-intercept.

K 76a KUMON

Quadratic Functions and Equations

For each set of functions, find the common points, and check your answer by drawing the graphs. If there are no common points write “No common points”.

Ex.

$y=x^2-4x+5,\quad y=2x$

[Sol]

Let $(x,\ y)$ be the coordinates of the common points.

$$\begin{cases} y=x^2-4x+5 & \cdots① \\ y=2x & \cdots② \end{cases}$$

From ① and ②,

$x^2-4x+5=2x$

$x^2-6x+5=0$

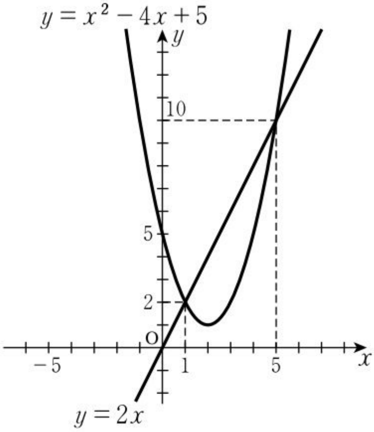
$(x-5)(x-1)=0$

$x=5,\ 1$

From ②, when $x=5$, $y=10$

From ②, when $x=1$, $y=2$

Therefore, the common points are $(5,\ 10),\ (1,\ 2)$.



(1) $y=x^2-4x+5,\quad y=2x-3$

[Sol] Let $(x,\ y)$ be the coordinates of the common points.

$$\begin{cases} y=x^2-4x+5 & \cdots① \\ y=2x-3 & \cdots② \end{cases}$$

From ① and ②,

$x^2-4x+5=2x-3$

$x^2-6x+8=0$

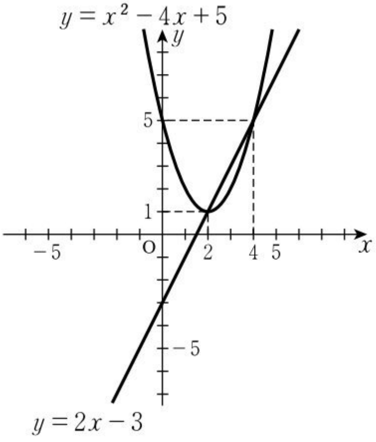
$(x-4)(x-2)=0$

$x=4,\ 2$

From ②, when $x=4$, $y=5$

From ②, when $x=2$, $y=1$

Therefore, the common points are **(4, 5), (2, 1)**.



K76b

（2） $y=x^2-4x+5,\ y=2x-4$

[Sol] Let $(x,\ y)$ be the coordinates of the common points.

$$\begin{cases} y=x^2-4x+5 & \cdots① \\ y=2x-4 & \cdots② \end{cases}$$

From ① and ②,

$$x^2-4x+5=2x-4$$

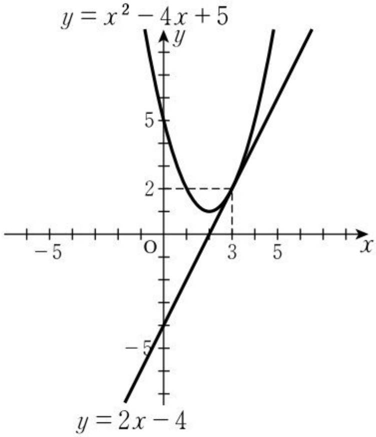
$$x^2-6x+9=0$$

$$(x-3)^2=0$$

$$x=3$$

From ②, when $x=3$, $y=2$

Therefore, the common point is $\mathbf{(3,\ 2)}$.



（3） $y=x^2-4x+5,\ y=2x-5$

[Sol] Let $(x,\ y)$ be the coordinates of the common points.

$$\begin{cases} y=x^2-4x+5 & \cdots① \\ y=2x-5 & \cdots② \end{cases}$$

From ① and ②,

$$x^2-4x+5=\boxed{\mathbf{2x-5}}$$

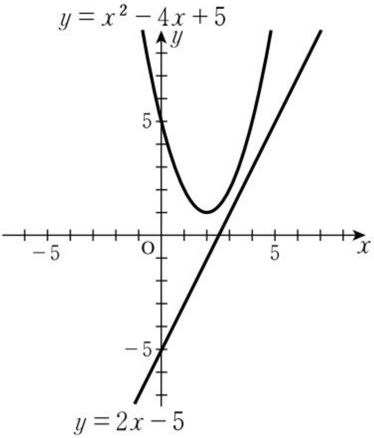
$$\boxed{\mathbf{x^2-6x+10}}=0 \quad \cdots③$$

Calculating the discriminant of ③,

$$\frac{D}{4}=\boxed{\mathbf{(-3)^2-10}}=\boxed{\mathbf{-1}}<0$$

Therefore, ③ has no solutions.

Ans. **No common points**



Note: Calculating *D*, the discriminant of a quadratic function, the number of common points is:

2 when $D>0$, 1 when $D=0$, 0 when $D<0$

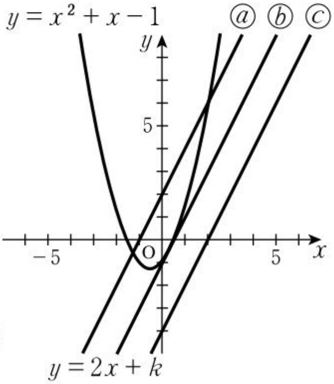
K 77a KUMON

Quadratic Functions and Equations

Ex.

Given the quadratic function $y = x^2 + x - 1$ and the linear function $y = 2x + k$, find the number of common points depending on the value of k.

[Sol]



$$\begin{cases} y = x^2 + x - 1 & \cdots ① \\ y = 2x + k & \cdots ② \end{cases}$$

From ① and ②,

$$x^2 + x - 1 = 2x + k$$
$$x^2 - x - 1 - k = 0 \quad \cdots ③$$

Calculating the discriminant, D, of ③,

$$D = (-1)^2 - 4 \times 1 \times (-1 - k) = 4k + 5$$

When $D > 0$, i.e. when $k > -\frac{5}{4}$,

there is/are $\boxed{2}$ common point(s). ☞ See graph ⓐ.

When $D = 0$, i.e. when $k = -\frac{5}{4}$,

there is/are $\boxed{1}$ common point(s). ☞ See graph ⓑ.

When $D < 0$, i.e. when $k < -\frac{5}{4}$,

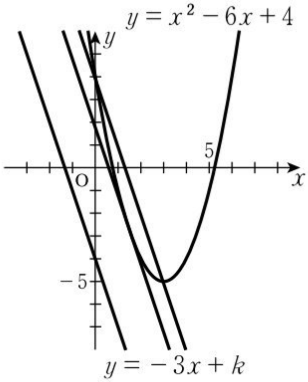
there is/are $\boxed{0}$ common point(s). ☞ See graph ⓒ.

Answers: in order 2, 1, 0

K 77b

1. Given the quadratic function $y = x^2 - 6x + 4$ and the linear function $y = -3x + k$, find the number of common points depending on the value of k.

[Sol]



$$\begin{cases} \boxed{\boldsymbol{y = x^2 - 6x + 4}} & \cdots ① \\ \boxed{\boldsymbol{y = -3x + k}} & \cdots ② \end{cases}$$

From ① and ②,

$$x^2 - 6x + 4 = -3x + k$$

$$\boxed{\boldsymbol{x^2 - 3x - k + 4 = 0}} \quad \cdots ③$$

Calculating the discriminant, D, of ③,

$$D = (-3)^2 - 4 \times 1 \times (-k + 4) = \boldsymbol{4k - 7}$$

When $\boxed{\boldsymbol{D > 0}}$, i.e. when $\boxed{\boldsymbol{k > \frac{7}{4}}}$, there is/are $\boxed{\mathbf{2}}$ common point(s).

When $\boxed{\boldsymbol{D = 0}}$, i.e. when $\boxed{\boldsymbol{k = \frac{7}{4}}}$, there is/are $\boxed{\mathbf{1}}$ common point(s).

When $\boxed{\boldsymbol{D < 0}}$, i.e. when $\boxed{\boldsymbol{k < \frac{7}{4}}}$, there is/are $\boxed{\mathbf{0}}$ common point(s).

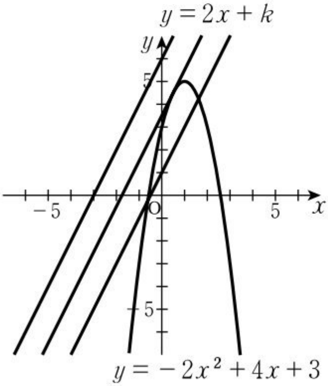
K 78a KUMON

Quadratic Functions and Equations

1. Given the quadratic function $y=-2x^2+4x+3$ and the linear function $y=2x+k$, find the number of common points depending on the value of k.

[Sol]

$$\begin{cases} y=-2x^2+4x+3 & \cdots① \\ y=2x+k & \cdots② \end{cases}$$



From ① and ②,

$$-2x^2+4x+3=2x+k$$

$$2x^2-2x+k-3=0 \quad \cdots③$$

Calculating the discriminant, D, of ③,

$$\frac{D}{4}=(-1)^2-2\times(k-3)=-2k+7$$

When $D>0$, i.e. **when $k<\frac{7}{2}$, there are 2 common points**.

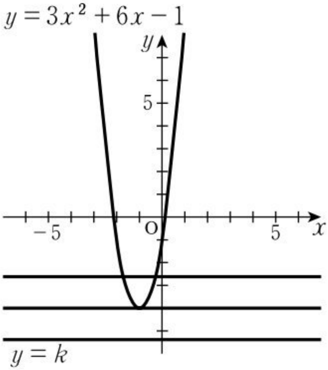
When $D=0$, i.e. **when $k=\frac{7}{2}$, there is 1 common point**.

When $D<0$, i.e. **when $k>\frac{7}{2}$, there are 0 common points**.

K 78b

2. Find the number of common points between the quadratic function $y=3x^2+6x-1$ and the linear function $y=k$.

[Sol] $\begin{cases} y=3x^2+6x-1 & \cdots① \\ y=k & \cdots② \end{cases}$



From ① and ②,

$3x^2+6x-1=k$

$3x^2+6x-k-1=0 \quad \cdots③$

Calculating the discriminant, *D*, of ③,

$$\frac{D}{4}=3^2-3\times(-k-1)=3k+12$$

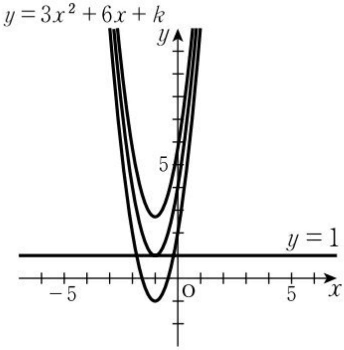
When $D>0$, i.e. **when $k>-4$, there are 2 common points**.

When $D=0$, i.e. **when $k=-4$, there is 1 common point**.

When $D<0$, i.e. **when $k<-4$, there are 0 common points**.

3. Find the number of common points between the quadratic function $y=3x^2+6x+k$ and the linear function $y=1$.

[Sol] $\begin{cases} y=3x^2+6x+k & \cdots① \\ y=1 & \cdots② \end{cases}$



From ① and ②,

$3x^2+6x+k=1$

$3x^2+6x+k-1=0 \quad \cdots③$

Calculating the discriminant, *D*, of ③,

$$\frac{D}{4}=3^2-3\times(k-1)=-3k+12$$

When $D>0$, i.e. **when $k<4$, there are 2 common points**.

When $D=0$, i.e. **when $k=4$, there is 1 common point**.

When $D<0$, i.e. **when $k>4$, there are 0 common points**.

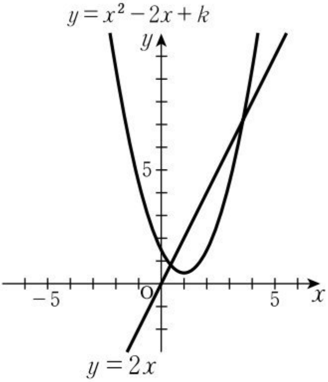
K79a KUMON

Quadratic Functions and Equations

1. Find the range of k for which the quadratic function $y = x^2 - 2x + k$ satisfies the following conditions:

(1) It crosses the line $y = 2x$ at two different points. *Hint*

[Sol] $\begin{cases} y = x^2 - 2x + k & \cdots ① \\ y = 2x & \cdots ② \end{cases}$



From ① and ②,

$x^2 - 2x + k = 2x$

$x^2 - 4x + k = 0 \quad \cdots ③$

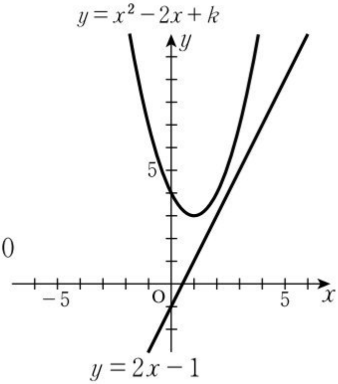
Calculating the discriminant, D, of ③,

$\frac{D}{4} = (-2)^2 - 1 \times k = -k + 4 > 0$

Therefore, $\boldsymbol{k < 4}$

(2) It has no common points with the line $y = 2x - 1$.

[Sol] $\begin{cases} y = x^2 - 2x + k & \cdots ① \\ y = 2x - 1 & \cdots ② \end{cases}$



From ① and ②,

$x^2 - 2x + k = 2x - 1$

$x^2 - 4x + k + 1 = 0 \quad \cdots ③$

Calculating the discriminant, D, of ③,

$\frac{D}{4} = (-2)^2 - 1 \times (k + 1) = -k + 3 < 0$

Therefore, $\boldsymbol{k > 3}$

Hint

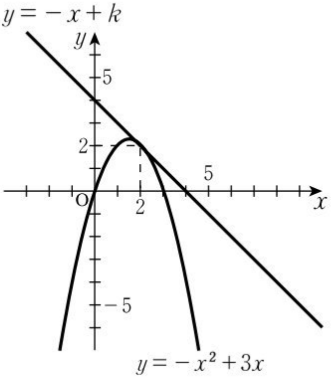
They have 2 common points.

K79b

2. Find the value of k for which the quadratic function $y=-x^2+3x$ touches the line $y=-x+k$. Then find the point of contact. *Hint*

[Sol] Let the point of contact be $(x,\ y)$.

$$\begin{cases} y=-x^2+3x & \cdots ① \\ y=-x+k & \cdots ② \end{cases}$$



From ① and ②,

$$-x^2+3x=-x+k$$

$$x^2-4x+k=0 \quad \cdots ③$$

Calculating the discriminant, D, of ③,

$$\frac{D}{4}=(-2)^2-1\times k=-k+4=0$$

Therefore, $\boldsymbol{k=4}$

Substituting into ③,

$$x^2-4x+4=0$$

$$(x-2)^2=0$$

$$x=2$$

From ①, when $x=2$, $y=2$

Therefore, the coordinates of the point of contact are **(2, 2)**.

Hint

They have 1 common point.

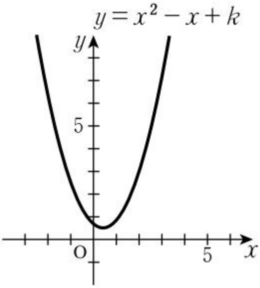
K 80a

KUMON

Quadratic Functions and Equations

1. Find the range of k for which the quadratic function $y=x^2-x+k$ satisfies the following conditions:

（1） It has no common points with the x-axis.

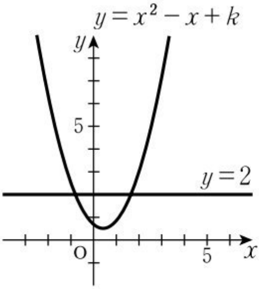


[Sol] $x^2-x+k=0$

$D=1-4k<0$

Therefore, $\boldsymbol{k>\frac{1}{4}}$

（2） It crosses the line $y=2$ at two different points.



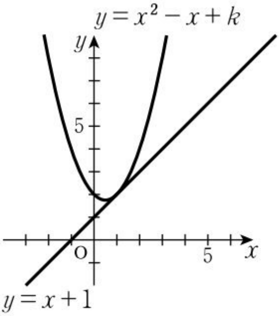
[Sol] $x^2-x+k=2$

$x^2-x+k-2=0$

$D=1-4(k-2)=-4k+9>0$

Therefore, $\boldsymbol{k<\frac{9}{4}}$

（3） It touches the line $y=x+1$.



[Sol] $x^2-x+k=x+1$

$x^2-2x+k-1=0$

$\frac{D}{4}=1-(k-1)=-k+2=0$

Therefore, $\boldsymbol{k=2}$

K 80b

2.* Given the quadratic function $y = x^2 - x + k - 1$ and the linear function $y = x + 1$, find the number of points that their graphs have in common. Assume $k > 4$.

[Sol] $x^2 - x + k - 1 = x + 1$

$x^2 - 2x + k - 2 = 0$

$\frac{D}{4} = 1 - (k - 2) = -k + 3$

Since $k > 4$, then $D < 0$

Therefore, **the graphs have no common points**.

Note Summary

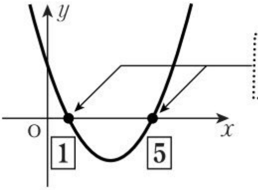
- Each real root of the quadratic equation $ax^2 + bx + c = 0$ corresponds to a common point of the quadratic function $y = ax^2 + bx + c$ and the x-axis.
- The number of real roots of $ax^2 + bx + c = 0$ is the same as the number of common points of $y = ax^2 + bx + c$ and the x-axis.
  (When $D > 0$ there are 2, when $D = 0$ there is 1, and when $D < 0$ there are 0.)
- Each real solution of the simultaneous equations $\begin{cases} y = ax^2 + bx + c \\ y = mx + n \end{cases}$ corresponds to a common point of the quadratic function $y = ax^2 + bx + c$ and the line $y = mx + n$.
- The number of real solutions of the equations $\begin{cases} y = ax^2 + bx + c \\ y = mx + n \end{cases}$ is the same as the number of common points of $y = ax^2 + bx + c$ and $y = mx + n$.
  This can be formed by calculating the discriminant of $ax^2 + (b - m)x + c - n = 0$.
  (When $D > 0$ there are 2, when $D = 0$ there is 1, and when $D < 0$ there are 0.)

K81a KUMON

Quadratic Functions and Inequalities

1. Given the quadratic function $y=x^2-6x+5$, complete the following questions.

(1) Find the values of x for which $y=0$.



$y=0$ for the values of x at the dots: ●

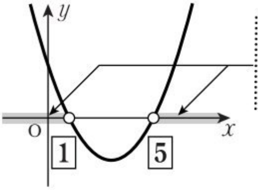
[Sol] $x^2-6x+5=0$

$(\ \boldsymbol{x-5}\ )(\ \boldsymbol{x-1}\ )=0$

Therefore,

$x=\boxed{5}, \boxed{1}$

(2) Find the values of x for which $y>0$.

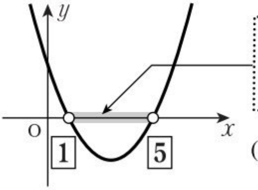


$y>0$ for the values of x in the shaded regions. ▨

[Sol] From the graph,

$x<\boxed{1}, x>\boxed{5}$

(3) Find the values of x for which $y<0$.



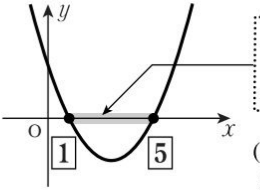
$y<0$ for the values of x in the shaded region. ▨

(between the circles)

[Sol] From the graph,

$\boxed{1}<x<\boxed{5}$

(4) Find the values of x for which $y\leqslant 0$.
(Find the values of x for which $y<0$ or $y=0$.)



$y\leqslant 0$ when the range of x in the shaded region. ▨

(including the dots and the values between them)

[Sol] From the graph,

$\boxed{1}\leqslant x\leqslant\boxed{5}$

When a particular point is included in a set of values, a dot (•) is used.
When the set of values goes up to, but does not include, that point, a circle (∘) is used.

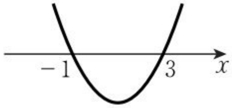
2. Solve the following inequalities by sketching.

Ex.

$(x+1)(x-3)>0$

[Sol] The sketch of

$y=(x+1)(x-3)$:



From the sketch, when $y>0$, the value of x is:

$x<-1,\ x>3$

Ex.

$(x+1)(x-3)<0$

[Sol] From the sketch above,

when $y<0$, the value of x is:

$-1<x<3$

Ex.

$(x+1)(x-3)\geqslant 0$

[Sol] From the sketch above,

when $y\geqslant 0$, the value of x is:

$x\leqslant -1,\ x\geqslant 3$

Ex.

$(x+1)(x-3)\leqslant 0$

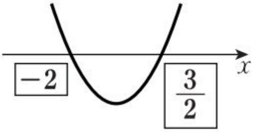
[Sol] From the sketch above,

when $y\leqslant 0$, the value of x is:

$-1\leqslant x\leqslant 3$

(1) $(x+2)(2x-3)>0$

[Sol]



From the sketch above,

$x<\boxed{-2},\ x>\boxed{\frac{3}{2}}$

(2) $(x+2)(2x-3)<0$

[Sol] From the sketch above,

$-2<x<\frac{3}{2}$

(3) $(x+2)(2x-3)\geqslant 0$

[Sol] From the sketch above,

$x\leqslant -2,\ x\geqslant \frac{3}{2}$

(4) $(x+2)(2x-3)\leqslant 0$

[Sol] From the sketch above,

$-2\leqslant x\leqslant \frac{3}{2}$

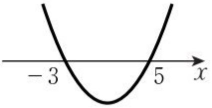
Quadratic Functions and Inequalities

1. Solve the following inequalities by sketching.

Ex.

$x^2-2x-15>0$

[Sol] $(x+3)(x-5)>0$

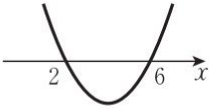


From the sketch,

$x<-3$, $x>5$

(1) $x^2-8x+12>0$

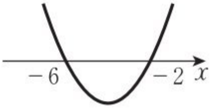
[Sol] $(x-2)(x-6)>0$



$x<2$, $x>6$

(2) $x^2+8x+12<0$

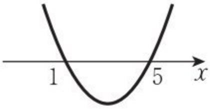
[Sol] $(x+6)(x+2)<0$



$-6<x<-2$

(3) $x^2-6x+5\geqq 0$

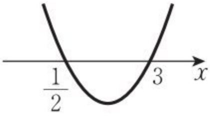
[Sol] $(x-1)(x-5)\geqq 0$



$x\leqq 1$, $x\geqq 5$

(4) $2x^2-7x+3>0$

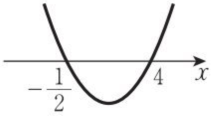
[Sol] $(2x-1)(x-3)>0$



$x<\frac{1}{2}$, $x>3$

(5) $2x^2-7x-4\leqq 0$

[Sol] $(2x+1)(x-4)\leqq 0$



$-\frac{1}{2}\leqq x\leqq 4$

K 82b

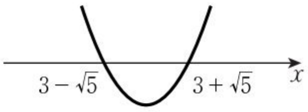
2. Solve the following inequalities by sketching.

Ex.

$x^2-6x+4<0$

[Sol] Using the quadratic formula to solve $x^2-6x+4=0$,

$x=3\pm\sqrt{5}$



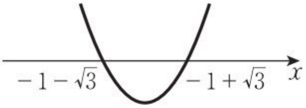
From the sketch,

$3-\sqrt{5}<x<3+\sqrt{5}$

(1) $x^2+2x-2>0$

[Sol] Solving $x^2+2x-2=0$,

$x=$ **$-1\pm\sqrt{3}$**

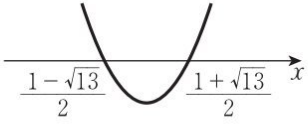


$x<-1-\sqrt{3},\ x>-1+\sqrt{3}$

(2) $x^2-x-3\leqslant 0$

[Sol] Solving $x^2-x-3=0$,

$x=\dfrac{1\pm\sqrt{13}}{2}$

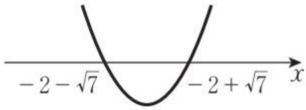


$\dfrac{1-\sqrt{13}}{2}\leqslant x\leqslant\dfrac{1+\sqrt{13}}{2}$

(3) $x^2+4x-3>0$

[Sol] Solving $x^2+4x-3=0$,

$x=-2\pm\sqrt{7}$

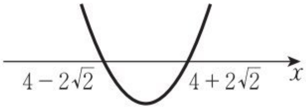


$x<-2-\sqrt{7},\ x>-2+\sqrt{7}$

(4) $x^2-8x+8<0$

[Sol] Solving $x^2-8x+8=0$,

$x=4\pm2\sqrt{2}$

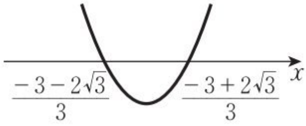


$4-2\sqrt{2}<x<4+2\sqrt{2}$

(5) $3x^2+6x-1>0$

[Sol] Solving $3x^2+6x-1=0$,

$x=\dfrac{-3\pm2\sqrt{3}}{3}$



$x<\dfrac{-3-2\sqrt{3}}{3},\ x>\dfrac{-3+2\sqrt{3}}{3}$

K 83a KUMON

Quadratic Functions and Inequalities

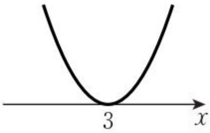
1. Solve the following inequalities by sketching.

Ex.

$x^2-6x+9>0$

[Sol] $y=x^2-6x+9=(x-3)^2$

Sketch



From the sketch,
$y>0$ for all values of x,
except when $x=3$.

$x<3,\ x>3$

$(x\neq 3)$

Ex.

$x^2-6x+9\geqslant 0$

[Sol] From the sketch above,
$y\geqslant 0$ for all values of x.

All real numbers

Ex.

$x^2-6x+9<0$

[Sol] From the sketch above,
when $y<0$ there are no values of x.

No solution

Ex.

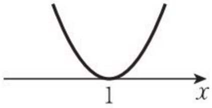
$x^2-6x+9\leqslant 0$

[Sol] From the sketch above,
when $y\leqslant 0$ the value of x is 3.

$x=3$

(1) $x^2-2x+1>0$

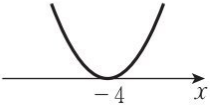
[Sol] $(\ \boldsymbol{x-1}\ )^2>0$



$x<\boxed{\mathbf{1}},\ x>\boxed{\mathbf{1}}$

(2) $x^2+8x+16\geqslant 0$

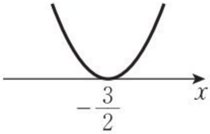
[Sol] $(x+4)^2\geqslant 0$



All real numbers

(3) $4x^2+12x+9<0$

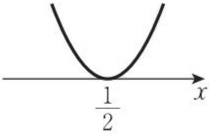
[Sol] $(2x+3)^2<0$



No solution

(4) $4x^2-4x+1\leqslant 0$

[Sol] $(2x-1)^2\leqslant 0$



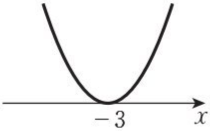
$\boldsymbol{x=\frac{1}{2}}$

K 83b

2. Solve the following inequalities by sketching.

(1) $x^2+6x+9\geqslant 0$

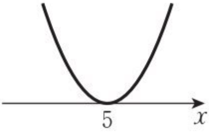
[Sol] $(x+3)^2\geqslant 0$



All real numbers

(2) $x^2-10x+25<0$

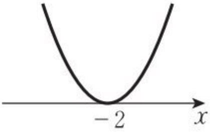
[Sol] $(x-5)^2<0$



No solution

(3) $x^2+4x+4\leqslant 0$

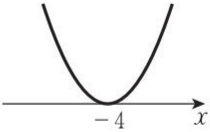
[Sol] $(x+2)^2\leqslant 0$



$\boldsymbol{x=-2}$

(4) $x^2+8x+16>0$

[Sol] $(x+4)^2>0$



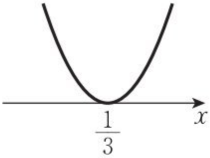
$\boldsymbol{x<-4,\ x>-4}$

$\boldsymbol{(x\neq -4)}$

(5) $9x^2<6x-1$

[Sol] $9x^2-6x+1<0$

$(3x-1)^2<0$

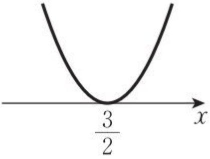


No solution

(6) $4x^2-6x+6\geqslant 6x-3$

[Sol] $4x^2-12x+9\geqslant 0$

$(2x-3)^2\geqslant 0$



All real numbers

K 84a KUMON

Quadratic Functions and Inequalities

Solve the following inequalities by sketching.

Ex.

$x^2-3x+4>0$

[Sol] Calculating the discriminant, D, of $x^2-3x+4=0$,

$D=9-16=-7<0$

Sketching $y=x^2-3x+4$

From the sketch,

$y>0$ for all values of x.

All real numbers

Ex.

$x^2-3x+4\geqslant 0$

[Sol] From the sketch above,

All real numbers

Ex.

$x^2-3x+4<0$

[Sol] From the sketch above,

when $y<0$ there are no values of x.

No solution

Ex.

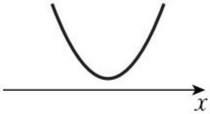
$x^2-3x+4\leqslant 0$

[Sol] From the sketch above,

No solution

(1) $x^2-x+2>0$

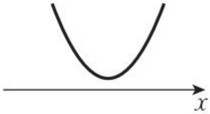
[Sol] $D=1-8=\mathbf{-7<0}$



All real numbers

(2) $x^2+5x+7\geqslant 0$

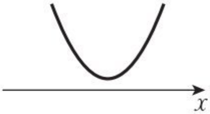
[Sol] $D=25-28=-3<0$



All real numbers

(3) $x^2-4x+5<0$

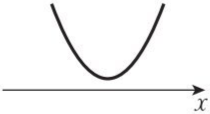
[Sol] $\frac{D}{4}=4-5=-1<0$



No solution

(4) $x^2+6x+10\leqslant 0$

[Sol] $\frac{D}{4}=9-10=-1<0$

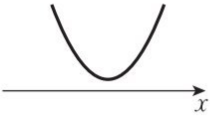


No solution

K 84b

(5) $x^2-x+1\geqq 0$

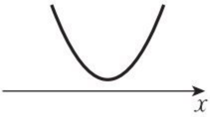
[Sol] $D=1-4=-3<0$



All real numbers

(6) $x^2+3x+3<0$

[Sol] $D=9-12=-3<0$

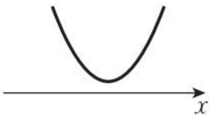


No solution

(7) $x^2\leqq 2x-3$

[Sol] $x^2-2x+3\leqq 0$

$\frac{D}{4}=1-3=-2<0$

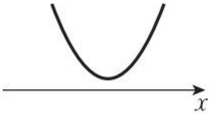


No solution

(8) $x(x-2)>-4$

[Sol] $x^2-2x+4>0$

$\frac{D}{4}=1-4=-3<0$

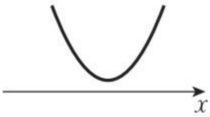


All real numbers

(9) $x^2+9>4(x+1)$

[Sol] $x^2-4x+5>0$

$\frac{D}{4}=4-5=-1<0$

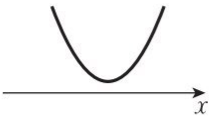


All real numbers

(10) $2x^2+4x-4<3(x-2)$

[Sol] $2x^2+x+2<0$

$D=1-16=-15<0$



No solution

K 85a

KUMON

Quadratic Functions and Inequalities

Solve the following inequalities by sketching.

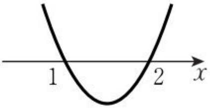
Ex.

$-x^2+3x-2 \leqslant 0$

[Sol] To solve $-x^2+3x-2 \leqslant 0$,

rewrite as $x^2-3x+2 \geqslant 0$

$(x-1)(x-2) \geqslant 0$



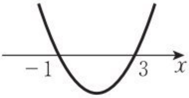
From the sketch,

$x \leqslant 1,\ x \geqslant 2$

(1) $-x^2+2x+3>0$

[Sol] $x^2-2x-3<0$

$(x+1)(x-3)<0$

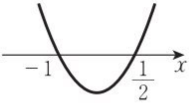


$-1<x<3$

(2) $-2x^2-x+1<0$

[Sol] $2x^2+x-1>0$

$(x+1)(2x-1)>0$



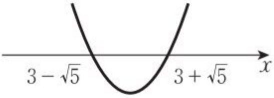
$x<-1,\ x>\frac{1}{2}$

(3) $-x^2+6x-4>0$

[Sol] $x^2-6x+4<0$

Solving $x^2-6x+4=0$,

$x=3 \pm \sqrt{5}$



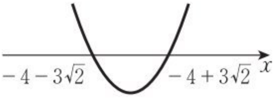
$3-\sqrt{5}<x<3+\sqrt{5}$

(4) $-x^2-8x+2 \leqslant 0$

[Sol] $x^2+8x-2 \geqslant 0$

Solving $x^2+8x-2=0$,

$x=-4 \pm 3\sqrt{2}$



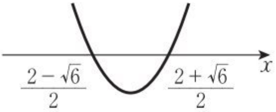
$x \leqslant -4-3\sqrt{2},\ x \geqslant -4+3\sqrt{2}$

(5) $-2x^2+4x+1 \geqslant 0$

[Sol] $2x^2-4x-1 \leqslant 0$

Solving $2x^2-4x-1=0$,

$x=\dfrac{2 \pm \sqrt{6}}{2}$



$\dfrac{2-\sqrt{6}}{2} \leqslant x \leqslant \dfrac{2+\sqrt{6}}{2}$

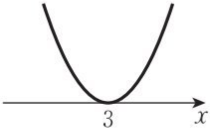
K 85b

(6) $-9 \leqq x^2-6x$

[Sol] $x^2-6x \geqq -9$

$x^2-6x+9 \geqq 0$

$(x-3)^2 \geqq 0$



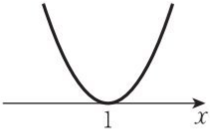
All real numbers

(7) $2x > x^2+1$

[Sol] $x^2+1 < 2x$

$x^2-2x+1 < 0$

$(x-1)^2 < 0$



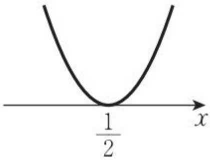
No solution

(8) $4x-1 < 4x^2$

[Sol] $4x^2 > 4x-1$

$4x^2-4x+1 > 0$

$(2x-1)^2 > 0$



$\boldsymbol{x < \frac{1}{2},\ x > \frac{1}{2}}$

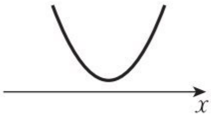
$\left[\boldsymbol{x \neq \frac{1}{2}}\right]$

(9) $3x < x^2+4$

[Sol] $x^2+4 > 3x$

$x^2-3x+4 > 0$

$D = 9-16 = -7 < 0$

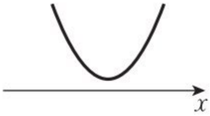


All real numbers

(10) $2x^2 < 3x-4$

[Sol] $2x^2-3x+4 < 0$

$D = 9-32 = -23 < 0$

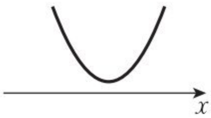


No solution

(11) $-3x^2 \leqq -2x+1$

[Sol] $3x^2-2x+1 \geqq 0$

$\frac{D}{4} = 1-3 = -2 < 0$



All real numbers

K 86a KUMON

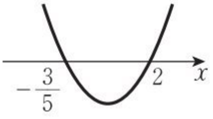
Quadratic Functions and Inequalities

1. Solve the following inequalities by sketching.

(1) $5x^2-6<7x$

[Sol] $5x^2-7x-6<0$

$(5x+3)(x-2)<0$



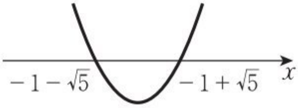
$$\boldsymbol{-\frac{3}{5}<x<2}$$

(2) $4-2x-x^2\leqslant 0$

[Sol] $x^2+2x-4\geqslant 0$

Solving $x^2+2x-4=0$,

$x=-1\pm\sqrt{5}$

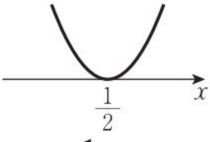


$$\boldsymbol{x\leqslant -1-\sqrt{5},\ x\geqslant -1+\sqrt{5}}$$

(3) $4x^2\leqslant 4x-1$

[Sol] $4x^2-4x+1\leqslant 0$

$(2x-1)^2\leqslant 0$

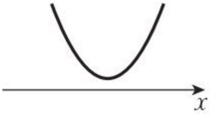


$$\boldsymbol{x=\frac{1}{2}}$$

(4) $3x<2x^2+4$

[Sol] $2x^2-3x+4>0$

$D=9-32=-23<0$

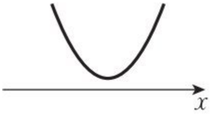


All real numbers

(5) $1-x^2>-3(x-2)$

[Sol] $x^2-3x+5<0$

$D=9-20=-11<0$



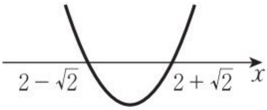
No solution

(6) $1-\frac{x}{2}\leqslant\frac{2}{3}+\frac{x(1-x)}{6}$

[Sol] $x^2-4x+2\leqslant 0$

Solving $x^2-4x+2=0$,

$x=2\pm\sqrt{2}$

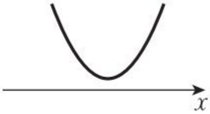


$$\boldsymbol{2-\sqrt{2}\leqslant x\leqslant 2+\sqrt{2}}$$

K 86b

2. Given the inequality $x^2-10x+c>0$, determine the range of values of the constant c that will make this inequality true for all real numbers.

[Sol] Calculating the discriminant, D, of the equation:



$$x^2-10x+c=0$$

$$\frac{D}{4}=25-c<0$$

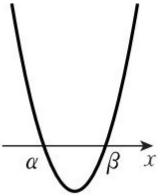
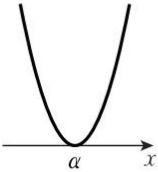
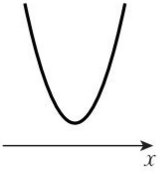
Therefore, $\boldsymbol{c>25}$

Note Summary

The relationship between the quadratic function $y=ax^2+bx+c$, and the quadratic inequalities:

$$ax^2+bx+c>0,\ ax^2+bx+c\geqslant 0$$
$$ax^2+bx+c<0,\ ax^2+bx+c\leqslant 0$$

can be summarized as follows. Assume $a>0$.

The sign of D	$D>0$	$D=0$	$D<0$
The graph of $y=ax^2+bx+c$			
Solution of:			
$ax^2+bx+c>0$	$x<\alpha,\ x>\beta$	$x<\alpha,\ x>\alpha$	All real numbers
$ax^2+bx+c\geqslant 0$	$x\leqslant\alpha,\ x\geqslant\beta$	All real numbers	All real numbers
$ax^2+bx+c<0$	$\alpha<x<\beta$	No solution	No solution
$ax^2+bx+c\leqslant 0$	$\alpha\leqslant x\leqslant\beta$	$x=\alpha$	No solution

K 87a

KUMON

Quadratic Functions and Inequalities

1. Find the values of b and c so that the following inequalities have the given solutions.

Ex.

$x^2+bx+c>0$ with solution $x<-2,\ x>3$

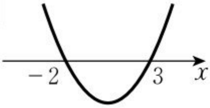
[Sol] $(x+2)(x-3)>0$ ☜ The solution is $x<-2,\ x>3$.

$x^2-x-6>0$

Therefore,

$b=-1,\ c=-6$

Compare the coefficients with those of $x^2+bx+c>0$.

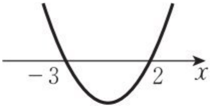


(1) $x^2+bx+c<0$ with solution $-3<x<2$

[Sol] $(x+3)(x-2)<0$

$x^2+x-6<0$

Therefore, $\boldsymbol{b=1,\ c=-6}$

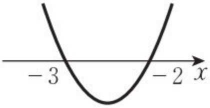


(2) $x^2+bx+c\geqq 0$ with solution $x\leqq -3,\ x\geqq -2$

[Sol] $(x+3)(x+2)\geqq 0$

$x^2+5x+6\geqq 0$

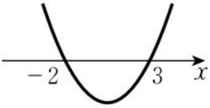
Therefore, $\boldsymbol{b=5,\ c=6}$



2. Find the values of a and b so that the following inequalities have the given solutions.

Ex.

$ax^2+bx-12>0$ with solution $x<-2,\ x>3$ (assume $a>0$).



[Sol] $a(x+2)(x-3)>0$

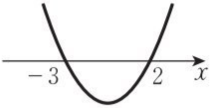
$ax^2-ax-6a>0$

Therefore,

$$\begin{cases} b=-a & \cdots ① \\ -12=-6a & \cdots ② \end{cases}$$

From ① and ②, $a=2,\ b=-2$

(1) $ax^2+bx-6\leqslant 0$ with solution $-3\leqslant x\leqslant 2$ (assume $a>0$).



[Sol] $a(x+3)(x-2)\leqslant 0$

$ax^2+ax-6a\leqslant 0$

Therefore,

$$\begin{cases} b=a & \cdots ① \\ -6=-6a & \cdots ② \end{cases}$$

From ① and ②, $\boldsymbol{a=1,\ b=1}$

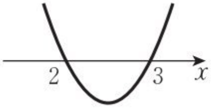
(2)* $ax^2+bx-12>0$ with solution $2<x<3$ (assume $a<0$).

[Sol] Since $a<0$,

$-ax^2-bx+12<0$

$-a(\ \mathbf{x-2}\ )(\ \mathbf{x-3}\ )<0$

$-ax^2+5ax-6a<0$



(Sketch of $y=-ax^2-bx+12$)

Therefore,

$$\begin{cases} -b=5a & \cdots ① \\ 12=-6a & \cdots ② \end{cases}$$

From ① and ②, $\boldsymbol{a=-2,\ b=10}$

K 88a KUMON

Quadratic Functions and Inequalities

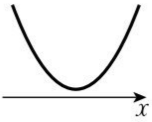
1. Find the range of k so that the solution of the following inequalities is "All real numbers".

Ex.

$x^2+kx+k+3>0$

[Sol] $D=k^2-4(k+3)=k^2-4k-12=(k+2)(k-6)<0$

Therefore, $-2<k<6$



(1) $x^2+(k+1)x+k+4>0$

[Sol] $D=(k+1)^2-4(k+4)$

$=k^2-2k-15$

$=(k+3)(k-5)<0$

Therefore, $\boldsymbol{-3<k<5}$

(2) $-x^2+(k+1)x-k^2<0$

[Sol] $x^2-(k+1)x+k^2>0$

$D=[-(k+1)]^2-4k^2$

$=-3k^2+2k+1$

$=-(3k+1)(k-1)<0$

$(3k+1)(k-1)>0$

Therefore, $\boldsymbol{k<-\frac{1}{3},\ \ k>1}$

2. Find the range of a so that the solution of $ax^2+2ax+3\geqslant 0$ is "All real numbers" (assume $a>0$).

[Sol] $\dfrac{D}{4}=a^2-3a=a(a-3)\leqslant 0$

Therefore, $\boldsymbol{0<a\leqslant 3}$

3. Find the range of m so that the function $y=x^2+2mx+m+2$ satisfies the following conditions.

(1) The graph is above the x-axis but not touching.

[Sol] $\dfrac{D}{4}=m^2-(m+2)=m^2-m-2=(m+1)(m-2)<0$

Therefore, $\boldsymbol{-1<m<2}$

(2) The graph crosses the x-axis at two different points.

[Sol] $\dfrac{D}{4}=(m+1)(m-2)>0$

Therefore, $\boldsymbol{m<-1,\ m>2}$

K 89a　KUMON

Quadratic Functions and Inequalities

In each question, find the range of x for which both inequalities hold true.
(Note: For some questions, the answer is "No solution".)

Ex.

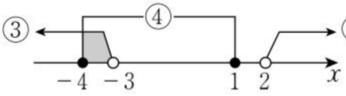
$$\begin{cases} x^2+x-6>0 & \cdots ① \\ x^2+3x-4\leqq 0 & \cdots ② \end{cases}$$

[Sol] From ①, $(x+3)(x-2)>0$

Therefore, $x<-3,\ x>2 \cdots ③$

From ②, $(x+4)(x-1)\leqq 0$

Therefore, $-4\leqq x\leqq 1 \quad \cdots ④$



From ③ and ④,

$-4\leqq x<-3$

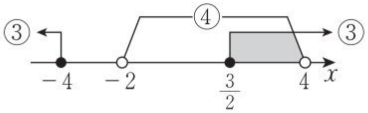
(1) $$\begin{cases} 2x^2+5x-12\geqq 0 & \cdots ① \\ x^2-2x-8<0 & \cdots ② \end{cases}$$

[Sol] From ①, $(x+4)(2x-3)\geqq 0$

Therefore, $x\leqq -4,\ x\geqq \dfrac{3}{2} \cdots ③$

From ②, $(x+2)(x-4)<0$

Therefore, $-2<x<4 \quad \cdots ④$



From ③ and ④,

$\boldsymbol{\dfrac{3}{2}\leqq x<4}$

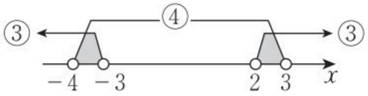
(2) $$\begin{cases} x^2+x-6>0 & \cdots ① \\ x^2+x-12<0 & \cdots ② \end{cases}$$

[Sol] From ①, $(x+3)(x-2)>0$

Therefore, $x<-3,\ x>2 \cdots ③$

From ②, $(x+4)(x-3)<0$

Therefore, $-4<x<3 \quad \cdots ④$



From ③ and ④,

$\boldsymbol{-4<x<-3,\ 2<x<3}$

(3) $$\begin{cases} 2x^2+5x+2\geqq 0 & \cdots ① \\ x^2<3-2x & \cdots ② \end{cases}$$

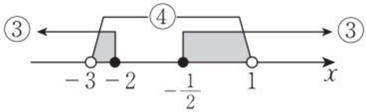
[Sol] From ①, $(x+2)(2x+1)\geqq 0$

Therefore,

$x\leqq -2,\ x\geqq -\dfrac{1}{2} \quad \cdots ③$

From ②, $(x+3)(x-1)<0$

Therefore, $-3<x<1 \quad \cdots ④$



From ③ and ④,

$\boldsymbol{-3<x\leqq -2,\ -\dfrac{1}{2}\leqq x<1}$

K 89b

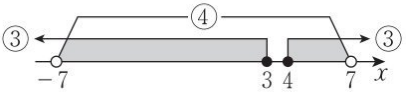
(4) $\begin{cases} x^2-7x+12 \geqq 0 & \cdots ① \\ x^2 < 49 & \cdots ② \end{cases}$

[Sol] From ①, $(x-3)(x-4) \geqq 0$

Therefore, $x \leqq 3,\ x \geqq 4$ ⋯③

From ②, $(x+7)(x-7) < 0$

Therefore, $-7 < x < 7$ ⋯④



From ③ and ④,

$\boldsymbol{-7 < x \leqq 3,\ \ 4 \leqq x < 7}$

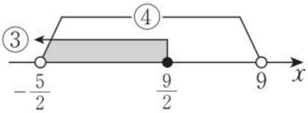
(5) $\begin{cases} x+1 \geqq 3x-8 & \cdots ① \\ 2x^2 < 13x+45 & \cdots ② \end{cases}$

[Sol] From ①, $2x \leqq 9$

Therefore, $x \leqq \dfrac{9}{2}$ ⋯③

From ②, $(2x+5)(x-9) < 0$

Therefore, $-\dfrac{5}{2} < x < 9$ ⋯④



From ③ and ④,

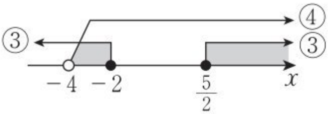
$\boldsymbol{-\dfrac{5}{2} < x \leqq \dfrac{9}{2}}$

(6) $\begin{cases} 3x^2-x-2 \geqq x^2+8 & \cdots ① \\ 3x+2 > 2x-2 & \cdots ② \end{cases}$

[Sol] From ①, $(x+2)(2x-5) \geqq 0$

Therefore, $x \leqq -2,\ x \geqq \dfrac{5}{2}$ ⋯③

From ②, $x > -4$ ⋯④



From ③ and ④,

$\boldsymbol{-4 < x \leqq -2,\ \ x \geqq \dfrac{5}{2}}$

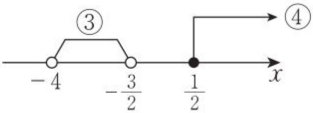
(7) $\begin{cases} 2(x+3)^2 < x+6 & \cdots ① \\ 3x+2 \geqq x+3 & \cdots ② \end{cases}$

[Sol] From ①, $(x+4)(2x+3) < 0$

Therefore, $-4 < x < -\dfrac{3}{2}$ ⋯③

From ②, $2x \geqq 1$

Therefore, $x \geqq \dfrac{1}{2}$ ⋯④



From ③ and ④,

No solution

K 90a

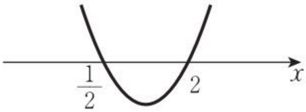
KUMON

Quadratic Functions and Inequalities

1. Solve the following inequalities.

(1) $2x^2-5x+2\leqslant 0$

[Sol] $(2x-1)(x-2)\leqslant 0$



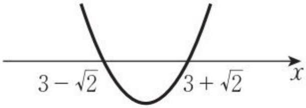
$$\boldsymbol{\frac{1}{2}\leqslant x\leqslant 2}$$

(2) $6x<x^2+7$

[Sol] $x^2-6x+7>0$

Solving $x^2-6x+7=0$,

$x=3\pm\sqrt{2}$

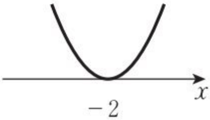


$$\boldsymbol{x<3-\sqrt{2},\ x>3+\sqrt{2}}$$

(3) $x^2+8<4(1-x)$

[Sol] $x^2+4x+4<0$

$(x+2)^2<0$

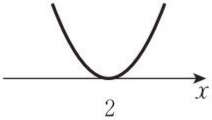


No solution

(4) $3x^2-2x+3>x^2+6x-5$

[Sol] $2x^2-8x+8>0$

$2(x-2)^2>0$



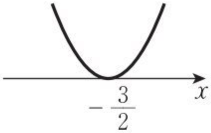
$$\boldsymbol{x<2,\ x>2}$$

〔$x\neq 2$〕

(5) $\dfrac{x(x+3)}{9}\geqslant -\dfrac{1}{4}$

[Sol] $4x^2+12x+9\geqslant 0$

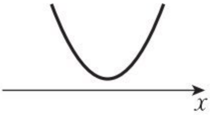
$(2x+3)^2\geqslant 0$



All real numbers

(6) $x^2+2x+3\geqslant 0$

[Sol] $\dfrac{D}{4}=1-3=-2<0$



All real numbers

2. Find the range of k for which the quadratic function $y = x^2 + (k+3)x + k + 3$ crosses the x-axis at two different points.

[Sol] Calculating the discriminant, D, of $x^2 + (k+3)x + k + 3 = 0$,

$$D = (k+3)^2 - 4(k+3) = (k+3)(k-1) > 0$$

Therefore, $\boldsymbol{k < -3,\ k > 1}$

3. Find the range of k so that the solution of $x^2 + (k+3)x + k + 3 > 0$ is "All real numbers".

[Sol] Calculating the discriminant, D, of $x^2 + (k+3)x + k + 3 = 0$,

$$D = (k+3)^2 - 4(k+3) = (k+3)(k-1) < 0$$

Therefore, $\boldsymbol{-3 < k < 1}$

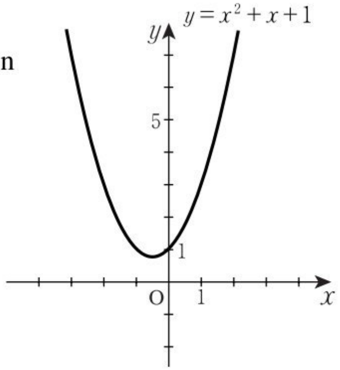
Consider this!

Given the graph of the quadratic function $y = x^2 + x + 1$, note that there are no common points with the x-axis.

Solving the quadratic function when $y = 0$, $x^2 + x + 1 = 0$ gives

$$x = \boxed{\dfrac{-1 \pm \sqrt{3}\, i}{2}},$$

i.e. complex solutions.

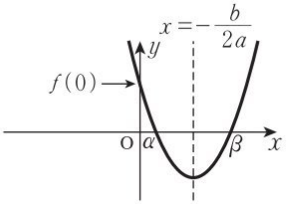


Answer: $\dfrac{-1 \pm \sqrt{3}\, i}{2}$

Quadratic Functions and Solutions of Quadratic Equations

Given that the equation $ax^2+bx+c=0$ $(a>0)$ has two solutions α and β (with $\alpha<\beta$), and with $f(x)=ax^2+bx+c$, determine the signs of ①~⑤ for the graph of $y=f(x)$ in each of the following cases.

Ex.



[Sol] ① $D>0$ ② $-\dfrac{b}{2a}>0$ ③ $f(0)>0$

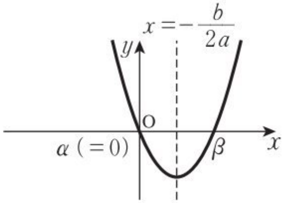
④ $\alpha>0$ ⑤ $\beta>0$

Note: D is the discriminant $(=b^2-4ac)$.

$x=-\dfrac{b}{2a}$ is the axis of symmetry.

$f(0)$ is the y-intercept.

(1)

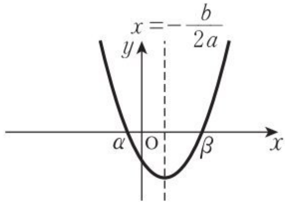


[Sol]

① $D\boxed{>}0$ ② $-\dfrac{b}{2a}\boxed{>}0$ ③ $f(0)\boxed{=}0$

④ $\alpha=0$ ⑤ $\beta\boxed{>}0$

(2)

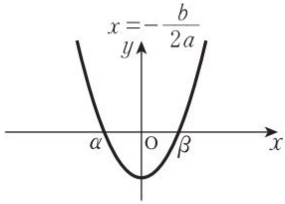


[Sol]

① $D>0$ ② $-\dfrac{b}{2a}>0$ ③ $f(0)<0$

④ $\alpha<0$ ⑤ $\beta>0$

(3)



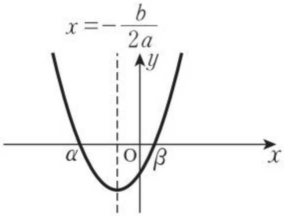
[Sol]

① $D>0$ ② $-\dfrac{b}{2a}=0$ ③ $f(0)<0$

④ $\alpha<0$ ⑤ $\beta>0$

K91b

(4)

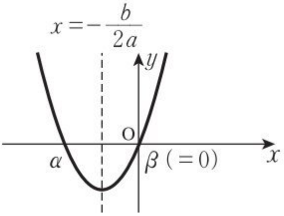


[Sol]

① $\boldsymbol{D>0}$ ② $\boldsymbol{-\frac{b}{2a}<0}$ ③ $\boldsymbol{f(0)<0}$

④ $\boldsymbol{\alpha<0}$ ⑤ $\boldsymbol{\beta>0}$

(5)

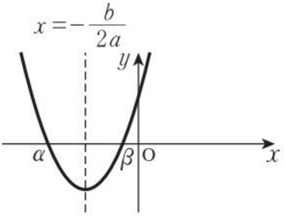


[Sol]

① $\boldsymbol{D>0}$ ② $\boldsymbol{-\frac{b}{2a}<0}$ ③ $\boldsymbol{f(0)=0}$

④ $\boldsymbol{\alpha<0}$ ⑤ $\boldsymbol{\beta=0}$

(6)



[Sol]

① $\boldsymbol{D>0}$ ② $\boldsymbol{-\frac{b}{2a}<0}$ ③ $\boldsymbol{f(0)>0}$

④ $\boldsymbol{\alpha<0}$ ⑤ $\boldsymbol{\beta<0}$

Given an equation $ax^2+bx+c=0$ $(a>0)$ with two solutions α and β $(\alpha<\beta)$, you can find the signs of α and β by determining the signs of D, $-\frac{b}{2a}$ and $f(0)$.

(i) When $D>0$, $-\frac{b}{2a}>0$, $f(0)>0$, then $\alpha>0$, $\beta>0$. ☞ As in the example

(ii) When $D>0$, $-\frac{b}{2a}<0$, $f(0)>0$, then $\alpha<0$, $\beta<0$. ☞ As in question (6)

(iii) When $f(0)<0$, then $\alpha<0$, $\beta>0$. ☞ As in questions (2)〜(4)

K 92a

Quadratic Functions and Solutions of Quadratic Equations

Find the range of k for which each quadratic equation has two different positive solutions ($\alpha > 0,\ \beta > 0,\ \alpha < \beta$).

Ex.

$x^2 - 2kx + k + 2 = 0$

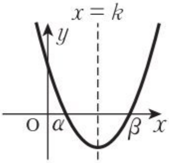
[Sol] Let $f(x) = x^2 - 2kx + k + 2$

$$\begin{cases} \dfrac{D}{4} = k^2 - (k+2) = (k+1)(k-2) > 0 & \cdots ① \\ k > 0 & \cdots ② \\ f(0) = k + 2 > 0 & \cdots ③ \end{cases}$$

① ☜ The discriminant > 0

② ☜ The axis of symmetry > 0

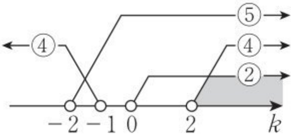
③ ☜ The y-intercept > 0



From ①, $k < -1,\ k > 2 \quad \cdots ④$

From ③, $k > -2 \quad \cdots ⑤$

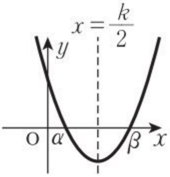
From ②, ④ and ⑤, $k > 2$



(1) $x^2 - kx + 3 - k = 0$

[Sol] Let $f(x) = x^2 - kx + 3 - k$

$$\begin{cases} D = k^2 - 4(3-k) = (k+6)(k-2) > 0 & \cdots ① \\ \dfrac{k}{2} > 0 & \cdots ② \\ f(0) = 3 - k > 0 & \cdots ③ \end{cases}$$

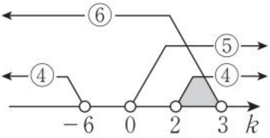


From ①, $k < -6,\ k > 2 \quad \cdots ④$

From ②, $k > 0 \quad \cdots ⑤$

From ③, $k < 3 \quad \cdots ⑥$

From ④, ⑤ and ⑥, $\mathbf{2 < k < 3}$

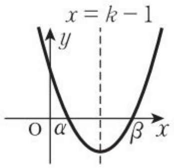


K 92b

(2) $x^2-2(k-1)x+3-k=0$

[Sol] Let $f(x)=x^2-2(k-1)x+3-k$

$$\begin{cases} \dfrac{D}{4}=(k-1)^2-(3-k)=(k+1)(k-2)>0 \cdots \text{①} \\ k-1>0 \quad \cdots \text{②} \\ f(0)=3-k>0 \quad \cdots \text{③} \end{cases}$$

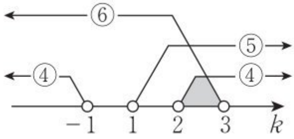


From ①, $k<-1,\ k>2$ $\cdots$④

From ②, $k>1$ $\cdots$⑤

From ③, $k<3$ $\cdots$⑥

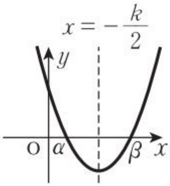
From ④, ⑤ and ⑥, $\boldsymbol{2<k<3}$



(3) $2x^2+2kx+k^2-k-12=0$

[Sol] Let $f(x)=2x^2+2kx+k^2-k-12$

$$\begin{cases} \dfrac{D}{4}=k^2-2(k^2-k-12)=-(k+4)(k-6)>0 \cdots \text{①} \\ -\dfrac{k}{2}>0 \quad \cdots \text{②} \\ f(0)=k^2-k-12=(k+3)(k-4)>0 \quad \cdots \text{③} \end{cases}$$

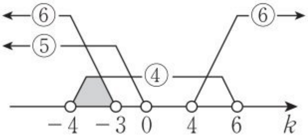


From ①, $-4<k<6$ $\cdots$④

From ②, $k<0$ $\cdots$⑤

From ③, $k<-3,\ k>4$ $\cdots$⑥

From ④, ⑤ and ⑥, $\boldsymbol{-4<k<-3}$



K 93a

Quadratic Functions and Solutions of Quadratic Equations

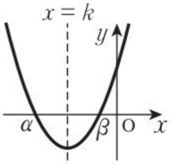
1. Find the range of k for which each given quadratic equation has two different negative solutions ($\alpha < 0$, $\beta < 0$, $\alpha < \beta$).

Ex.

$x^2 - 2kx + k + 2 = 0$

[Sol] Let $f(x) = x^2 - 2kx + k + 2$

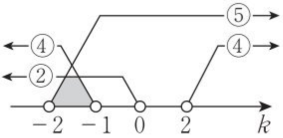
$$\begin{cases} \dfrac{D}{4} = k^2 - (k+2) = (k+1)(k-2) > 0 \cdots ① & ☜ \text{ The discriminant } > 0 \\ k < 0 \quad \cdots ② & ☜ \text{ The axis of symmetry} < 0 \\ f(0) = k + 2 > 0 \quad \cdots ③ & ☜ \text{ The } y\text{-intercept} > 0 \end{cases}$$



From ①, $k < -1$, $k > 2$ $\cdots$④

From ③, $k > -2$ $\cdots$⑤

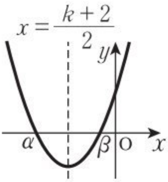
From ②, ④ and ⑤, $-2 < k < -1$



(1) $x^2 - (k+2)x + k + 5 = 0$

[Sol] Let $f(x) = x^2 - (k+2)x + k + 5$

$$\begin{cases} D = (k+2)^2 - 4(k+5) = (k+4)(k-4) > 0 \quad \cdots ① \\ \dfrac{k+2}{2} < 0 \quad \cdots ② \\ f(0) = k + 5 > 0 \quad \cdots ③ \end{cases}$$

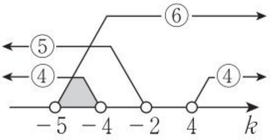


From ①, $k < -4$, $k > 4$ $\cdots$④

From ②, $k < -2$ $\cdots$⑤

From ③, $k > -5$ $\cdots$⑥

From ④, ⑤ and ⑥, $\boldsymbol{-5 < k < -4}$



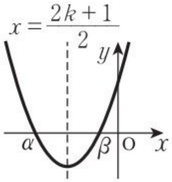
K 93b

2. Find the range of k for which the quadratic equation $x^2-(2k+1)x+k^2-k-2=0$ has the following solutions.

(1) Two different negative solutions

[Sol] Let $f(x)=x^2-(2k+1)x+k^2-k-2$

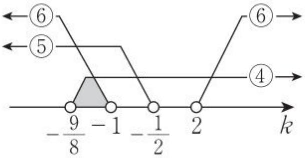
$$\begin{cases} D=(2k+1)^2-4(k^2-k-2)=8k+9>0 \quad \cdots\text{①} \\ \dfrac{2k+1}{2}<0 \quad \cdots\text{②} \\ f(0)=k^2-k-2=(k+1)(k-2)>0 \quad \cdots\text{③} \end{cases}$$



From ①, $k>-\dfrac{9}{8}$ $\cdots$④

From ②, $k<-\dfrac{1}{2}$ $\cdots$⑤

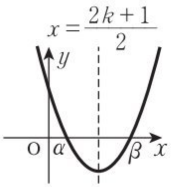
From ③, $k<-1,\ k>2$ $\cdots$⑥



From ④, ⑤ and ⑥, $\boldsymbol{-\dfrac{9}{8}<k<-1}$

(2) Two different positive solutions

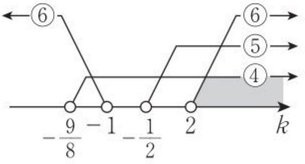
[Sol] $$\begin{cases} D=8k+9>0 \quad \cdots\text{①} \\ \dfrac{2k+1}{2}>0 \quad \cdots\text{②} \\ f(0)=(k+1)(k-2)>0 \quad \cdots\text{③} \end{cases}$$



From ①, $k>-\dfrac{9}{8}$ $\cdots$④

From ②, $k>-\dfrac{1}{2}$ $\cdots$⑤

From ③, $k<-1,\ k>2$ $\cdots$⑥



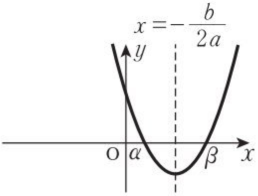
From ④, ⑤ and ⑥, $\boldsymbol{k>2}$

K 94a KUMON

Quadratic Functions and Solutions of Quadratic Equations

1. Given that the quadratic equation $ax^2+bx+c=0$ $(a>0)$ has two solutions α and β, and with $f(x)=ax^2+bx+c$, determine the signs of ①～④ for each of the following graph of $y=f(x)$.

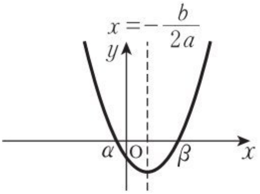
(1)



[Sol] ① $-\frac{b}{2a}\boxed{>}0$ ② $f(0)\boxed{>}0$

③ $\alpha\boxed{>}0$ ④ $\beta\boxed{>}0$

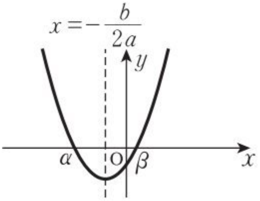
(2)



[Sol] ① $-\frac{b}{2a}>0$ ② $f(0)<0$

③ $\alpha<0$ ④ $\beta>0$

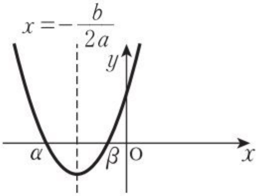
(3)



[Sol] ① $-\frac{b}{2a}<0$ ② $f(0)<0$

③ $\alpha<0$ ④ $\beta>0$

(4)



[Sol] ① $-\frac{b}{2a}<0$ ② $f(0)>0$

③ $\alpha<0$ ④ $\beta<0$

When a quadratic equation $(a>0)$ has one positive and one negative solution, as in (2) and (3), the y-intercept must be negative, i.e. $f(0)<0$.

($f(0)<0$ regardless of the sign of $-\frac{b}{2a}$.)

K 94b

2. Find the range of k for which each quadratic equation has one positive and one negative solution.

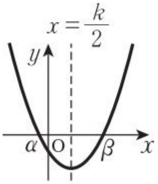
Ex.

$x^2-kx+3-k=0$

[Sol] Let $f(x)=x^2-kx+3-k$

$$f(0)=3-k<0$$

Therefore, $k>3$ ☜ It has one positive and one negative solution if and only if $f(0)<0$.

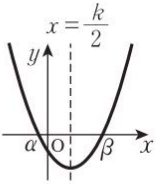


(1) $x^2-kx+k^2-k-2=0$

[Sol] Let $f(x)=x^2-kx+k^2-k-2$

$$f(0)=k^2-k-2=(k+1)(k-2)<0$$

Therefore, **$-1<k<2$**

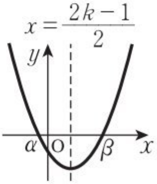


(2) $x^2-(2k-1)x+k^2-k=0$

[Sol] Let $f(x)=x^2-(2k-1)x+k^2-k$

$$f(0)=k^2-k=k(k-1)<0$$

Therefore, **$0<k<1$**



K 95a KUMON

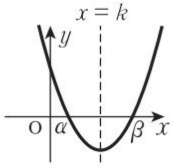
Quadratic Functions and Solutions of Quadratic Equations

1. Find the range of k for which the quadratic equation $x^2-2kx+k+2=0$ has the following solutions.

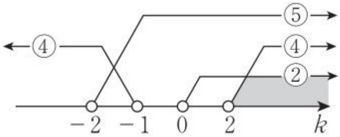
(1) Two different positive solutions

[Sol] Let $f(x)=x^2-2kx+k+2$

$$\begin{cases} \dfrac{D}{4}=k^2-(k+2)=(k+1)(k-2)>0 \cdots ① \\ k>0 \quad \cdots ② \\ f(0)=k+2>0 \quad \cdots ③ \end{cases}$$

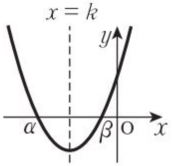


From ①, $k<-1,\ k>2 \quad \cdots ④$
From ③, $k>-2 \quad \cdots ⑤$
From ②, ④ and ⑤, $\boldsymbol{k>2}$

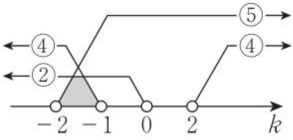


(2) Two different negative solutions

[Sol]
$$\begin{cases} \dfrac{D}{4}=(k+1)(k-2)>0 \cdots ① \\ k<0 \quad \cdots ② \\ f(0)=k+2>0 \quad \cdots ③ \end{cases}$$

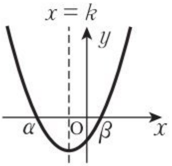


From ①, $k<-1,\ k>2 \quad \cdots ④$
From ③, $k>-2 \quad \cdots ⑤$
From ②, ④ and ⑤, $\boldsymbol{-2<k<-1}$



(3) One positive and one negative solution

[Sol] $f(0)=k+2<0$
$\boldsymbol{k<-2}$

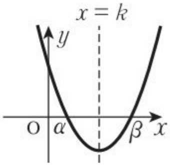


K95b

2. Find the range of k for which the quadratic equation $x^2-2kx+(k-1)(k-2)=0$ has two different positive solutions.

[Sol] Let $f(x)=x^2-2kx+(k-1)(k-2)$

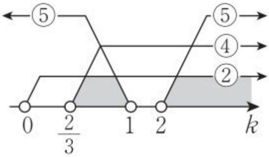
$$\begin{cases} \dfrac{D}{4}=k^2-(k-1)(k-2)=3k-2>0 & \cdots ① \\ k>0 & \cdots ② \\ f(0)=(k-1)(k-2)>0 & \cdots ③ \end{cases}$$



From ①, $k>\dfrac{2}{3}$ $\cdots$④

From ③, $k<1$, $k>2$ $\cdots$⑤

From ②, ④ and ⑤, $\boldsymbol{\dfrac{2}{3}<k<1,\ k>2}$



Consider this!

We can solve question 2 using the ***root-coefficient relationships*** from J131b. In order for the quadratic equation $ax^2+bx+c=0$ $(a>0)$ to have two different positive solutions, α and β, the following conditions must be true:

(i) α and β are two different real roots $\Rightarrow D=b^2-4ac>0$

(ii) $\alpha+\beta>0 \Rightarrow \alpha+\beta=-\dfrac{b}{a}>0$, therefore, $b<0$

(iii) $\alpha\beta>0 \Rightarrow \alpha\beta=\dfrac{c}{a}>0$, therefore, $c>0$

Applying these to $x^2-2kx+(k-1)(k-2)=0$ in question 2 above.

Since $a=1$, $b=-2k$, $c=(k-1)(k-2)$,

(i) $4k^2-4(k-1)(k-2)=4(3k-2)>0$, therefore, $k>\boxed{\dfrac{2}{3}}$ $\cdots$①

(ii) $-2k<0$, therefore, $k>\boxed{0}$ $\cdots$②

(iii) $(k-1)(k-2)>0$, therefore, $k<\boxed{1}$, $k>\boxed{2}$ $\cdots$③

From ①, ② and ③, $\boxed{\dfrac{2}{3}<k<1}$, $\boxed{k>2}$

Answers: in order (i) $\dfrac{2}{3}$ (ii) 0 (iii) 1, 2, $\dfrac{2}{3}<k<1$, $k>2$

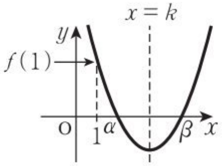
K 96a

Quadratic Functions and Solutions of Quadratic Equations

1. Find the range of k for which each quadratic equation has two different solutions greater than 1 ($\alpha > 1,\ \beta > 1,\ \alpha < \beta$).

Ex.

$x^2 - 2kx + k + 2 = 0$



[Sol] Let $f(x) = x^2 - 2kx + k + 2$

$$\begin{cases} \dfrac{D}{4} = k^2 - (k+2) = (k+1)(k-2) > 0 & \cdots ① \\ k > 1 & \cdots ② \\ f(1) = 1 - 2k + k + 2 = -k + 3 > 0 & \cdots ③ \end{cases}$$

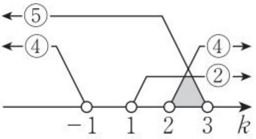
☜ From the graph, the axis of symmetry > 1.

☜ From the graph, $f(1) > 0$.

From ①, $k < -1,\ k > 2 \quad \cdots ④$

From ③, $k < 3 \quad \cdots ⑤$

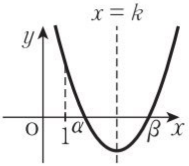
From ②, ④ and ⑤, $2 < k < 3$



(1) $x^2 - 2kx + k + 6 = 0$

[Sol] Let $f(x) = x^2 - 2kx + k + 6$

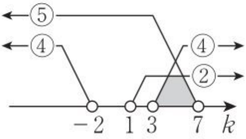
$$\begin{cases} \dfrac{D}{4} = k^2 - (k+6) = (k+2)(k-3) > 0 & \cdots ① \\ k > 1 & \cdots ② \\ f(1) = 1 - 2k + k + 6 = -k + 7 > 0 & \cdots ③ \end{cases}$$



From ①, $k < -2,\ k > 3 \quad \cdots ④$

From ③, $k < 7 \quad \cdots ⑤$

From ②, ④ and ⑤, $\mathbf{3 < k < 7}$

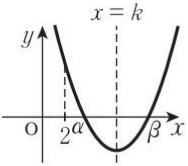


K96b

2. Find the range of k for which the quadratic equation $x^2-2kx+2k+8=0$ has two different solutions greater than 2.

[Sol] Let $f(x)=x^2-2kx+2k+8$

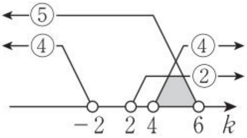
$$\begin{cases} \dfrac{D}{4}=k^2-(2k+8)=(k+2)(k-4)>0 \cdots \text{①} \\ k>2 \quad \cdots \text{②} \\ f(2)=4-4k+2k+8=-2k+12>0 \cdots \text{③} \end{cases}$$



From ①, $k<-2,\ k>4 \quad \cdots$④

From ③, $k<6 \quad \cdots$⑤

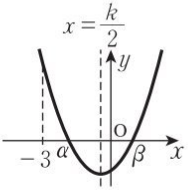
From ②, ④ and ⑤, $\boldsymbol{4<k<6}$



3. Find the range of k for which the quadratic equation $x^2-kx+k+3=0$ has two different solutions greater than -3.

[Sol] Let $f(x)=x^2-kx+k+3$

$$\begin{cases} D=k^2-4(k+3)=(k+2)(k-6)>0 \quad \cdots \text{①} \\ \dfrac{k}{2}>-3 \quad \cdots \text{②} \\ f(-3)=9+3k+k+3=4k+12>0 \quad \cdots \text{③} \end{cases}$$

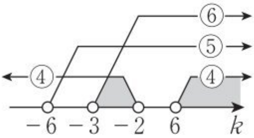


From ①, $k<-2,\ k>6 \quad \cdots$④

From ②, $k>-6 \quad \cdots$⑤

From ③, $k>-3 \quad \cdots$⑥

From ④, ⑤ and ⑥, $\boldsymbol{-3<k<-2,\ k>6}$



Quadratic Functions and Solutions of Quadratic Equations

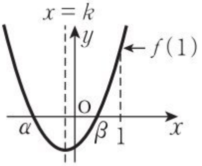
1. Find the range of k for which each quadratic equation has two different solutions less than 1 ($\alpha<1,\ \beta<1,\ \alpha<\beta$).

Ex.

$x^2-2kx+k+2=0$

[Sol] Let $f(x)=x^2-2kx+k+2$

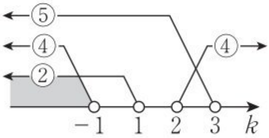
$$\begin{cases}\dfrac{D}{4}=k^2-(k+2)=(k+1)(k-2)>0 \cdots ① \\ k<1 \quad \cdots ② \\ f(1)=1-2k+k+2=-k+3>0 \cdots ③\end{cases}$$



From ①, $k<-1,\ k>2 \quad \cdots ④$

From ③, $k<3 \quad \cdots ⑤$

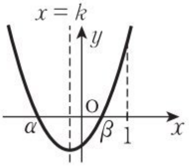
From ②, ④ and ⑤, $k<-1$



(1) $x^2-2kx+k+6=0$

[Sol] Let $f(x)=x^2-2kx+k+6$

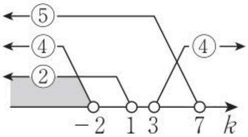
$$\begin{cases}\dfrac{D}{4}=k^2-(k+6)=(k+2)(k-3)>0 \cdots ① \\ k<1 \quad \cdots ② \\ f(1)=1-2k+k+6=-k+7>0 \cdots ③\end{cases}$$



From ①, $k<-2,\ k>3 \quad \cdots ④$

From ③, $k<7 \quad \cdots ⑤$

From ②, ④ and ⑤, $\boldsymbol{k<-2}$



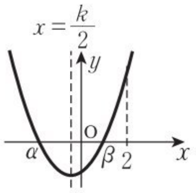
K 97b

2. Find the range of k for which the quadratic equation $x^2-kx+k+3=0$ has the following solutions.

（1） Two different solutions less than 2

[Sol] Let $f(x)=x^2-kx+k+3$

$$\begin{cases} D=k^2-4(k+3)=(k+2)(k-6)>0 & \cdots① \\ \dfrac{k}{2}<2 & \cdots② \\ f(2)=4-2k+k+3=-k+7>0 & \cdots③ \end{cases}$$

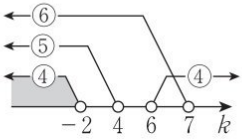


From ①, $k<-2,\ k>6$ $\cdots$④

From ②, $k<4$ $\cdots$⑤

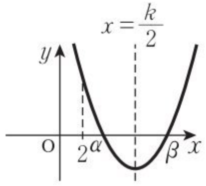
From ③, $k<7$ $\cdots$⑥

From ④, ⑤ and ⑥, $\boldsymbol{k<-2}$



（2） Two different solutions greater than 2

[Sol]
$$\begin{cases} D=(k+2)(k-6)>0 & \cdots① \\ \dfrac{k}{2}>2 & \cdots② \\ f(2)=-k+7>0 & \cdots③ \end{cases}$$

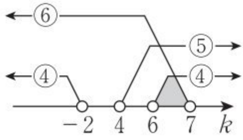


From ①, $k<-2,\ k>6$ $\cdots$④

From ②, $k>4$ $\cdots$⑤

From ③, $k<7$ $\cdots$⑥

From ④, ⑤ and ⑥, $\boldsymbol{6<k<7}$

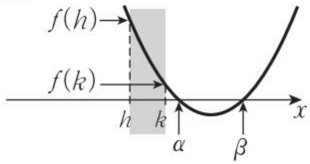


K 98a

Quadratic Functions and Solutions of Quadratic Equations

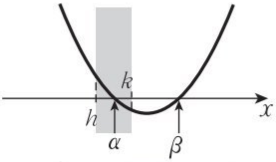
1. Given the quadratic function $f(x)=ax^2+bx+c$ $(a>0)$, find the signs of $f(h)$ and $f(k)$ for each of the following graphs.

Ex.



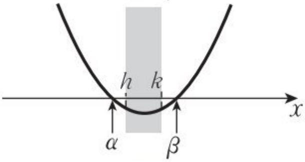
[Sol] $f(h)>0,\quad f(k)>0$

(1)



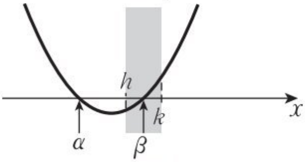
[Sol] $f(h)\boxed{>}0,\quad f(k)\boxed{<}0$

(2)



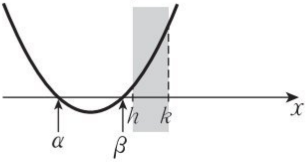
[Sol] $\boldsymbol{f(h)<0},\quad \boldsymbol{f(k)<0}$

(3)



[Sol] $\boldsymbol{f(h)<0},\quad \boldsymbol{f(k)>0}$

(4)



[Sol] $\boldsymbol{f(h)>0},\quad \boldsymbol{f(k)>0}$

2. Fill in the blank box.

From the graphs above, note that $f(x)$ crosses the x-axis in the domain $h<x<k$, in graphs (1) and $\boxed{(3)}$. In these cases, since $f(h)$ and $f(k)$ have opposite signs, $f(h)\cdot f(k)<0$.

Therefore, we can conclude that, when $f(h)\cdot f(k)<0$, the equation $ax^2+bx+c=0$ has exactly one solution between h and k ('exactly one' means 'one and only one').

3. Find the range of k for which the quadratic equation $x^2-kx+4k-3=0$ has two different solutions, exactly one of which is within the given domain.

Ex.

$1<x<2$

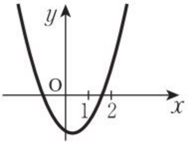
[Sol] Let $f(x)=x^2-kx+4k-3$

$f(1)=1-k+4k-3=3k-2$

$f(2)=4-2k+4k-3=2k+1$

$f(1)\cdot f(2)=(3k-2)(2k+1)<0$

Therefore, $-\frac{1}{2}<k<\frac{2}{3}$



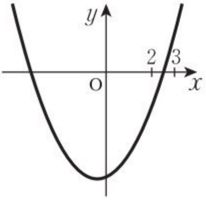
(1) $2<x<3$

[Sol] $f(2)=4-2k+4k-3=2k+1$

$f(3)=9-3k+4k-3=k+6$

$f(2)\cdot f(3)=(2k+1)(k+6)<0$

Therefore, $\mathbf{-6<k<-\frac{1}{2}}$



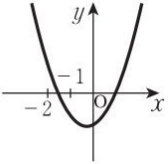
(2) $-2<x<-1$

[Sol] $f(-2)=4+2k+4k-3=6k+1$

$f(-1)=1+k+4k-3=5k-2$

$f(-2)\cdot f(-1)=(6k+1)(5k-2)<0$

Therefore, $\mathbf{-\frac{1}{6}<k<\frac{2}{5}}$



K 99a KUMON

Quadratic Functions and Solutions of Quadratic Equations

1. Find the range of k for which the quadratic equation $x^2-kx+3k-2=0$ has two different solutions satisfying the following conditions.

(1) Exactly one solution is within $1<x<2$.

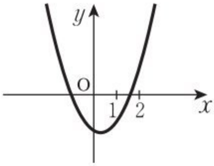
[Sol] Let $f(x)=x^2-kx+3k-2$

$f(1)=1-k+3k-2=2k-1$

$f(2)=4-2k+3k-2=k+2$

$f(1)\cdot f(2)=(2k-1)(k+2)<0$

Therefore, $\boldsymbol{-2<k<\frac{1}{2}}$



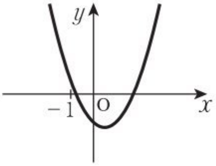
(2) Exactly one solution is within $-1<x<0$.

[Sol] $f(-1)=1+k+3k-2=4k-1$

$f(0)=3k-2$

$f(-1)\cdot f(0)=(4k-1)(3k-2)<0$

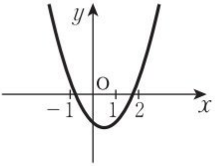
Therefore, $\boldsymbol{\frac{1}{4}<k<\frac{2}{3}}$

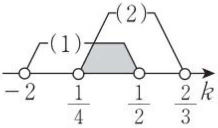


(3) One solution is within $-1<x<0$, and the other solution is within $1<x<2$.

[Sol] From (1) and (2),

$\boldsymbol{\frac{1}{4}<k<\frac{1}{2}}$





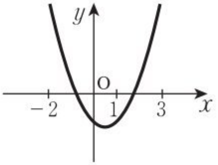
K99b

2. Find the range of k for which the solutions α and β of the quadratic equation $x^2-kx+5k-4=0$ satisfy $-2<\alpha<0$ and $1<\beta<3$.

[Sol] Let $f(x)=x^2-kx+5k-4$

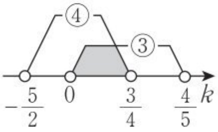
$f(-2)=7k,\ f(0)=5k-4$

$f(1)=4k-3,\ f(3)=2k+5$



Therefore,

$$\begin{cases} f(-2)\cdot f(0)=7k(5k-4)<0 & \cdots ① \\ f(1)\cdot f(3)=(4k-3)(2k+5)<0 & \cdots ② \end{cases}$$



From ①, $0<k<\frac{4}{5}$ $\cdots$③

From ②, $-\frac{5}{2}<k<\frac{3}{4}$ $\cdots$④

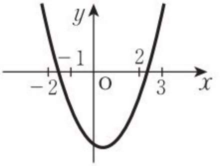
From ③ and ④, $\boldsymbol{0<k<\frac{3}{4}}$

3. Find the range of k for which the solutions α and β of the quadratic equation $x^2-(k+1)x+4k-5=0$ satisfy $-2<\alpha<-1$ and $2<\beta<3$.

[Sol] Let $f(x)=x^2-(k+1)x+4k-5$

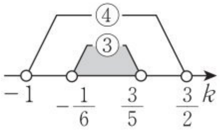
$f(-2)=6k+1,\ f(-1)=5k-3$

$f(2)=2k-3,\ f(3)=k+1$



Therefore,

$$\begin{cases} f(-2)\cdot f(-1)=(6k+1)(5k-3)<0 & \cdots ① \\ f(2)\cdot f(3)=(2k-3)(k+1)<0 & \cdots ② \end{cases}$$



From ①, $-\frac{1}{6}<k<\frac{3}{5}$ $\cdots$③

From ②, $-1<k<\frac{3}{2}$ $\cdots$④

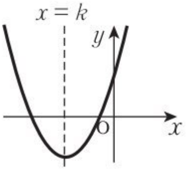
From ③ and ④, $\boldsymbol{-\frac{1}{6}<k<\frac{3}{5}}$

K 100a KUMON

Quadratic Functions and Solutions of Quadratic Equations

1. Find the range of k for which the quadratic equation $x^2-2kx+k+6=0$ has two different negative solutions.

[Sol] Let $f(x)=x^2-2kx+k+6$

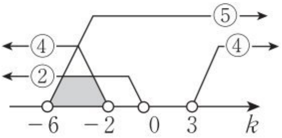


$$\begin{cases} \dfrac{D}{4}=k^2-(k+6)=(k+2)(k-3)>0 \cdots ① \\ k<0 \quad \cdots ② \\ f(0)=k+6>0 \quad \cdots ③ \end{cases}$$

From ①, $k<-2, k>3 \quad \cdots ④$

From ③, $k>-6 \quad \cdots ⑤$

From ②, ④ and ⑤, $\boldsymbol{-6<k<-2}$



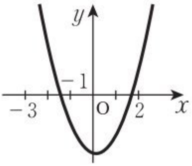
2. Find the range of k for which the solutions α and β of the quadratic equation $x^2-kx+3k-4=0$ satisfy $-3<\alpha<-1$ and $0<\beta<2$.

[Sol] Let $f(x)=x^2-kx+3k-4$

$f(-3)=6k+5, \ f(-1)=4k-3$

$f(0)=3k-4, \ f(2)=k$

Therefore,

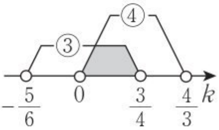


$$\begin{cases} f(-3)\cdot f(-1)=(6k+5)(4k-3)<0 \quad \cdots ① \\ f(0)\cdot f(2)=(3k-4)k<0 \quad \cdots ② \end{cases}$$

From ①, $-\dfrac{5}{6}<k<\dfrac{3}{4} \quad \cdots ③$

From ②, $0<k<\dfrac{4}{3} \quad \cdots ④$

From ③ and ④, $\boldsymbol{0<k<\dfrac{3}{4}}$

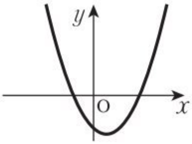


3. Find the range of k for which the quadratic equation $x^2-2kx+2k^2-k-3=0$ has one positive and one negative solution.

[Sol] Let $f(x)=x^2-2kx+2k^2-k-3$

$f(0)=2k^2-k-3=(k+1)(2k-3)<0$

Therefore, $\mathbf{-1<k<\frac{3}{2}}$



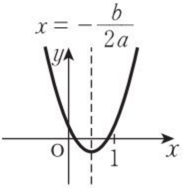
Consider this!

Given that a quadratic equation $ax^2+bx+c=0$ $(a>0)$ has solutions α and β, and with $f(x)=ax^2+bx+c$, consider the conditions for which $0<\alpha<1$ and $0<\beta<1$. These conditions are

(i) $D>0$

(ii) $f(0)>0$, $f(1)>0$

(iii) $0<-\dfrac{b}{2a}<1$



Let's try this!

Find the range of k for which the quadratic equation $x^2-2kx+k+2=0$ has two different solutions within $0<x<3$.

[Sol] (i) $\dfrac{D}{4}=(k+1)(k-2)>0$

Therefore, $k<\boxed{\mathbf{-1}}$, $k>\boxed{\mathbf{2}}$ …①

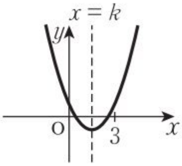
(ii) $f(0)=k+2>0$

Therefore, $k>\boxed{\mathbf{-2}}$ …②

$f(3)=-5k+11>0$

Therefore, $k<\boxed{\mathbf{\frac{11}{5}}}$ …③

(iii) $\boxed{\mathbf{0}}<k<\boxed{\mathbf{3}}$ …④



From ①, ②, ③ and ④, $\boxed{\mathbf{2<k<\frac{11}{5}}}$

Answers: in order (i) -1, 2 (ii) -2, $\frac{11}{5}$ (iii) 0, 3, $2<k<\frac{11}{5}$

K 101a KUMON

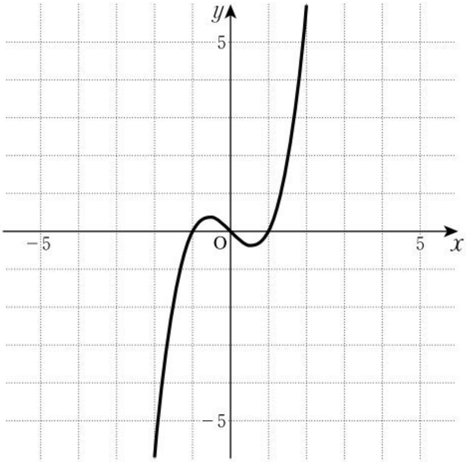
Higher Degree Functions

1. Graph the following *cubic function*.

Ex.

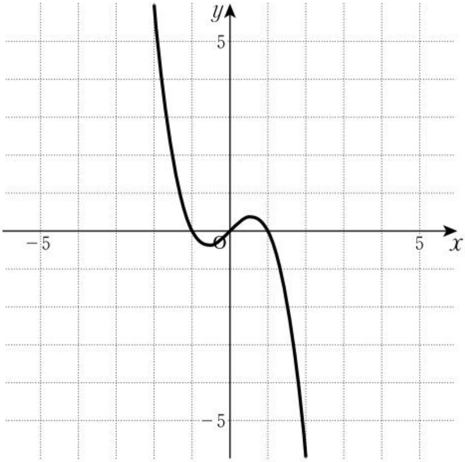
$y = x(x-1)(x+1)$

x	y
-2	-6
-1	0
$-\frac{1}{2}$	$\frac{3}{8}$
0	0
$\frac{1}{2}$	$-\frac{3}{8}$
1	0
2	6



(1) $y = -x(x-1)(x+1)$

x	y
-2	$\mathbf{6}$
-1	$\mathbf{0}$
$-\frac{1}{2}$	$\mathbf{-\frac{3}{8}}$
0	$\mathbf{0}$
$\frac{1}{2}$	$\mathbf{\frac{3}{8}}$
1	$\mathbf{0}$
2	$\mathbf{-6}$

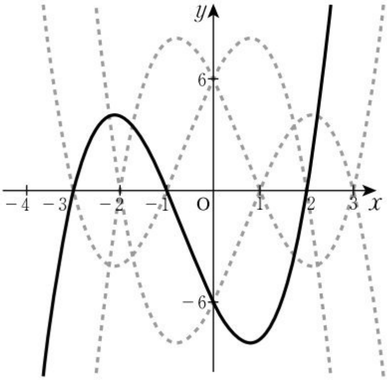


A function that can be written as $y = ax^3 + bx^2 + cx + d$ $(a \neq 0)$ is called a ***cubic function***.
$y = x(x-1)(x+1)$ is a ***cubic function***, since it becomes $y = x^3 - x$ when expanded.

2. Trace the graphs of the following cubic functions.

Ex.

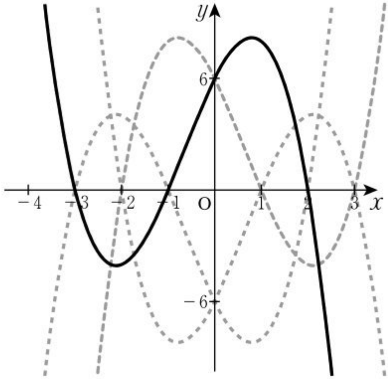
$y=(x+3)(x+1)(x-2)$



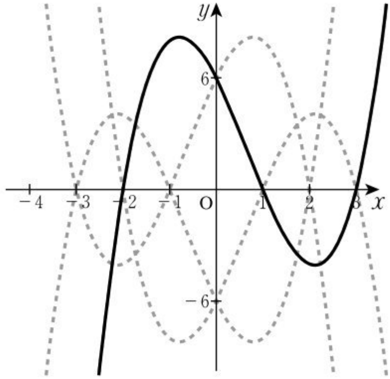
Note: If $-3<x<-1$, then $y>0$.

(For example, if you substitute $x=-2$, you get $y=4>0$.)

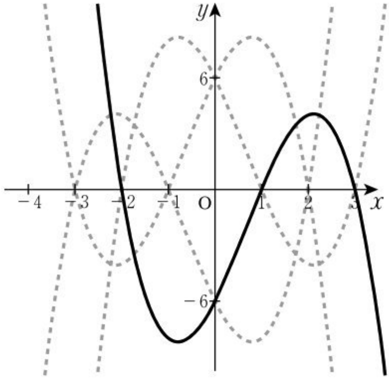
(2) $y=-(x+3)(x+1)(x-2)$



(1) $y=(x+2)(x-1)(x-3)$



(3) $y=-(x+2)(x-1)(x-3)$

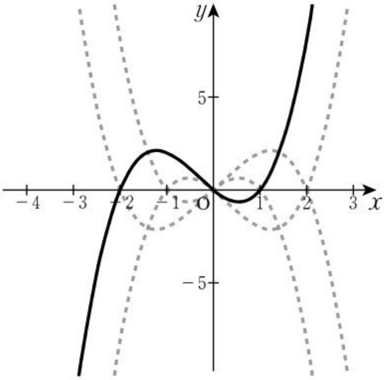


K 102a

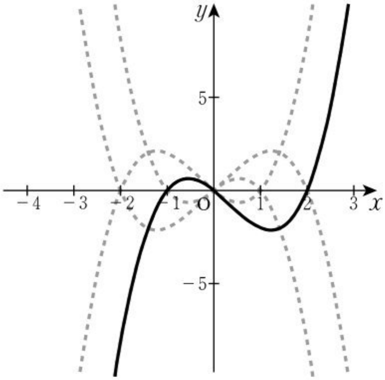
Higher Degree Functions

1. Trace the graphs of the following cubic functions.

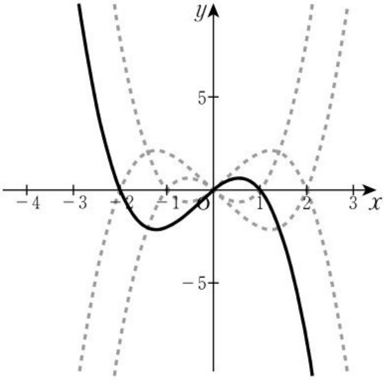
(1) $y=x(x+2)(x-1)$



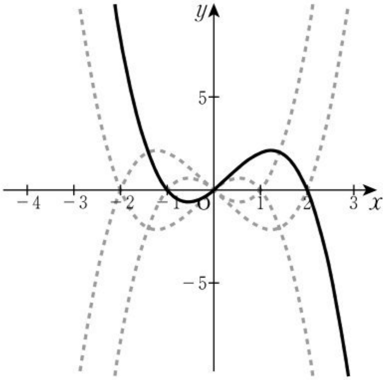
(3) $y=x(x-2)(x+1)$



(2) $y=-x(x+2)(x-1)$



(4) $y=-x(x-2)(x+1)$

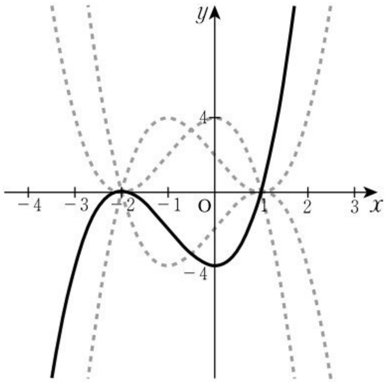


K 102b

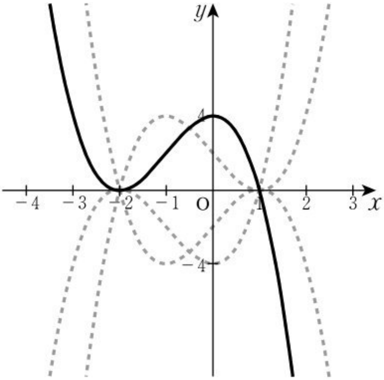
2. Trace the graphs of the following cubic functions.

Ex.

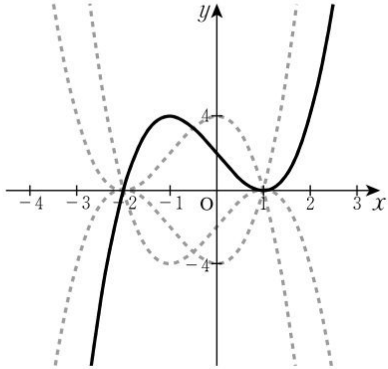
$y=(x+2)^2(x-1)$



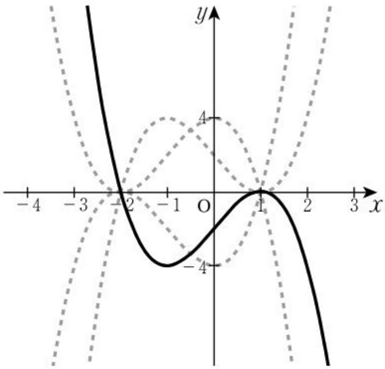
(2) $y=-(x+2)^2(x-1)$



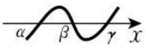
(1) $y=(x+2)(x-1)^2$



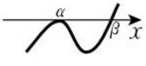
(3) $y=-(x+2)(x-1)^2$



- Consider the cubic function $y=a(x-\overset{\text{alpha}}{\alpha})(x-\overset{\text{beta}}{\beta})(x-\overset{\text{gamma}}{\gamma})$ $(a>0,\ \alpha<\beta<\gamma)$.
  When $y=0$, $x=\alpha,\ \beta,\ \gamma$. The graph crosses the x-axis at three points.



- Consider the cubic function $y=a(x-\alpha)^2(x-\beta)$ $(a>0,\ \alpha<\beta)$.
  When $y=0$, $x=\alpha,\ \beta$. The graph touches the x-axis at $x=\alpha$
  and crosses it at $x=\beta$.

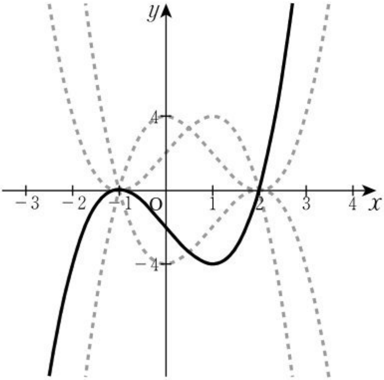


K 103a KUMON

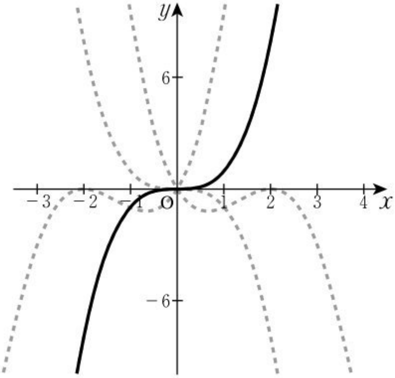
Higher Degree Functions

1. Trace the graphs of the following cubic functions.

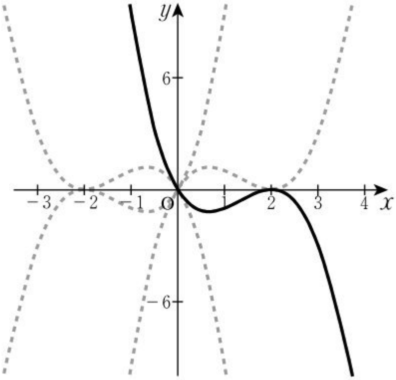
(1) $y=(x+1)^2(x-2)$



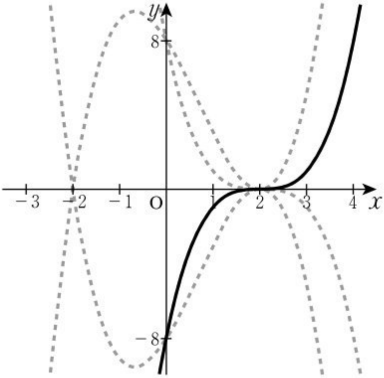
(3) $y=x^3$



(2) $y=-x(x-2)^2$



(4) $y=(x-2)^3$



2. Trace the graph of the following cubic function.

Ex.

$y=x^3-3x-2$

[Sol] Let $f(x)=x^3-3x-2$

From $f(-1)=0$,

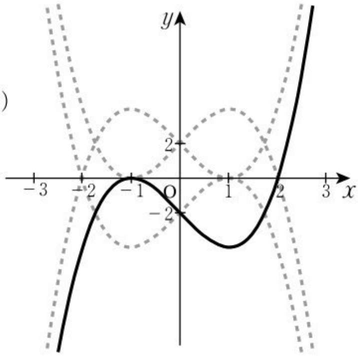
$$f(x)=(x+1)(x^2-x-2)$$

$$=(x+1)^2(x-2)$$

By the *Factor Theorem*, $(x+1)$ is a *factor*.

Note: If $f(x)$ is divided by $x+1$ the quotient is x^2-x-2. (Refer to the *Factor Theorem* on J172.)

$$\begin{array}{r r}
 & x^2-\ x\ -2 \\ \hline
x+1\,\big) & x^3+0x^2-3x-2 \\
 & x^3+\ x^2\qquad\quad \\ \hline
 & -\ x^2-3x\quad \\
 & -\ x^2-\ x\quad \\ \hline
 & -2x-2 \\
 & -2x-2 \\ \hline
 & 0
\end{array}$$



(1) $y=x^3-3x+2$

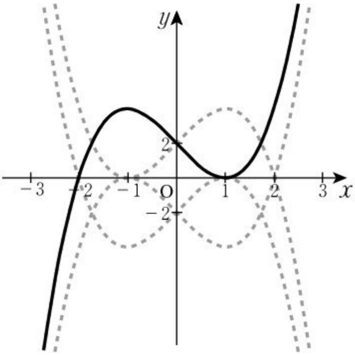
[Sol] Let $f(x)=x^3-3x+2$

From $f(1)=0$,

$$f(x)=(x-1)(x^2+x-2)$$

$$=(x-1)^2(x+2)$$

$$\begin{array}{r r}
 & x^2+\ x\ -2 \\ \hline
x-1\,\big) & x^3+0x^2-3x+2 \\
 & x^3-\ x^2\qquad\quad \\ \hline
 & x^2-3x\quad \\
 & x^2-\ x\quad \\ \hline
 & -2x+2 \\
 & -2x+2 \\ \hline
 & 0
\end{array}$$



K 104a KUMON

Higher Degree Functions

1. For each cubic function, write the letter (A)～(F) of the corresponding sketch.

(1) $y=x(x+2)(x-3)$ ··· $\boxed{\text{(E)}}$

(2) $y=x(x+3)(x-2)$ ··· $\boxed{\text{(B)}}$

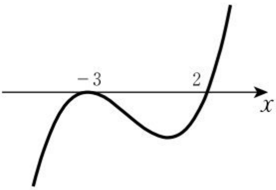
(3) $y=-x(x+3)(x-2)$ ··· $\boxed{\text{(D)}}$

(4) $y=(x+3)(x-2)^2$ ··· $\boxed{\text{(F)}}$

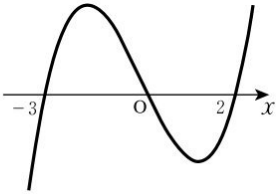
(5) $y=(x-2)(x+3)^2$ ··· $\boxed{\text{(A)}}$

(6) $y=-(x+3)^2(x-2)$ ··· $\boxed{\text{(C)}}$

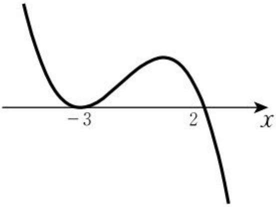
(A)



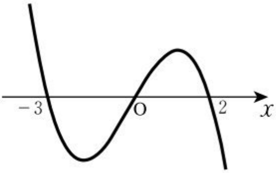
(B)



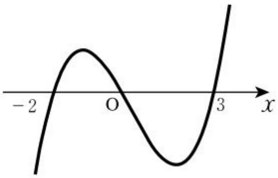
(C)



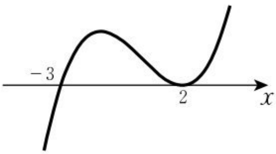
(D)



(E)



(F)



K 104b

2. For each of the two functions, write the letter (A)〜(F) of the corresponding sketch.

(1) $y=x^3-3x^2-x+3 \quad \cdots \boxed{\textbf{(E)}}$ (2) $y=x^3+2x^2+x \quad \cdots \boxed{\textbf{(D)}}$

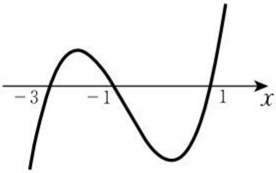
[Sol] From the *Factor Theorem*,

$$\begin{aligned} y &= (x-1)(x^2-2x-3) \\ &= (x-1)(x-3)(x+1) \end{aligned}$$

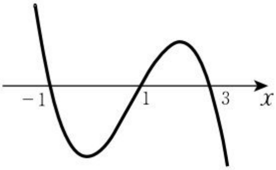
[Sol] $y=x(x^2+2x+1)$

$$= x(x+1)^2$$

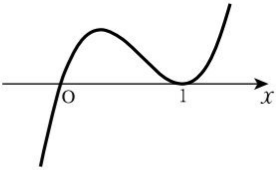
(A)



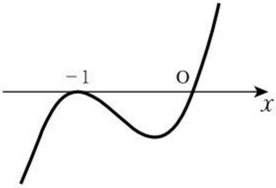
(B)



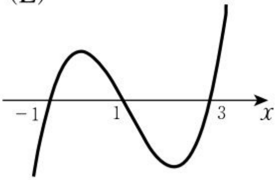
(C)



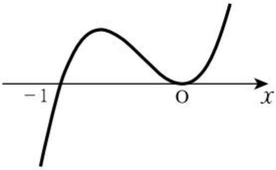
(D)



(E)



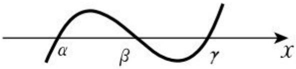
(F)



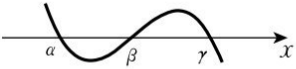
Note Summary

- The possible sketches of the cubic function $y=a(x-\alpha)(x-\beta)(x-\gamma)$ $(\alpha<\beta<\gamma)$ are:

When $a>0$

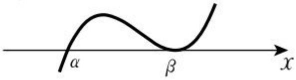


When $a<0$

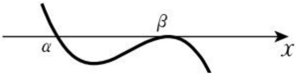


- The possible sketches of the cubic function $y=a(x-\alpha)(x-\beta)^2$ $(\alpha<\beta)$ are:

When $a>0$

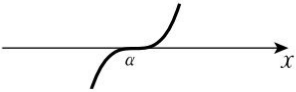


When $a<0$

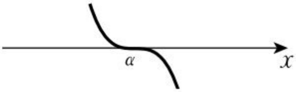


- The possible sketches of the cubic function $y=a(x-\alpha)^3$ are:

When $a>0$



When $a<0$



K 105a KUMON

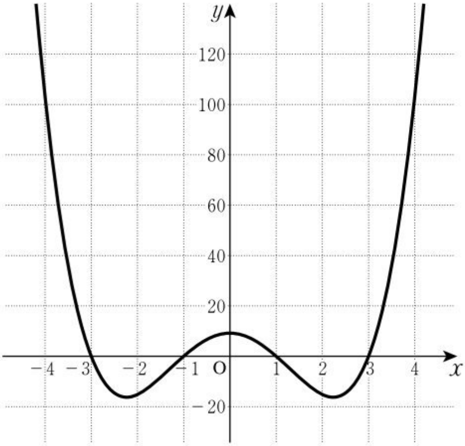
Higher Degree Functions

1. Graph the following ***quartic functions***.

Ex.

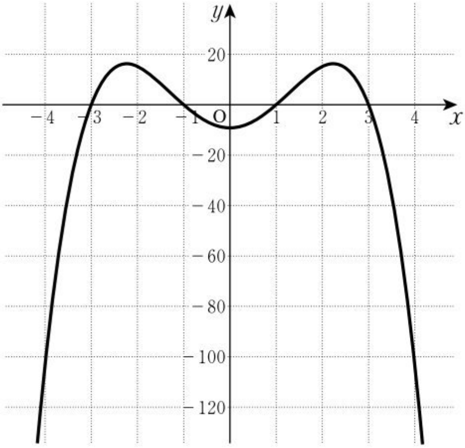
$y=(x+3)(x+1)(x-1)(x-3)$

x	y
-4	105
-3	0
-2	-15
-1	0
0	9
1	0
2	-15
3	0
4	105



(1) $y=-(x+3)(x+1)(x-1)(x-3)$

x	y
-4	**−105**
-3	**0**
-2	**15**
-1	**0**
0	**−9**
1	**0**
2	**15**
3	**0**
4	**−105**



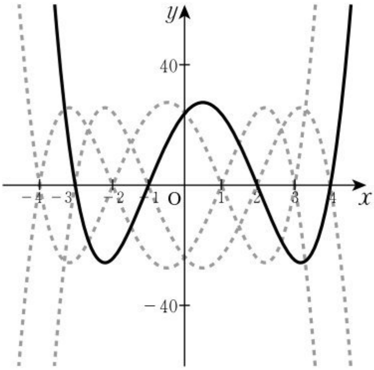
A function that can be written as $y=ax^4+bx^3+cx^2+dx+e$ $(a\neq0)$ is called a ***quartic function***. A ***quartic function*** is a 4th degree polynomial function.

K 105b

2. Trace the graphs of the following quartic functions.

Ex.

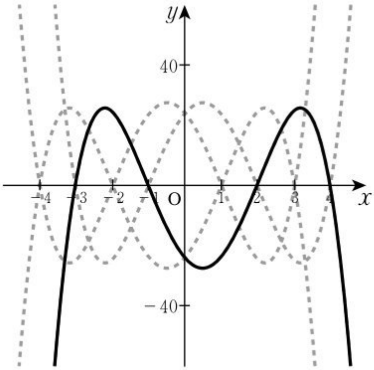
$y=(x+3)(x+1)(x-2)(x-4)$



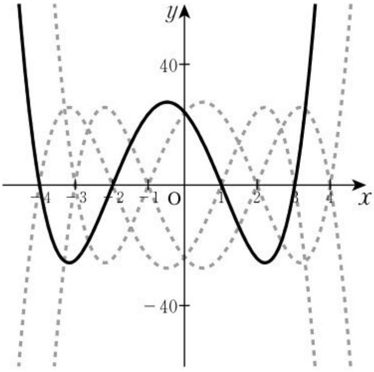
Note: If $-3<x<-1$, then $y<0$.

(For example, if you substitute $x=-2$, you get $y=-24<0$.)

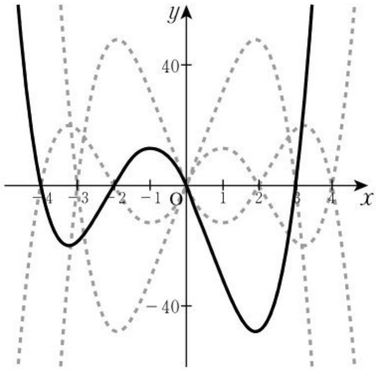
(2) $y=-(x+3)(x+1)(x-2)(x-4)$



(1) $y=(x+4)(x+2)(x-1)(x-3)$



(3) $y=x(x+4)(x+2)(x-3)$



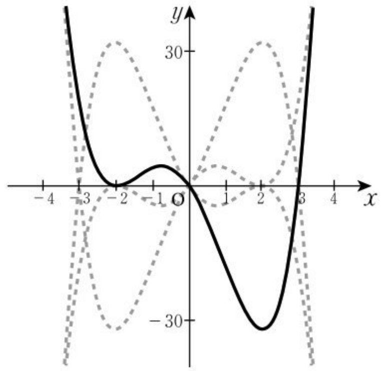
K 106a KUMON

Higher Degree Functions

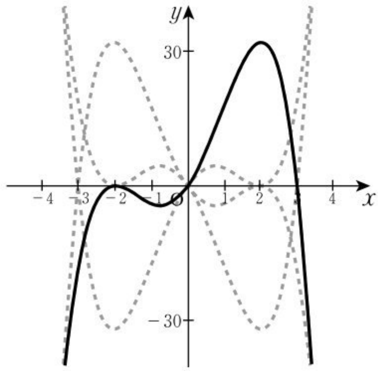
Trace the graphs of the following quartic functions.

Ex.

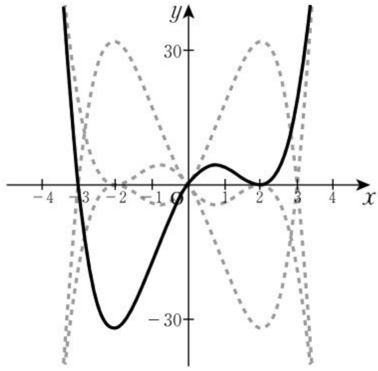
$y = x(x+2)^2(x-3)$



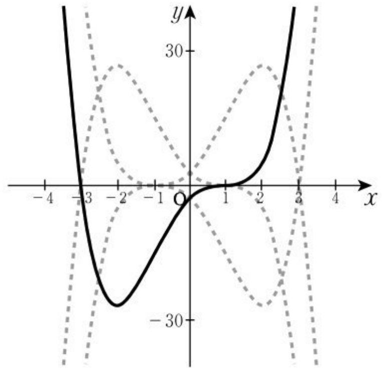
(2) $y = -x(x+2)^2(x-3)$



(1) $y = x(x+3)(x-2)^2$

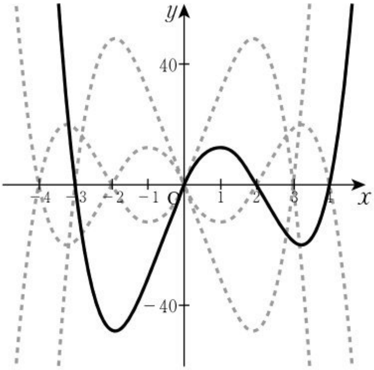


(3)* $y = (x+3)(x-1)^3$

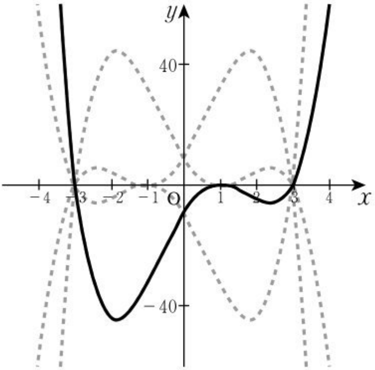


K 106b

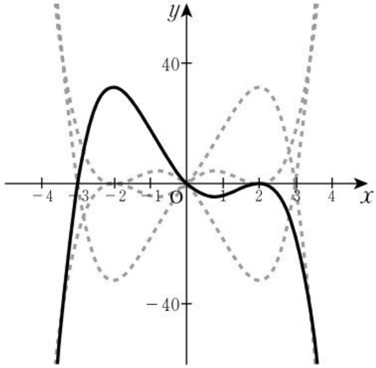
(4)　$y=x(x+3)(x-2)(x-4)$



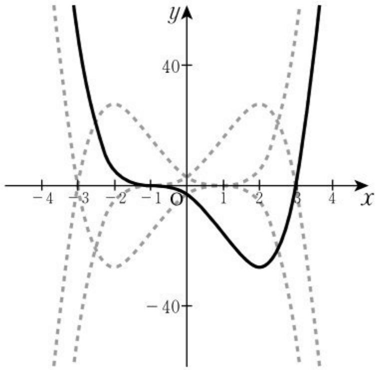
(5)　$y=(x+3)(x-1)^2(x-3)$



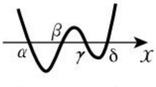
(6)　$y=-x(x+3)(x-2)^2$



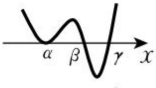
(7)　$y=(x+1)^3(x-3)$



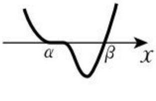
- Consider the quartic function $y=a(x-\overset{\textit{alpha}}{\alpha})(x-\overset{\textit{beta}}{\beta})(x-\overset{\textit{gamma}}{\gamma})(x-\overset{\textit{delta}}{\delta})$ $(a>0,\ \alpha<\beta<\gamma<\delta)$.
  When $y=0$, $x=\alpha,\ \beta,\ \gamma,\ \delta$. The graph crosses the x-axis at four points.



- Consider the quartic function $y=a(x-\alpha)^2(x-\beta)(x-\gamma)$ $(a>0,\ \alpha<\beta<\gamma)$.
  When $y=0$, $x=\alpha,\ \beta,\ \gamma$. The graph touches the x-axis at $x=\alpha$
  and crosses it at $x=\beta,\ \gamma$.



- Consider the quartic function $y=a(x-\alpha)^3(x-\beta)$ $(a>0,\ \alpha<\beta)$.
  When $y=0$, $x=\alpha,\ \beta$. The graph crosses the x-axis at $x=\alpha$ and at $x=\beta$.



K 107a KUMON

Higher Degree Functions

1. For each quartic function, write the letter (A)～(F) of the corresponding sketch.

(1) $y=(x+4)(x+2)(x-1)(x-4)$ ··· **(B)**

(2) $y=(x+3)(x-1)(x-3)^2$ ··· **(F)**

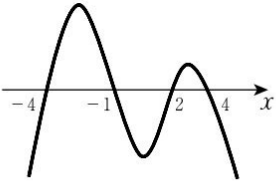
(3) $y=(x+3)(x-1)^2(x-3)$ ··· **(D)**

(4) $y=-(x+4)(x+1)(x-2)(x-4)$ ··· **(A)**

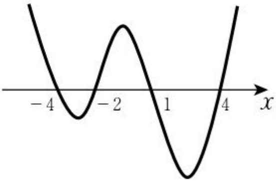
(5) $y=-(x+3)^2(x-1)(x-3)$ ··· **(E)**

(6) $y=-(x+3)(x-3)^3$ ··· **(C)**

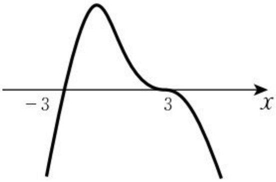
(A)



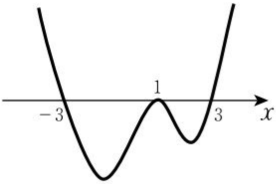
(B)



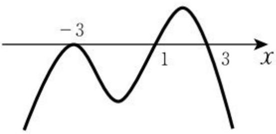
(C)



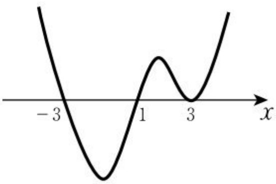
(D)



(E)



(F)



K 107b

2. For each of the three functions, write the letter (A)～(D) of the corresponding sketch.

(1) $y=-(x+3)^2(x-3)^2$

… $\boxed{\textbf{(B)}}$

(2) $y=(x+3)^4$

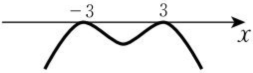
… $\boxed{\textbf{(C)}}$

(3) $y=-(x-3)^4$

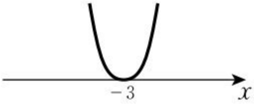
… $\boxed{\textbf{(D)}}$

(A)

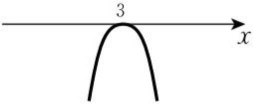
(B)



(C)



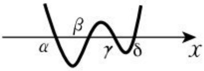
(D)



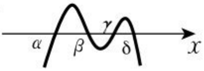
Note Summary

- The possible sketches of the quartic function $y=a(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$ $(\alpha<\beta<\gamma<\delta)$ are:

When $a>0$

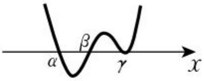


When $a<0$

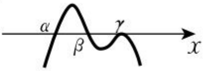


- The possible sketches of the quartic function $y=a(x-\alpha)(x-\beta)(x-\gamma)^2$ $(\alpha<\beta<\gamma)$ are:

When $a>0$

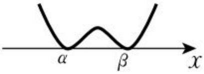


When $a<0$

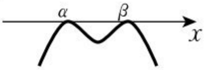


- The possible sketches of the quartic function $y=a(x-\alpha)^2(x-\beta)^2$ $(\alpha<\beta)$ are:

When $a>0$

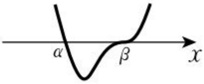


When $a<0$

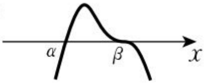


- The possible sketches of the quartic function $y=a(x-\alpha)(x-\beta)^3$ $(\alpha<\beta)$ are:

When $a>0$

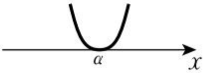


When $a<0$

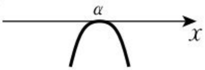


- The possible sketches of the quartic function $y=a(x-\alpha)^4$ are:

When $a>0$



When $a<0$



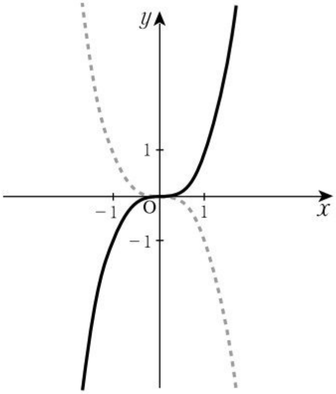
K 108a KUMON

Higher Degree Functions

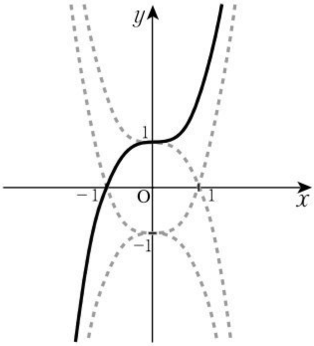
1. Trace the graphs of the following cubic functions.

Ex.

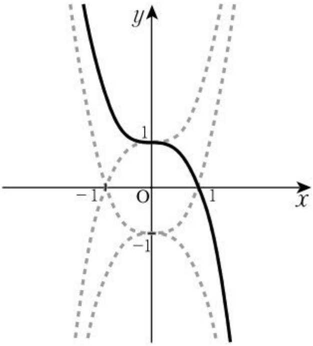
$y=x^3$



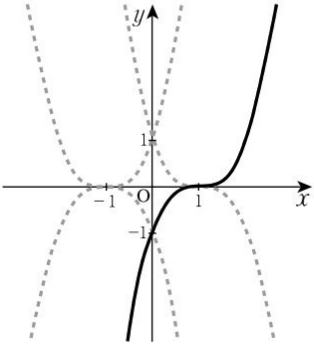
(1) $y=x^3+1$



(2) $y=-x^3+1$



(3) $y=(x-1)^3$



➢ The graph of $y=x^3+1$ is a translation of the graph of $y=x^3$, 1 unit along the y-axis.

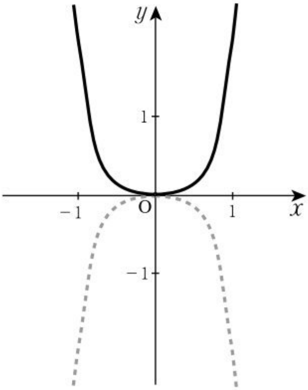
➢ The graph of $y=(x-1)^3$ is a translation of the graph of $y=x^3$, 1 unit along the x-axis.

K 108b

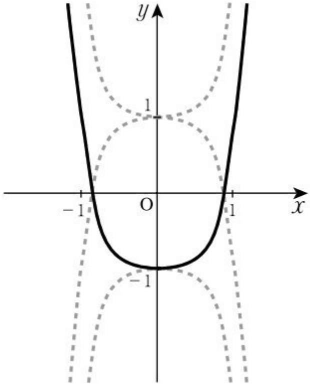
2. Trace the graphs of the following quartic functions.

Ex.

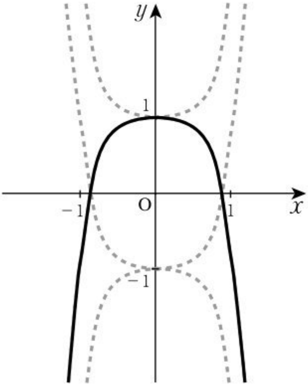
$y=2x^4$



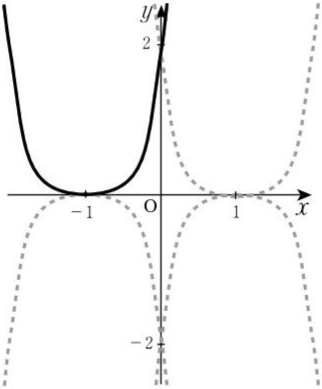
（1） $y=2x^4-1$



（2） $y=-2x^4+1$



（3） $y=2(x+1)^4$



3. Complete the following using the above graphs.

（1） The graph of $y=2x^4-1$ is a translation of the graph of $y=2x^4$,

$\boxed{\mathbf{-1}}$ unit(s) along the $\boxed{\boldsymbol{y}}$-axis.

（2） The graph of $y=2(x+1)^4$ is a translation of the graph of $y=2x^4$,

$\boxed{\mathbf{-1}}$ unit(s) along the $\boxed{\boldsymbol{x}}$-axis.

K 109a KUMON

Higher Degree Functions

1. Find the number of units and the direction in which the following graphs have been translated from the graph of $y=x^3$.

(1) $y=x^3+1$ $\boxed{1}$ unit(s) along the $\boxed{y}$-axis

(2) $y=x^3-2$ **−2 units along the *y*-axis**

(3) $y=(x-1)^3$ **1 unit along the *x*-axis**

(4) $y=(x+2)^3$ **−2 units along the *x*-axis**

2. Find the number of units and the direction in which the following graphs have been translated from the graph of $y=-2x^4$.

(1) $y=-2x^4+3$ **3 units along the *y*-axis**

(2) $y=-2x^4-1$ **−1 unit along the *y*-axis**

(3) $y=-2(x-3)^4$ **3 units along the *x*-axis**

(4) $y=-2(x+1)^4$ **−1 unit along the *x*-axis**

3. Find the equation of the graph that has been translated from the graph of the cubic function $y = -x^3$ as follows.

(1) 2 units along the x-axis $y = -(x - \boxed{\mathbf{2}})^3$

(2) 2 units along the y-axis $\boldsymbol{y = -x^3 + 2}$

(3) -1 unit along the y-axis $\boldsymbol{y = -x^3 - 1}$

4. Find the equation of the graph that has been translated from the graph of the quartic function $y = 3x^4$ as follows.

(1) 1 unit along the x-axis $\boldsymbol{y = 3(x-1)^4}$

(2) -3 units along the x-axis $\boldsymbol{y = 3(x+3)^4}$

(3) 1 unit along the y-axis $\boldsymbol{y = 3x^4 + 1}$

(4) -3 units along the y-axis $\boldsymbol{y = 3x^4 - 3}$

Consider this!

The graph of $y = 2(x-1)^3 + 4$ is a translation of $y = 2x^3$.

Think about how the graph has been translated.

Answer: 1 unit along the x-axis, and 4 units along the y-axis

K110a KUMON

Higher Degree Functions

1. For each function, write the letter (A)～(F) of the corresponding sketch.

(1) $y=(x+1)(x-1)(x-4)$ $\cdots$ **(E)**

(2) $y=(x+4)(x+1)(x-1)$ $\cdots$ **(A)**

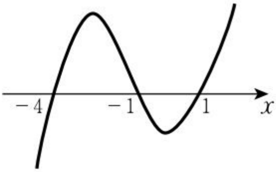
(3) $y=-(x+1)(x-1)(x-4)$ $\cdots$ **(C)**

(4) $y=(x+3)(x+1)(x-1)(x-4)$ $\cdots$ **(B)**

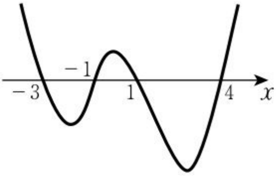
(5) $y=-(x+3)(x+1)^2(x-4)$ $\cdots$ **(D)**

(6) $y=-(x+3)(x+1)(x-4)^2$ $\cdots$ **(F)**

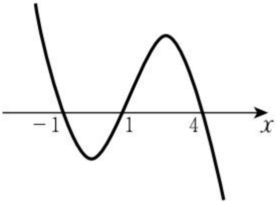
(A)



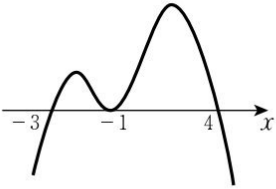
(B)



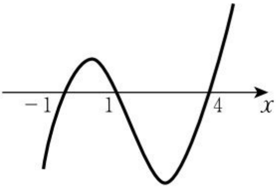
(C)



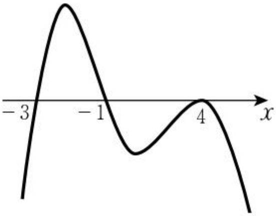
(D)



(E)

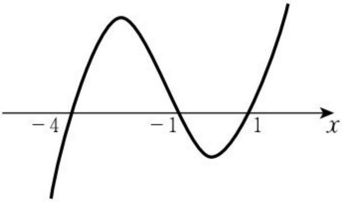


(F)

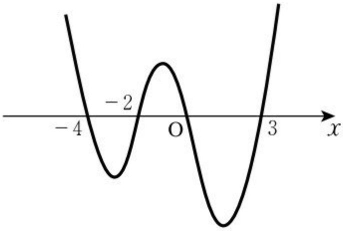


2. Sketch the following functions.

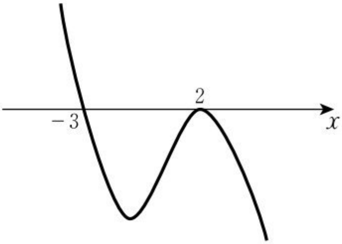
(1) $y=(x+4)(x+1)(x-1)$



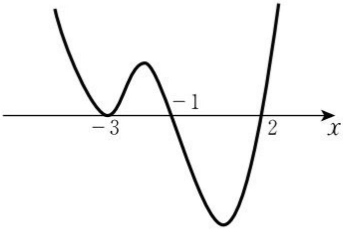
(3) $y=x(x+4)(x+2)(x-3)$



(2) $y=-(x+3)(x-2)^2$

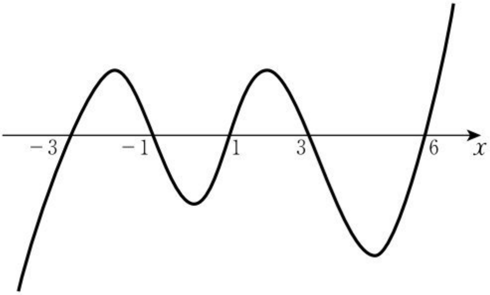


(4) $y=(x+3)^2(x+1)(x-2)$



3. Sketch the following ***quintic function***.

$y=(x+3)(x+1)(x-1)(x-3)(x-6)$ *Hint*



Hint

If $y=f(x)$, then $f(-4)<0$, $f(-3)=0$, $f(-2)>0$, …, $f(7)>0$

Terminology: A ***quintic function*** is a 5th degree polynomial function.

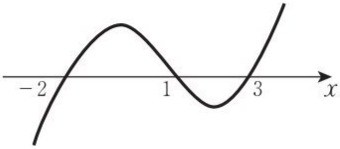
K111a KUMON

Higher Degree Equations and Inequalities

For each function, sketch the graph and find the values of x at which $y=0$.

Ex.

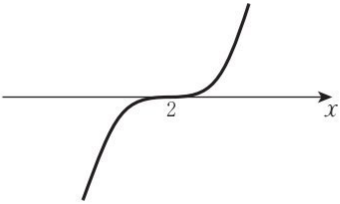
$y=(x+2)(x-1)(x-3)$



$(x+2)(x-1)(x-3)=0$

$x=-2,\ 1,\ 3$

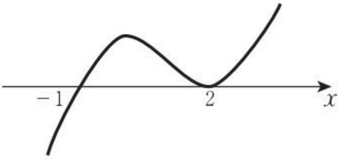
(2) $y=(x-2)^3$



$(x-2)^3=0$

$\boldsymbol{x=2}$

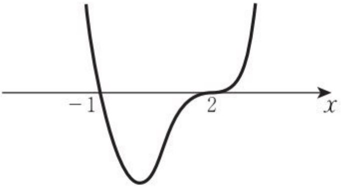
(1) $y=(x+1)(x-2)^2$



$(x+1)(x-2)^2=0$

$\boldsymbol{x=-1,\ 2}$

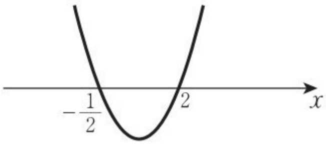
(3) $y=(x+1)(x-2)^3$



$(x+1)(x-2)^3=0$

$\boldsymbol{x=-1,\ 2}$

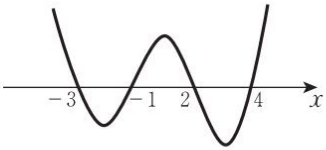
(4) $y=(2x+1)(x-2)$



$$(2x+1)(x-2)=0$$

$$\boldsymbol{x=-\frac{1}{2},\ 2}$$

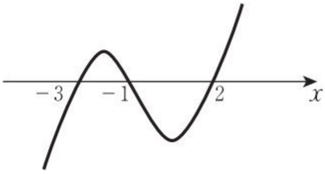
(6) $y=(x+3)(x+1)(x-2)(x-4)$



$$(x+3)(x+1)(x-2)(x-4)=0$$

$$\boldsymbol{x=-3,\ -1,\ 2,\ 4}$$

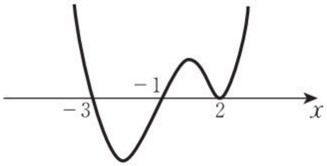
(5) $y=(x+3)(x+1)(x-2)$



$$(x+3)(x+1)(x-2)=0$$

$$\boldsymbol{x=-3,\ -1,\ 2}$$

(7) $y=(x+3)(x+1)(x-2)^2$



$$(x+3)(x+1)(x-2)^2=0$$

$$\boldsymbol{x=-3,\ -1,\ 2}$$

The solutions of the quadratic equation $(x-\alpha)(x-\beta)=0$ correspond to the points where the function $y=(x-\alpha)(x-\beta)$ intersects the x-axis.

Similarly, the solutions of the cubic equation $(x-\alpha)(x-\beta)(x-\gamma)=0$ correspond to the points where the function $y=(x-\alpha)(x-\beta)(x-\gamma)$ intersects the x-axis.

The solutions of the quartic equation $(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)=0$ correspond to the points where the function $y=(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$ intersects the x-axis.

Higher Degree Equations and Inequalities

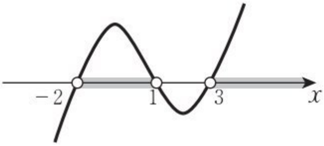
1. Solve the following inequalities by sketching.

Ex.

$(x+2)(x-1)(x-3)>0$

[Sol] Sketching

$y=(x+2)(x-1)(x-3)$,



Therefore,

$-2<x<1,\ x>3$

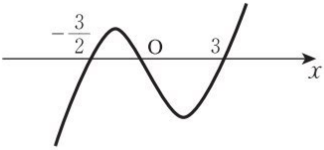
y is positive for the x-values in the shaded regions.

Note: For example, when substituting $x=0$, we get $y=6$. Thus, we can see that y is positive for $-2<x<1$.

(2) $x(2x+3)(x-3)>0$

[Sol] Sketching

$y=x(2x+3)(x-3)$,



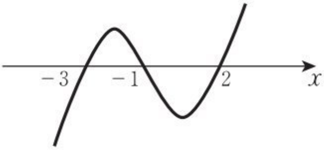
Therefore,

$$-\frac{3}{2}<x<0,\ x>3$$

(1) $(x+3)(x+1)(x-2)<0$

[Sol] Sketching

$y=(x+3)(x+1)(x-2)$,



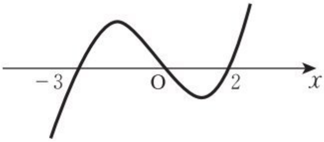
Therefore,

$$x<-3,\ -1<x<2$$

(3) $x(x-2)(x+3)<0$

[Sol] Sketching

$y=x(x-2)(x+3)$,



Therefore,

$$x<-3,\ 0<x<2$$

2. Solve the following inequalities by sketching.

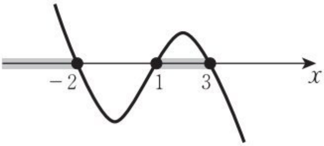
Ex.

$(x+2)(x-1)(3-x) \geqq 0$

[Sol] $-(x+2)(x-1)(x-3) \geqq 0$

Sketching

$y = -(x+2)(x-1)(x-3)$,



Therefore, $x \leqq -2$, $1 \leqq x \leqq 3$

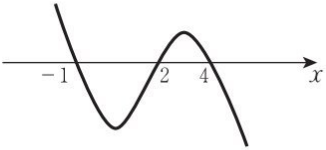
Note: For example, when substituting $x=2$, we get $y=4$. Thus, we can see that y is positive for $1<x<3$.

(1) $(x+1)(2-x)(x-4) \leqq 0$

[Sol] $-(x+1)(x-2)(x-4) \leqq 0$

Sketching

$y = -(x+1)(x-2)(x-4)$,



Therefore,

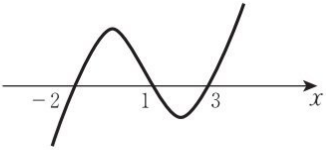
$\boldsymbol{-1 \leqq x \leqq 2,\ x \geqq 4}$

(2) $(x+2)(1-x)(3-x) < 0$

[Sol] $(x+2)(x-1)(x-3) < 0$

Sketching

$y = (x+2)(x-1)(x-3)$,



Therefore,

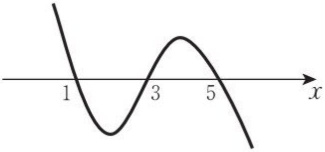
$\boldsymbol{x < -2,\ 1 < x < 3}$

(3) $(1-x)(3-x)(5-x) \geqq 0$

[Sol] $-(x-1)(x-3)(x-5) \geqq 0$

Sketching

$y = -(x-1)(x-3)(x-5)$,



Therefore,

$\boldsymbol{x \leqq 1,\ 3 \leqq x \leqq 5}$

K 1 1 3a KUMON

Higher Degree Equations and Inequalities

1. Solve the following inequalities by sketching.

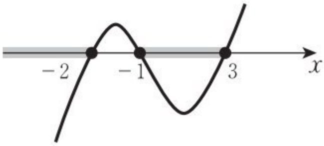
Ex.

$(x+2)(x^2-2x-3) \leqslant 0$

[Sol] $(x+2)(x-3)(x+1) \leqslant 0$

Sketching

$y=(x+2)(x-3)(x+1)$,



Therefore,

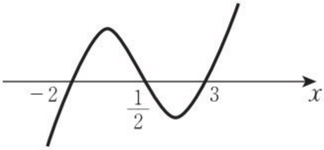
$x \leqslant -2, \quad -1 \leqslant x \leqslant 3$

(2) $(2x-1)(x^2-x-6)<0$

[Sol] $(2x-1)(x-3)(x+2)<0$

Sketching

$y=(2x-1)(x-3)(x+2)$,



Therefore,

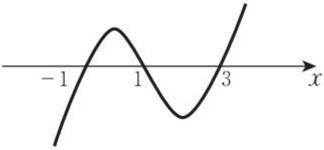
$\boldsymbol{x<-2, \quad \frac{1}{2}<x<3}$

(1) $(x+1)(x^2-4x+3)>0$

[Sol] $(x+1)(x-3)(x-1)>0$

Sketching

$y=(x+1)(x-3)(x-1)$,



Therefore,

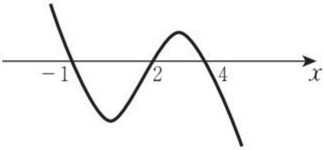
$\boldsymbol{-1<x<1, \quad x>3}$

(3) $(2-x)(x^2-3x-4) \geqslant 0$

[Sol] $-(x-2)(x-4)(x+1) \geqslant 0$

Sketching

$y=-(x-2)(x-4)(x+1)$,



Therefore,

$\boldsymbol{x \leqslant -1, \quad 2 \leqslant x \leqslant 4}$

K113b

2. Solve the following inequalities by sketching.

Ex.

$(x-1)(x^2-4x+2)>0$

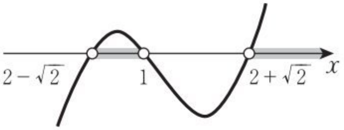
[Sol] Solving the equation

$$x^2-4x+2=0,$$

$$x=2\pm\sqrt{2}$$

Sketching

$$y=(x-1)(x^2-4x+2),$$



Therefore,

$$2-\sqrt{2}<x<1,\ x>2+\sqrt{2}$$

(1) $(x+1)(x^2+2x-2)\leqslant 0$

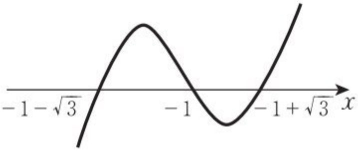
[Sol] Solving the equation

$$x^2+2x-2=0,$$

$$x=-1\pm\sqrt{3}$$

Sketching

$$y=(x+1)(x^2+2x-2),$$



Therefore,

$$\boldsymbol{x\leqslant -1-\sqrt{3},\ -1\leqslant x\leqslant -1+\sqrt{3}}$$

(2) $(3-x)(x^2-2x-1)>0$

[Sol] $-(x-3)(x^2-2x-1)>0$

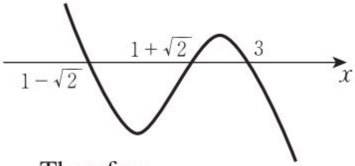
Solving the equation

$$x^2-2x-1=0,$$

$$x=1\pm\sqrt{2}$$

Sketching

$$y=-(x-3)(x^2-2x-1),$$



Therefore,

$$\boldsymbol{x<1-\sqrt{2},\ 1+\sqrt{2}<x<3}$$

(3) $(x-2)(x^2-3x+1)\geqslant 0$

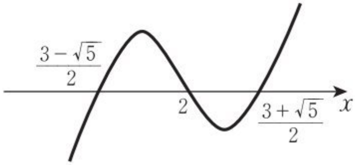
[Sol] Solving the equation

$$x^2-3x+1=0,$$

$$x=\frac{3\pm\sqrt{5}}{2}$$

Sketching

$$y=(x-2)(x^2-3x+1),$$



[**Note:** $\sqrt{5}=2.23\ldots$]

Therefore,

$$\boldsymbol{\frac{3-\sqrt{5}}{2}\leqslant x\leqslant 2,\ x\geqslant\frac{3+\sqrt{5}}{2}}$$

Higher Degree Equations and Inequalities

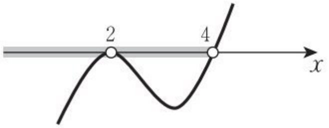
1. Solve the following inequalities by sketching.

Ex.

$(x-2)^2(x-4)<0$

[Sol] Sketching

$y=(x-2)^2(x-4)$,



Therefore,

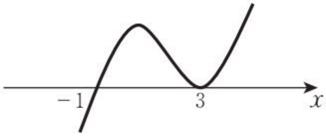
$x<2,\ 2<x<4$ ☜ $x=2$ is not included in the answer.

(2) $(x+1)(x^2-6x+9)<0$

[Sol] $(x+1)(x-3)^2<0$

Sketching

$y=(x+1)(x-3)^2$,

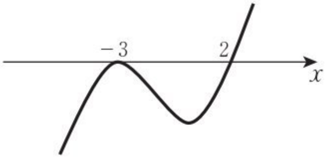


Therefore, $x<\boxed{\mathbf{-1}}$

(1) $(x+3)^2(x-2)<0$

[Sol] Sketching

$y=(x+3)^2(x-2)$,



Therefore,

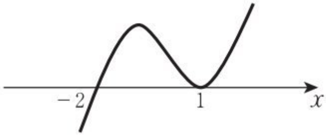
$\boldsymbol{x<-3,\ -3<x<2}$

(3) $(x-1)(x^2+x-2)<0$

[Sol] $(x-1)^2(x+2)<0$

Sketching

$y=(x-1)^2(x+2)$,



Therefore, $\boldsymbol{x<-2}$

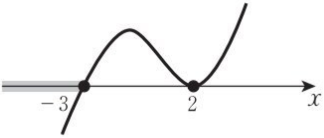
2. Solve the following inequalities by sketching.

Ex.

$(x+3)(x-2)^2 \leqq 0$

[Sol] Sketching

$y=(x+3)(x-2)^2$,



Therefore,

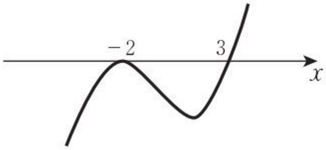
$x \leqq -3,\ x=2$ ☜ $x=2$ is included in the answer.

(2) $(x+2)(x^2-x-6) \leqq 0$

[Sol] $(x+2)^2(x-3) \leqq 0$

Sketching

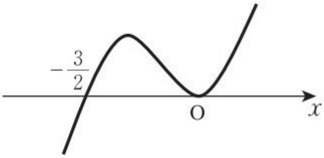
$y=(x+2)^2(x-3)$,



Therefore, $x \leqq \boxed{3}$

(1) $x^2(2x+3) \leqq 0$

[Sol] Sketching $y=x^2(2x+3)$,



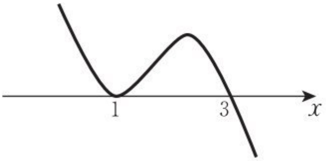
Therefore, $\boldsymbol{x \leqq -\frac{3}{2},\ x=0}$

(3) $(1-x)(x^2-4x+3) \geqq 0$

[Sol] $-(x-1)^2(x-3) \geqq 0$

Sketching

$y=-(x-1)^2(x-3)$,



Therefore, $\boldsymbol{x \leqq 3}$

K115a KUMON

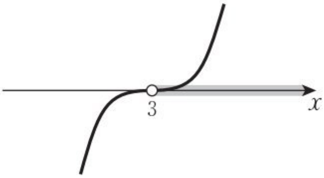
Higher Degree Equations and Inequalities

1. Solve the following inequalities by sketching.

Ex.

$(x-3)^3>0$

[Sol] Sketching $y=(x-3)^3$,

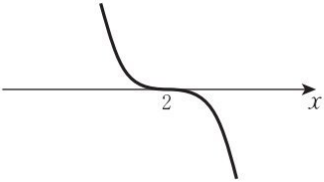


Therefore, $x>3$

(2) $(2-x)^3<0$

[Sol] $-(x-2)^3<0$

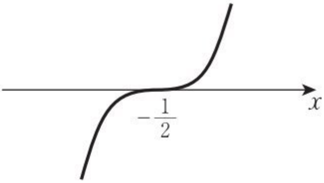
Sketching $y=-(x-2)^3$,



Therefore, $\boldsymbol{x>2}$

(1) $(2x+1)^3>0$

[Sol] Sketching $y=(2x+1)^3$,

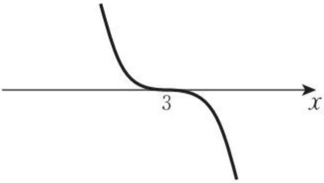


Therefore, $\boldsymbol{x>-\frac{1}{2}}$

(3) $(3-x)(x^2-6x+9)\geqq 0$

[Sol] $-(x-3)^3\geqq 0$

Sketching $y=-(x-3)^3$,



Therefore, $\boldsymbol{x\leqq 3}$

2. Solve the following inequalities.

Ex.

$(x+2)(x^2+3x+5)>0$

[Sol] Let D be the discriminant of the equation $x^2+3x+5=0$.

As $D=9-20=-11<0$,

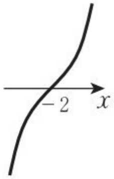
so $x^2+3x+5>0$ for all x.

Therefore,

$x+2>0$ ☜ Since x^2+3x+5 is always positive.

Thus,

$x>-2$



[Reference]
The sketch is shown on the right.

(1) $(2x+1)(x^2-x+2)\leqslant 0$

[Sol] Let D be the discriminant of the equation $x^2-x+2=0$.

As $D=1-8=-7<0$,

so $x^2-x+2>0$ for all x.

Therefore, $2x+1\leqslant 0$

Thus, $\boldsymbol{x\leqslant -\frac{1}{2}}$

(2) $(1-2x)(x^2-2x+3)\geqslant 0$

[Sol] $-(2x-1)(x^2-2x+3)\geqslant 0$

Let D be the discriminant of the equation

$x^2-2x+3=0$.

As $\frac{D}{4}=1-3=-2<0$,

so $x^2-2x+3>0$ for all x.

Therefore, $-(2x-1)\geqslant 0$

Thus, $\boldsymbol{x\leqslant \frac{1}{2}}$

(3) $x^3-x^2+x-1\geqslant 0$

[Sol] $(x-1)(x^2+1)\geqslant 0$

$x^2+1>0$ for all x.

Therefore, $x-1\geqslant 0$

Thus, $\boldsymbol{x\geqslant 1}$

K 116a KUMON

Higher Degree Equations and Inequalities

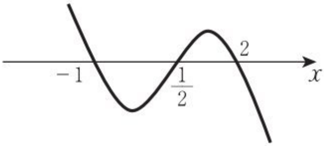
Solve the following inequalities by sketching.

(1) $(1-2x)(x^2-x-2)\leq 0$

[Sol] $-(2x-1)(x-2)(x+1)\leq 0$

Sketching

$y=-(2x-1)(x-2)(x+1)$,



Therefore,

$$\boldsymbol{-1\leq x\leq \frac{1}{2},\ \ x\geq 2}$$

(2) $(x-2)(x^2-4x-1)\geq 0$

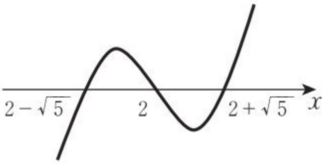
[Sol] Solving the equation

$x^2-4x-1=0$,

$x=2\pm\sqrt{5}$

Sketching

$y=(x-2)(x^2-4x-1)$,



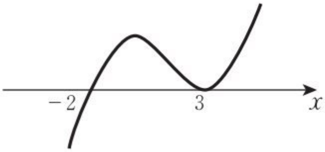
Therefore,

$$\boldsymbol{2-\sqrt{5}\leq x\leq 2,\ \ x\geq 2+\sqrt{5}}$$

(3) $(x-3)(x^2-x-6)>0$

[Sol] $(x-3)^2(x+2)>0$

Sketching $y=(x-3)^2(x+2)$,

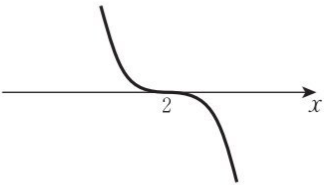


Therefore, $\boldsymbol{-2<x<3,\ \ x>3}$

(4) $(2-x)(x^2-4x+4)>0$

[Sol] $-(x-2)^3>0$

Sketching $y=-(x-2)^3$,



Therefore, $\boldsymbol{x<2}$

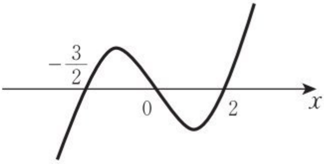
K116b

(5) $2x^3-x^2-6x<0$

[Sol] $x(2x+3)(x-2)<0$

Sketching

$y=x(2x+3)(x-2)$,



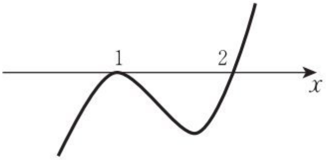
Therefore,

$$\boldsymbol{x<-\frac{3}{2},\ \ 0<x<2}$$

(6) $x^3-4x^2+5x-2\leqslant 0$

[Sol] $(x-1)^2(x-2)\leqslant 0$

Sketching $y=(x-1)^2(x-2)$,

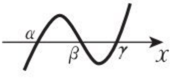


Therefore, $\boldsymbol{x\leqslant 2}$

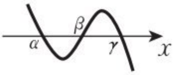
Note Summary

- Solving the inequality $a(x-\alpha)(x-\beta)(x-\gamma)>0\ \ (\alpha<\beta<\gamma)$,

When $a>0$: $\alpha<x<\beta,\ x>\gamma$

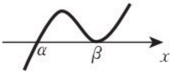


When $a<0$: $x<\alpha,\ \beta<x<\gamma$

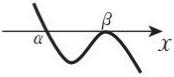


- Solving the inequality $a(x-\alpha)(x-\beta)^2>0\ \ (\alpha<\beta)$,

When $a>0$: $\alpha<x<\beta,\ x>\beta$

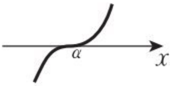


When $a<0$: $x<\alpha$

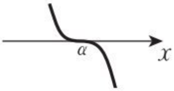


- Solving the inequality $a(x-\alpha)^3>0$,

When $a>0$: $x>\alpha$



When $a<0$: $x<\alpha$



K 117a KUMON

Higher Degree Equations and Inequalities

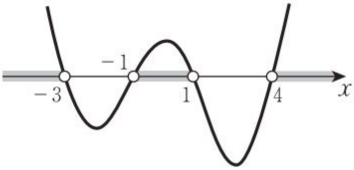
1. Solve the following inequalities by sketching.

Ex.

$(x+3)(x+1)(x-1)(x-4)>0$

[Sol] Sketching

$y=(x+3)(x+1)(x-1)(x-4)$,



Therefore,

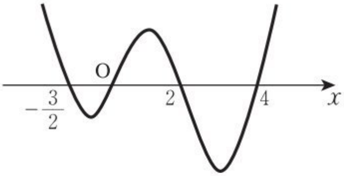
$x<-3,\ -1<x<1,\ x>4$

Note: For example, when substituting $x=0$, we get $y=12$. Thus, we can see that y is positive for $-1<x<1$.

(1) $x(2x+3)(x-2)(x-4)<0$

[Sol] Sketching

$y=x(2x+3)(x-2)(x-4)$,



Therefore,

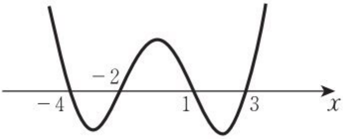
$\boldsymbol{-\frac{3}{2}<x<0,\ 2<x<4}$

(2) $(x+4)(x-3)(x^2+x-2)\leqq 0$

[Sol] $(x+4)(x-3)(x+2)(x-1)\leqq 0$

Sketching

$y=(x+4)(x-3)(x+2)(x-1)$,



Therefore,

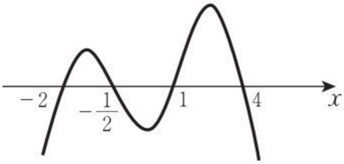
$\boldsymbol{-4\leqq x\leqq -2,\ 1\leqq x\leqq 3}$

(3) $(2x+1)(1-x)(x^2-2x-8)\leqq 0$

[Sol] $-(2x+1)(x-1)(x-4)(x+2)\leqq 0$

Sketching

$y=-(2x+1)(x-1)(x-4)(x+2)$,



Therefore,

$\boldsymbol{x\leqq -2,\ -\frac{1}{2}\leqq x\leqq 1,\ x\geqq 4}$

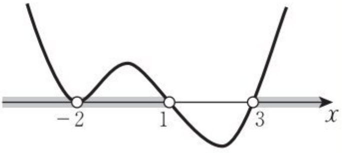
2. Solve the following inequalities by sketching.

Ex.

$(x+2)^2(x-1)(x-3)>0$

[Sol] Sketching

$y=(x+2)^2(x-1)(x-3)$,



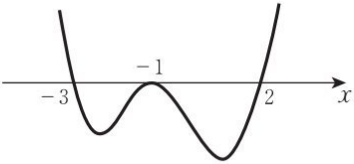
Therefore,

$x<-2,\ -2<x<1,\ x>3$

(1) $(x+3)(x+1)^2(x-2)<0$

[Sol] Sketching

$y=(x+3)(x+1)^2(x-2)$,



Therefore,

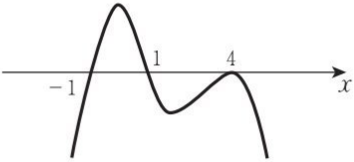
$-3<x<-1,\ -1<x<2$

(2) $(x+1)(1-x)(x-4)^2\geqslant 0$

[Sol] $-(x+1)(x-1)(x-4)^2\geqslant 0$

Sketching

$y=-(x+1)(x-1)(x-4)^2$,



Therefore,

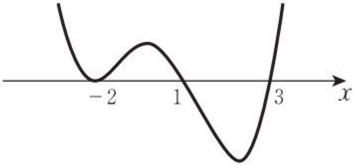
$-1\leqslant x\leqslant 1,\ x=4$

(3) $(x+2)(x-3)(x^2+x-2)\geqslant 0$

[Sol] $(x+2)^2(x-3)(x-1)\geqslant 0$

Sketching

$y=(x+2)^2(x-3)(x-1)$,



Therefore, **$x\leqslant 1,\ x\geqslant 3$**

K118a KUMON

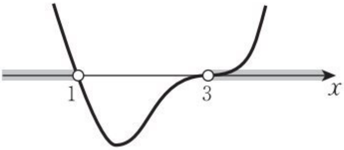
Higher Degree Equations and Inequalities

1. Solve the following inequalities by sketching.

Ex.

$(x-1)(x-3)^3>0$

[Sol] Sketching $y=(x-1)(x-3)^3$,

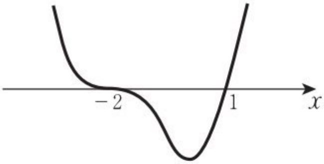


Therefore, $x<1$, $x>3$

(2) $(x^2+4x+4)(x^2+x-2)\geqq 0$

[Sol] $(x+2)^3(x-1)\geqq 0$

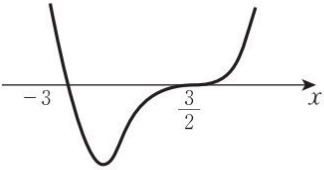
Sketching $y=(x+2)^3(x-1)$,



Therefore, $\boldsymbol{x\leqq -2,\ x\geqq 1}$

(1) $(x+3)(2x-3)^3<0$

[Sol] Sketching $y=(x+3)(2x-3)^3$,

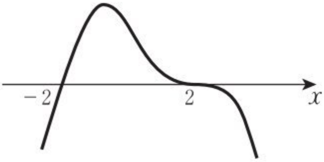


Therefore, $\boldsymbol{-3<x<\frac{3}{2}}$

(3) $(2-x)(x^3-2x^2-4x+8)\geqq 0$

[Sol] $-(x-2)^3(x+2)\geqq 0$

Sketching $y=-(x-2)^3(x+2)$,



Therefore, $\boldsymbol{-2\leqq x\leqq 2}$

2. Solve the following inequalities.

Ex.

$(x+2)(x-1)(x^2+3x+3)<0$

[Sol]

Calculating the discriminant, D, of

$x^2+3x+3=0$,

$D=9-12=-3<0$.

So $x^2+3x+3>0$ for all x.

Therefore,

$(x+2)(x-1)<0$ ☜

Since x^2+3x+3 is always positive.

Thus, $-2<x<1$

(1) $(x+2)(2x+3)(x^2-3x+4)\geqslant 0$

[Sol] Calculating the discriminant, D, of

$x^2-3x+4=0$,

$D=9-16=-7<0$.

So $x^2-3x+4>0$ for all x.

Therefore,

$(x+2)(2x+3)\geqslant 0$

Thus, $\boldsymbol{x\leqslant -2,\ x\geqslant -\frac{3}{2}}$

(2) $(x^2-x+2)(x^2-x-6)<0$

[Sol] $(x^2-x+2)(x-3)(x+2)<0$

Calculating the discriminant, D, of $x^2-x+2=0$,

$D=1-8=-7<0$.

So $x^2-x+2>0$ for all x.

Therefore,

$(x-3)(x+2)<0$

Thus, $\boldsymbol{-2<x<3}$

(3) $(2-3x)(x^3+4x)\leqslant 0$

[Sol] $-x(3x-2)(x^2+4)\leqslant 0$

$x^2+4>0$ for all x.

Therefore,

$-x(3x-2)\leqslant 0$

Thus, $\boldsymbol{x\leqslant 0,\ x\geqslant \frac{2}{3}}$

K119a KUMON

Higher Degree Equations and Inequalities

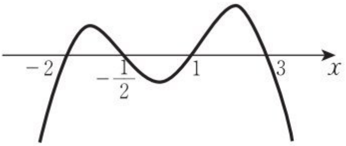
1. Solve the following inequalities by sketching.

(1) $(2x+1)(1-x)(x^2-x-6)<0$

[Sol] $-(2x+1)(x-1)(x-3)(x+2)<0$

Sketching

$y=-(2x+1)(x-1)(x-3)(x+2)$,



Therefore,

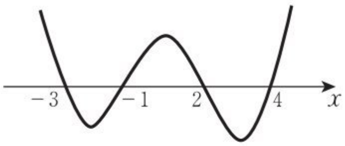
$\boldsymbol{x<-2,\ -\frac{1}{2}<x<1,\ x>3}$

(2) $(x^2-3x-4)(x^2+x-6)>0$

[Sol] $(x-4)(x+1)(x+3)(x-2)>0$

Sketching

$y=(x-4)(x+1)(x+3)(x-2)$,



Therefore,

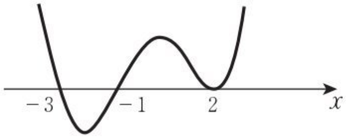
$\boldsymbol{x<-3,\ -1<x<2,\ x>4}$

(3) $(x^2-x-2)(x^2+x-6)>0$

[Sol] $(x-2)^2(x+1)(x+3)>0$

Sketching

$y=(x-2)^2(x+1)(x+3)$,



Therefore,

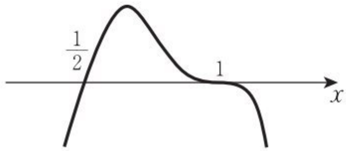
$\boldsymbol{x<-3,\ -1<x<2,\ x>2}$

(4) $(1-2x)(x^3-3x^2+3x-1)\leqq 0$

[Sol] $-(2x-1)(x-1)^3\leqq 0$

Sketching

$y=-(2x-1)(x-1)^3$,



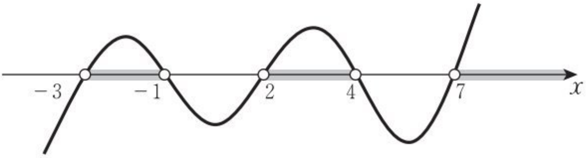
Therefore, $\boldsymbol{x\leqq\frac{1}{2},\ x\geqq 1}$

2. Solve the following inequalities by sketching.

Ex.

$(x+3)(x+1)(x-2)(x-4)(x-7)>0$

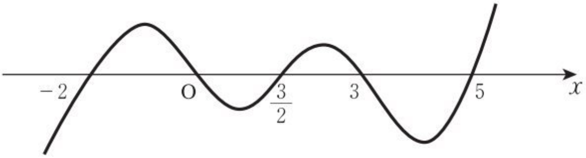
Sketching $y=(x+3)(x+1)(x-2)(x-4)(x-7)$,



Therefore, $-3<x<-1$, $2<x<4$, $x>7$

(1) $x(x+2)(2x-3)(x-3)(x-5)<0$

[Sol] Sketching $y=x(x+2)(2x-3)(x-3)(x-5)$,

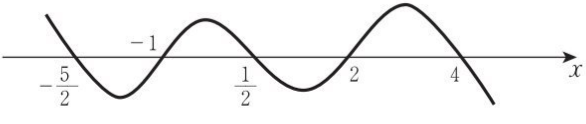


Therefore, $\boldsymbol{x<-2,\ 0<x<\frac{3}{2},\ 3<x<5}$

(2) $(x+1)(2x+5)(1-2x)(x-2)(x-4)\geqq 0$

[Sol] $-(x+1)(2x+5)(2x-1)(x-2)(x-4)\geqq 0$

Sketching $y=-(x+1)(2x+5)(2x-1)(x-2)(x-4)$,



Therefore, $\boldsymbol{x\leqq -\frac{5}{2},\ -1\leqq x\leqq \frac{1}{2},\ 2\leqq x\leqq 4}$

K 120a KUMON

Higher Degree Equations and Inequalities

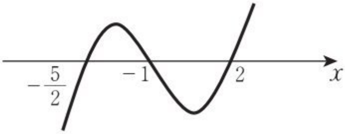
Solve the following inequalities.

(1) $(2x+5)(x^2-x-2)<0$

[Sol] $(2x+5)(x-2)(x+1)<0$

Sketching

$y=(2x+5)(x-2)(x+1)$,



Therefore,

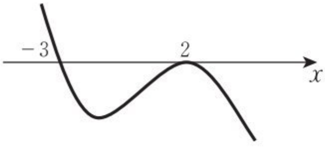
$\boldsymbol{x<-\frac{5}{2},\ -1<x<2}$

(2) $(2-x)(x^2+x-6)\geqq 0$

[Sol] $-(x-2)^2(x+3)\geqq 0$

Sketching

$y=-(x-2)^2(x+3)$,



Therefore,

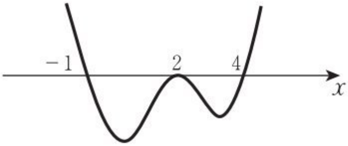
$\boldsymbol{x\leqq -3,\ x=2}$

(3) $(x^2-x-2)(x^2-6x+8)<0$

[Sol] $(x-2)^2(x+1)(x-4)<0$

Sketching

$y=(x-2)^2(x+1)(x-4)$,



Therefore,

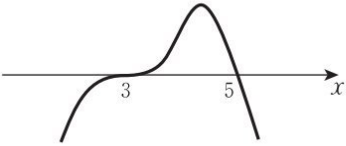
$\boldsymbol{-1<x<2,\ 2<x<4}$

(4) $(3-x)(x-5)(x^2-6x+9)\leqq 0$

[Sol] $-(x-3)^3(x-5)\leqq 0$

Sketching

$y=-(x-3)^3(x-5)$,



Therefore,

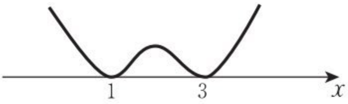
$\boldsymbol{x\leqq 3,\ x\geqq 5}$

K120b

(5)　$(x^2-4x+3)^2>0$

[Sol] $(x-1)^2(x-3)^2>0$

Sketching $y=(x-1)^2(x-3)^2$,



Therefore,

$\boldsymbol{x<1,\ 1<x<3,\ x>3}$

$\boldsymbol{(x\neq 1,\ x\neq 3)}$

(7)　$(x^2+x-6)(x^2+x+6)\leqq 0$

[Sol] $(x+3)(x-2)(x^2+x+6)\leqq 0$

Calculating the discriminant, D, of $x^2+x+6=0$,

$D=1-24=-23<0$.

So $x^2+x+6>0$ for all x.

Therefore,

$(x+3)(x-2)\leqq 0$

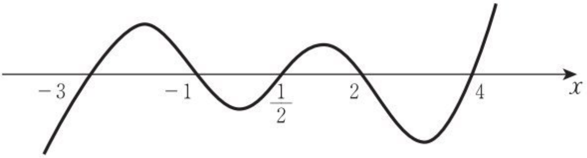
Thus, $\boldsymbol{-3\leqq x\leqq 2}$

(6)　$(2x-1)(x^2-3x-4)(x^2+x-6)\leqq 0$

[Sol] $(2x-1)(x-4)(x+1)(x+3)(x-2)\leqq 0$

Sketching

$y=(2x-1)(x-4)(x+1)(x+3)(x-2)$,



Therefore, $\boldsymbol{x\leqq -3,\ -1\leqq x\leqq \frac{1}{2},\ 2\leqq x\leqq 4}$

Consider this!

$-x(x-1)(x-2)>0$ ⋯①

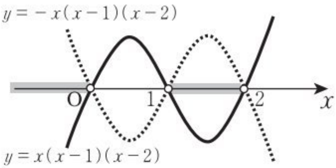
You can solve ① in the following way:

Multiplying both sides by -1,

$x(x-1)(x-2)<0$ Notice the direction of the inequality sign.

From the sketch of $y=x(x-1)(x-2)$,

$x<\boxed{0}$, $\boxed{1}<x<\boxed{2}$



Answers: in order 0, 1, 2

K 121 a KUMON

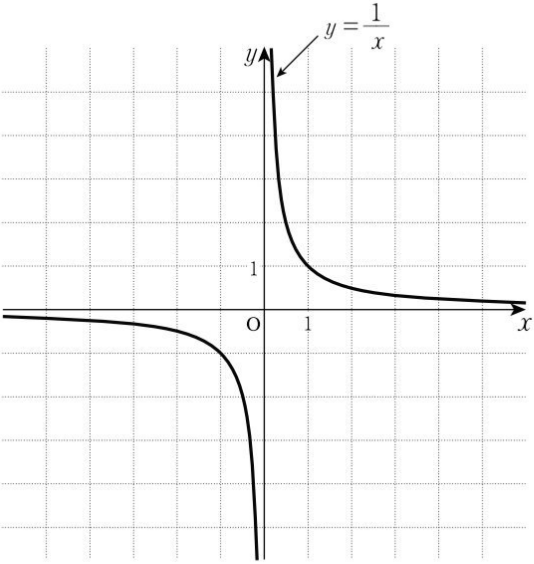
Graphs of Fractional Functions I

Graph the following fractional functions.

Ex.

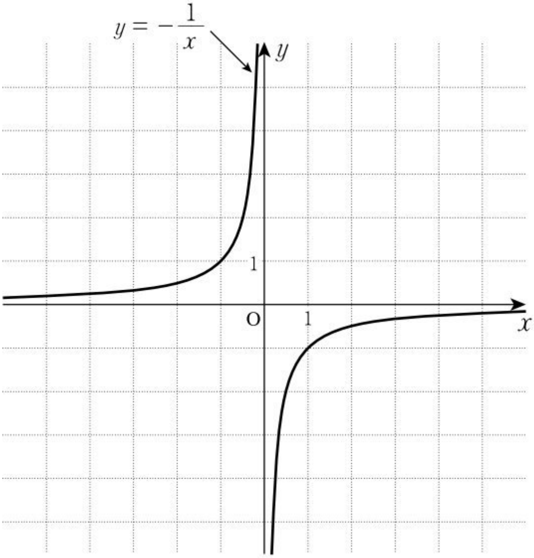
$y = \frac{1}{x}$

x	y
-3	$-\frac{1}{3}$
-2	$-\frac{1}{2}$
-1	-1
$-\frac{1}{2}$	-2
$-\frac{1}{3}$	-3
0	╳
$\frac{1}{3}$	3
$\frac{1}{2}$	2
1	1
2	$\frac{1}{2}$
3	$\frac{1}{3}$



(1) $y = -\frac{1}{x}$

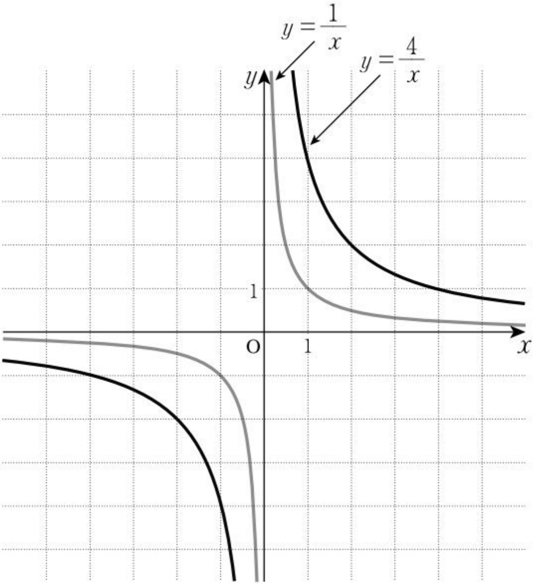
x	y
-3	$\mathbf{\frac{1}{3}}$
-2	$\mathbf{\frac{1}{2}}$
-1	**1**
$-\frac{1}{2}$	**2**
$-\frac{1}{3}$	**3**
0	╳
$\frac{1}{3}$	**−3**
$\frac{1}{2}$	**−2**
1	**−1**
2	$\mathbf{-\frac{1}{2}}$
3	$\mathbf{-\frac{1}{3}}$



K121b

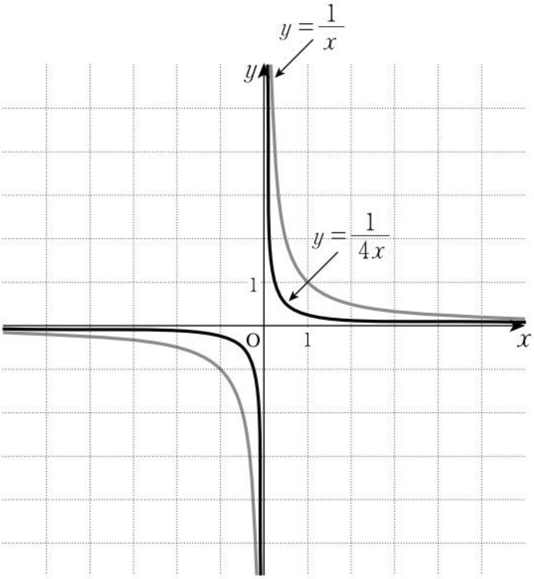
（2）　$y=\dfrac{4}{x}$

x	y
-5	$\mathbf{-\frac{4}{5}}$
-4	$\mathbf{-1}$
-3	$\mathbf{-\frac{4}{3}}$
-2	$\mathbf{-2}$
-1	$\mathbf{-4}$
0	
1	$\mathbf{4}$
2	$\mathbf{2}$
3	$\mathbf{\frac{4}{3}}$
4	$\mathbf{1}$
5	$\mathbf{\frac{4}{5}}$



（3）　$y=\dfrac{1}{4x}$

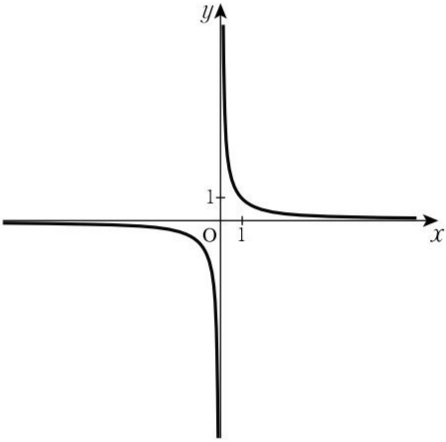
x	y
-3	$-\frac{1}{12}$
-2	$-\frac{1}{8}$
-1	$-\frac{1}{4}$
$-\frac{1}{2}$	$-\frac{1}{2}$
$-\frac{1}{4}$	-1
0	
$\frac{1}{4}$	$\mathbf{1}$
$\frac{1}{2}$	$\mathbf{\frac{1}{2}}$
1	$\mathbf{\frac{1}{4}}$
2	$\mathbf{\frac{1}{8}}$
3	$\mathbf{\frac{1}{12}}$



K I22a KUMON

Graphs of Fractional Functions I

1. Complete the following, using the graph of $y = \frac{1}{x}$.



Find the value of y for each of the following values of x.

When $x = 10$, $y = \boxed{\frac{1}{10}}$ When $x = 1$, $y = \boxed{1}$

When $x = 100$, $y = \boxed{\frac{1}{100}}$ When $x = \frac{1}{10}$, $y = \boxed{10}$

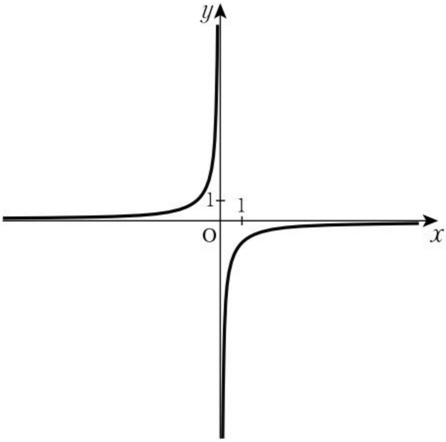
When $x = 1000$, $y = \boxed{\frac{1}{1000}}$ When $x = \frac{1}{100}$, $y = \boxed{100}$

$\vdots$ $\vdots$

From the graph of $y = \frac{1}{x}$,

- As x gets further away from zero, the graph gets closer to the x-axis. However, it never crosses the x-axis.
- As x approaches zero, the graph gets closer to the y-axis. However, it never crosses the y-axis.

2. Complete the following, using the graph of $y = -\dfrac{1}{x}$.



Find the value of y for each of the following values of x.

When $x = 10$, $y = \boxed{-\frac{1}{10}}$ When $x = 1$, $y = \boxed{-1}$

When $x = 100$, $y = \boxed{-\frac{1}{100}}$ When $x = \frac{1}{10}$, $y = \boxed{-10}$

When $x = 1000$, $y = \boxed{-\frac{1}{1000}}$ When $x = \frac{1}{100}$, $y = \boxed{-100}$

⋮ ⋮

From the graph of $y = -\dfrac{1}{x}$,

- As x gets further away from zero, the graph gets closer to the x-axis. However, it never crosses the x-axis.
- As x approaches zero, the graph gets closer to the y-axis. However, it never crosses the y-axis.

Generally, a line which a graph approaches indefinitely without crossing is called an ***asymptote*** of the graph. The asymptotes of $y = \dfrac{1}{x}$ and $y = -\dfrac{1}{x}$ are $y = 0$ (the x-axis) and $x = 0$ (the y-axis).

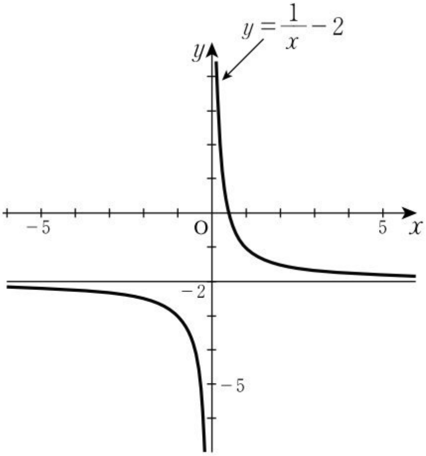
K 123a KUMON

Graphs of Fractional Functions I

Graph the following fractional functions.

(1) $y=\dfrac{1}{x}-2$

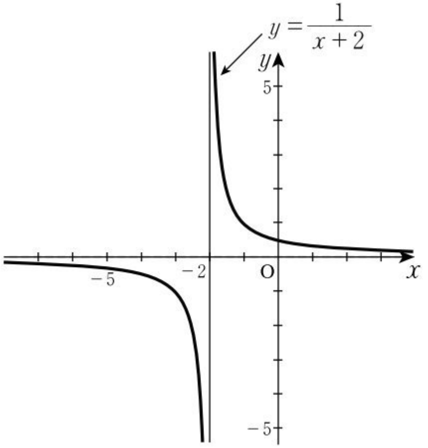
x	y
-3	$-\frac{7}{3}$
-2	$-\frac{5}{2}$
-1	-3
$-\frac{1}{2}$	-4
$-\frac{1}{3}$	-5
0	╳
$\frac{1}{3}$	1
$\frac{1}{2}$	0
1	-1
2	$-\frac{3}{2}$
3	$-\frac{5}{3}$



Asymptotes: $x=\boxed{0}$, $y=-2$

(2) $y=\dfrac{1}{x+2}$

x	y
-5	$-\frac{1}{3}$
-4	$-\frac{1}{2}$
-3	-1
$-\frac{5}{2}$	-2
$-\frac{7}{3}$	-3
-2	╳
$-\frac{5}{3}$	3
$-\frac{3}{2}$	2
-1	1
0	$\frac{1}{2}$
1	$\frac{1}{3}$

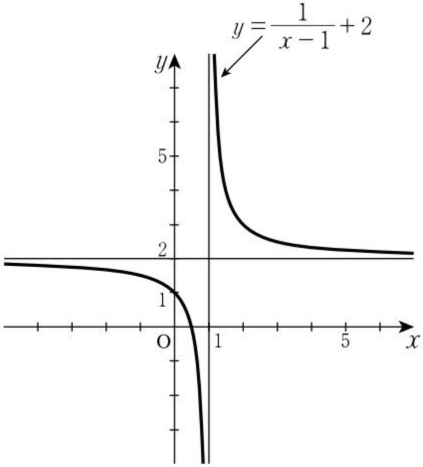


Asymptotes: $x=-2$, $y=\boxed{0}$

K123b

(3) $y=\dfrac{1}{x-1}+2$

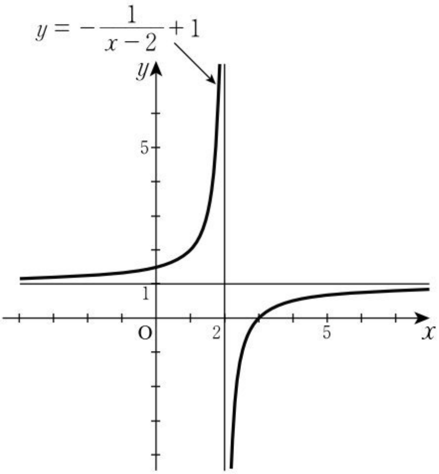
x	y
-2	$\frac{5}{3}$
-1	$\frac{3}{2}$
0	1
$\frac{1}{2}$	0
$\frac{2}{3}$	-1
1	╳
$\frac{4}{3}$	5
$\frac{3}{2}$	4
2	3
3	$\frac{5}{2}$
4	$\frac{7}{3}$



Asymptotes: $x=\boxed{\mathbf{1}}$, $y=\boxed{\mathbf{2}}$

(4) $y=-\dfrac{1}{x-2}+1$

x	y
-1	$\frac{4}{3}$
0	$\frac{3}{2}$
1	2
$\frac{3}{2}$	3
$\frac{5}{3}$	4
2	╳
$\frac{7}{3}$	-2
$\frac{5}{2}$	-1
3	0
4	$\frac{1}{2}$
5	$\frac{2}{3}$



Asymptotes: $x=\boxed{\mathbf{2}}$, $y=\boxed{\mathbf{1}}$

From questions (1)〜(4),
the asymptotes of $y=\dfrac{k}{x-p}+q\ (k\neq 0)$ are $x=p$, $y=q$.

K 124a KUMON

Graphs of Fractional Functions I

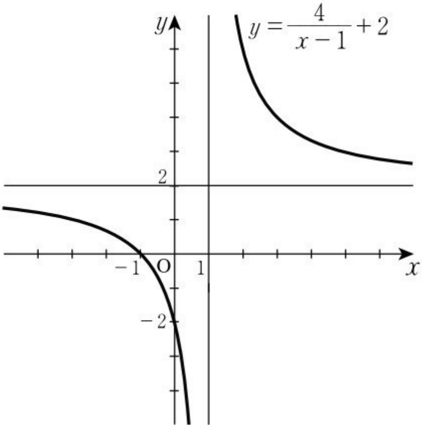
The asymptotes of the graph of $y=\dfrac{k}{x-p}+q\ (k\neq 0)$ are $x=p,\ y=q$.

Find the asymptotes of the following fractional functions, find the intercepts, and draw the graphs.

Ex.

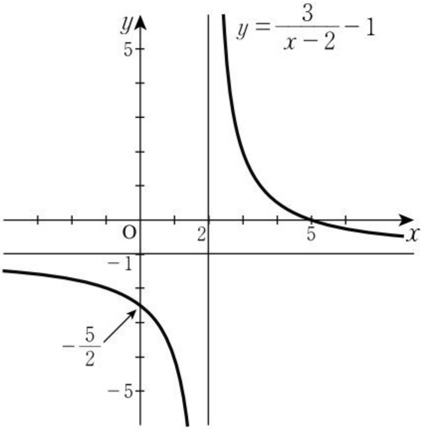
$y=\dfrac{4}{x-1}+2$

- Asymptotes: $x=1,\ y=2$
- Intercepts:
  x-axis: $(-1,\ 0)$ ☜ When $x=-1$, $y=0$
  y-axis: $(0,\ -2)$ ☜ When $x=0$, $y=-2$
- Graph
  (Draw the asymptotes first.)



(1) $y=\dfrac{3}{x-2}-1$

- Asymptotes: $\boldsymbol{x=2,\ y=-1}$
- Intercepts:
  x-axis: $(5,\ 0)$
  y-axis: $\left(0,\ -\dfrac{5}{2}\right)$
- Graph



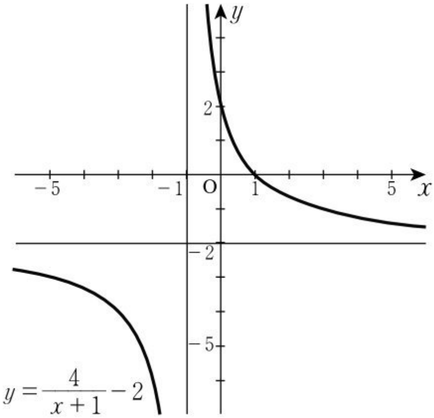
K 124b

(2) $y=\dfrac{4}{x+1}-2$

- Asymptotes: $\boldsymbol{x=-1,\ y=-2}$

- Intercepts:
  x-axis: (1, 0)
  y-axis: (0, 2)

- Graph

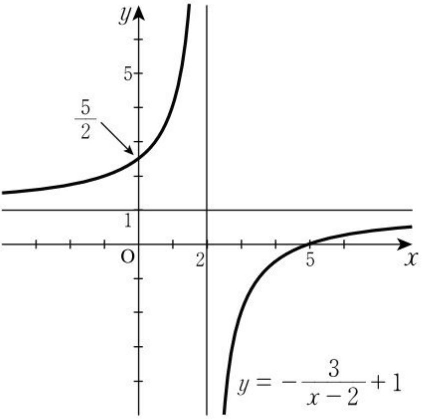


(3) $y=-\dfrac{3}{x-2}+1$

- Asymptotes: $\boldsymbol{x=2,\ y=1}$

- Intercepts:
  x-axis: (5, 0)
  y-axis: $\left(0,\ \dfrac{5}{2}\right)$

- Graph



K I25a KUMON

Graphs of Fractional Functions I

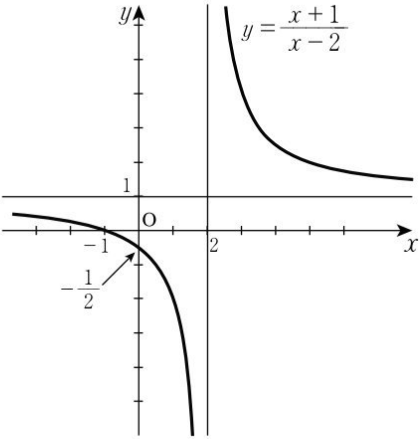
Find the asymptotes of the following fractional functions, find the intercepts, and draw the graphs.

Ex.

$$y=\frac{x+1}{x-2}$$

$$y=1+\frac{3}{x-2}$$ ☞ $$\begin{array}{r} 1 \\ x-2\,\overline{)\,x+1} \\ x-2 \\ \hline 3 \end{array}$$

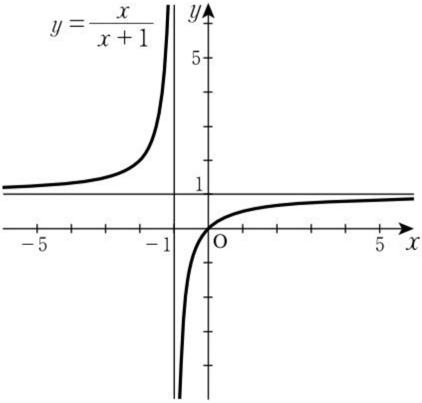
- Asymptotes: $x=2,\ y=1$ *Note
- Intercepts:
  x-axis: $(-1,\ 0)$
  y-axis: $\left(0,\ -\frac{1}{2}\right)$
- Graph



(1) $y=\dfrac{x}{x+1}$

$$y=1-\frac{1}{x+1}$$

- Asymptotes: $\boldsymbol{x=-1,\ y=1}$
- Intercepts:
  $(0,\ 0)$
- Graph



Note: In order to identify the asymptotes easily, you can rewrite each function as follows:

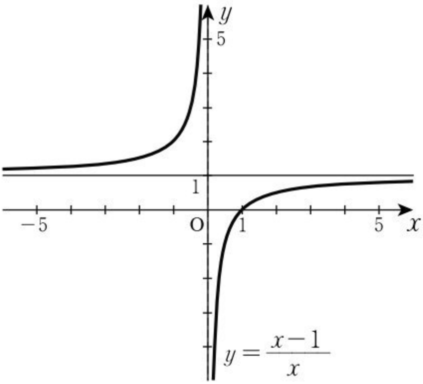
$$y=\frac{x+1}{x-2}=\frac{(x-2)+3}{x-2}=1+\frac{3}{x-2}$$

K 125b

(2) $y=\frac{x-1}{x}$

$y=1-\frac{1}{x}$

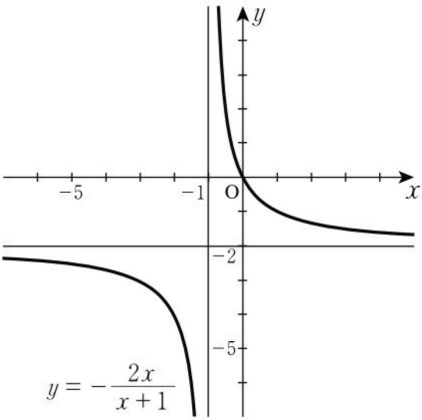
- Asymptotes: $\boldsymbol{x=0,\ y=1}$
- Intercepts:
  x-axis: (1, 0)
- Graph



(3) $y=-\frac{2x}{x+1}$

$y=-2+\frac{2}{x+1}$

- Asymptotes: $\boldsymbol{x=-1,\ y=-2}$
- Intercepts:
  (0, 0)
- Graph



To find the x-intercept of $y=\frac{ax+b}{cx+d}$, solve the equation $ax+b=0$ (i.e. the numerator $=0$).

For example, given $y=\frac{x+1}{x-2}$, solving $x+1=0$ we find $x=-1$, which is the x-intercept.

K 126a KUMON

Graphs of Fractional Functions I

Find the asymptotes of the following fractional functions, find the intercepts, and draw the graphs.

Ex.

$$y=\frac{2x+2}{2x-1}$$

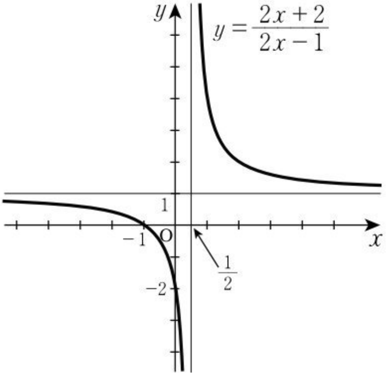
$$y=1+\frac{3}{2x-1}$$ ☜ $$\begin{array}{r} 1 \\ 2x-1\,\overline{)\,2x+2} \\ \underline{2x-1} \\ 3 \end{array}$$

*Note

- Asymptotes: $x=\frac{1}{2}$, $y=1$

- Intercepts:
  x-axis: $(-1,\ 0)$
  y-axis: $(0,\ -2)$

- Graph



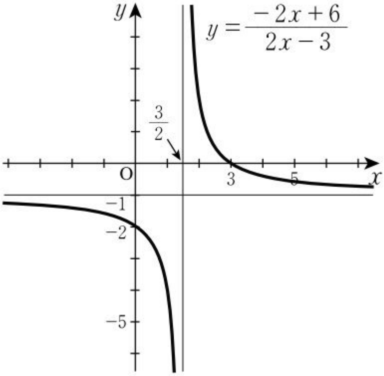
(1) $$y=\frac{-2x+6}{2x-3}$$

$$y=-1+\frac{3}{2x-3}$$

- Asymptotes: $\boldsymbol{x=\frac{3}{2}}$, $\boldsymbol{y=-1}$

- Intercepts:
  x-axis: $(3,\ 0)$
  y-axis: $(0,\ -2)$

- Graph



Note: In order to identify the asymptotes easily, you can rewrite each function as follows:

$$y=\frac{2x+2}{2x-1}=\frac{(2x-1)+3}{2x-1}=1+\frac{3}{2x-1}$$

K 126b

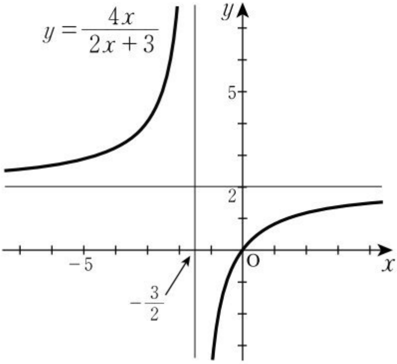
（2） $y=\dfrac{4x}{2x+3}$

$y=2-\dfrac{6}{2x+3}$

- Asymptotes: $\boldsymbol{x=-\dfrac{3}{2}}$**,** $\boldsymbol{y=2}$

- Intercepts:
  (0, 0)

- Graph



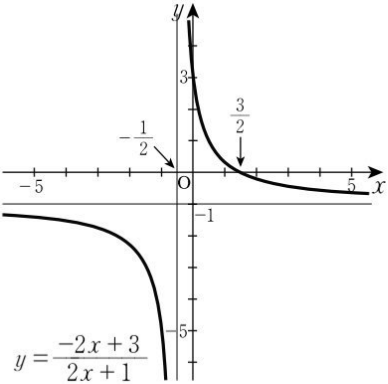
（3） $y=\dfrac{-2x+3}{2x+1}$

$y=-1+\dfrac{4}{2x+1}$

- Asymptotes: $\boldsymbol{x=-\dfrac{1}{2}}$**,** $\boldsymbol{y=-1}$

- Intercepts:
  (The x-intercept is a fraction.)
  ***x*-axis:** $\boldsymbol{\left(\dfrac{3}{2},\ 0\right)}$
  ***y*-axis: (0, 3)**

- Graph



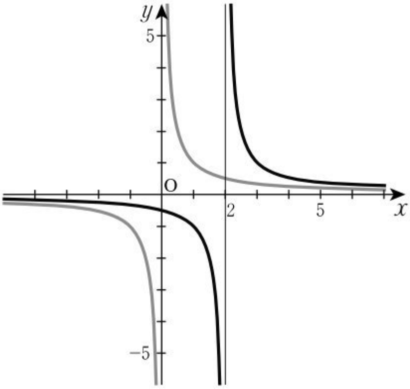
K I27a KUMON

Graphs of Fractional Functions I

For each fractional function, find the equations of the asymptotes, then draw them on the graph. (If the asymptote is the x or y-axis, there is no need to draw it again.) Then, state how each graph has been translated from $y=\dfrac{1}{x}$.

(1) $y=\dfrac{1}{x-2}$

Asymptotes: $\boldsymbol{x=2,\ y=0}$

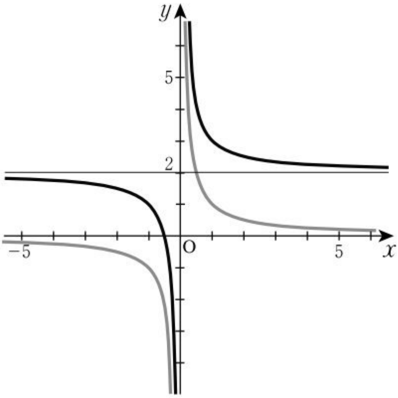


Translation:

$\boxed{\mathbf{2}}$ unit(s) along the x-axis

(2) $y=\dfrac{1}{x}+2$

Asymptotes: $\boldsymbol{x=0,\ y=2}$



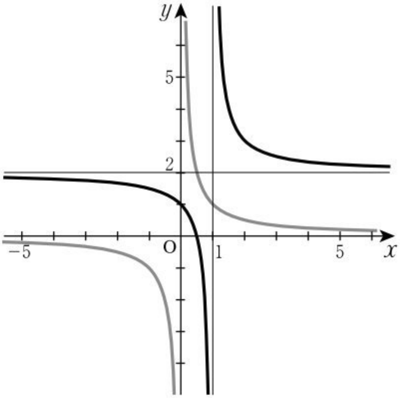
Translation:

$\boxed{\mathbf{2}}$ unit(s) along the y-axis

K127b

(3) $y=\dfrac{1}{x-1}+2$

Asymptotes: $\boldsymbol{x=1,\ y=2}$



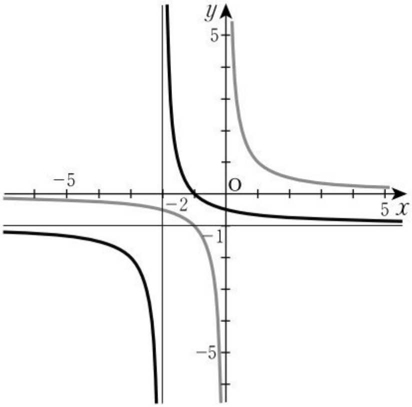
Translation:

$\boxed{\mathbf{1}}$ unit(s) along the x-axis

$\boxed{\mathbf{2}}$ unit(s) along the y-axis

(4) $y=\dfrac{1}{x+2}-1$

Asymptotes: $\boldsymbol{x=-2,\ y=-1}$



Translation:

$\boxed{\mathbf{-2}}$ unit(s) along the x-axis

$\boxed{\mathbf{-1}}$ unit(s) along the y-axis

Note: Given a fractional function $y=\dfrac{k}{x-p}+q\ (k\neq 0)$, the graph's asymptotes are $x=p$ and $y=q$. This graph has been translated from $y=\dfrac{k}{x}$, p units along the x-axis and q units along the y-axis.

K 128a KUMON

Graphs of Fractional Functions I

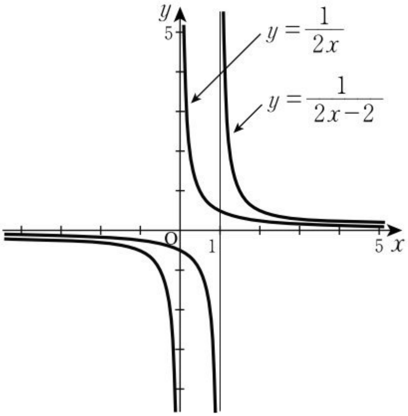
For each fractional function, find the equations of the asymptotes, then draw them on the graph. Then, state how each graph has been translated from $y=\frac{1}{2x}$.

(1) $y=\frac{1}{2x-2}$

$=\frac{1}{2(x-\boxed{1})}$

Asymptotes:

$x=\boxed{1}$, $y=\boxed{0}$

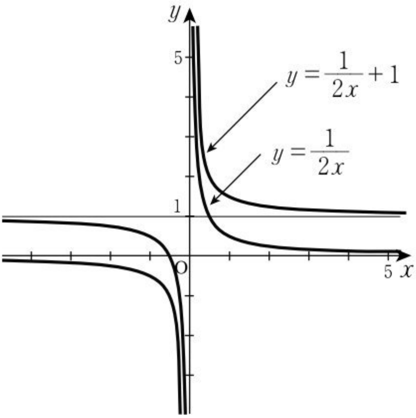


Translation:

$\boxed{1}$ unit(s) along the x-axis

(2) $y=\frac{1}{2x}+1$

Asymptotes: $\boldsymbol{x=0}$, $\boldsymbol{y=1}$



Translation:

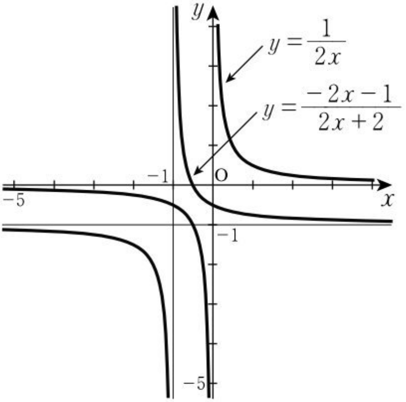
$\boxed{1}$ unit(s) along the y-axis

K 128b

(3) $y=\dfrac{-2x-1}{2x+2}$

$=\dfrac{1}{2(x+\boxed{1})}-\boxed{1}$

Asymptotes: $\boldsymbol{x=-1,\ y=-1}$



Translation:

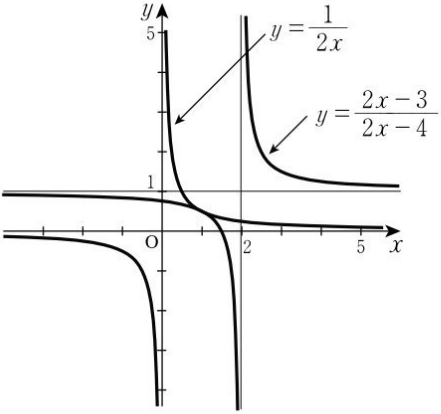
$\boxed{-1}$ unit(s) along the x-axis

$\boxed{-1}$ unit(s) along the y-axis

(4) $y=\dfrac{2x-3}{2x-4}$

$=\dfrac{1}{2(x-2)}+1$

Asymptotes: $\boldsymbol{x=2,\ y=1}$



Translation:

$\boxed{2}$ unit(s) along the x-axis

$\boxed{1}$ unit(s) along the y-axis

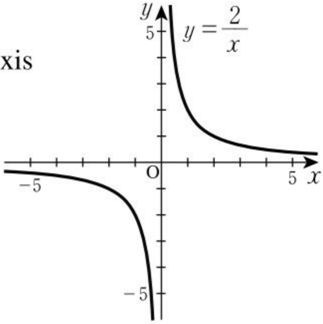
K 129a KUMON

Graphs of Fractional Functions I

1. State how each of the following graphs has been translated from $y=\frac{2}{x}$.

(1) $y=\frac{2}{x-1}$

[Sol] Translation: $\boxed{\mathbf{1}}$ unit(s) along the x-axis



(2) $y=\frac{2}{x}+3$

[Sol] Translation: **3 units along the y-axis**

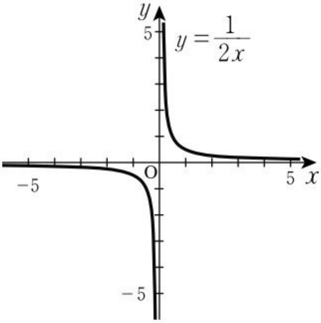
(3) $y=\frac{2}{x+3}-1$

[Sol] Translation: **-3 units along the x-axis,**
and -1 unit along the y-axis

2. State how each of the following graphs has been translated from $y=\frac{1}{2x}$.

(1) $y=\frac{1}{2x-2}=\frac{1}{2(x-\boxed{\mathbf{1}})}$

[Sol] Translation: **1 unit along the x-axis**



(2) $y=\frac{1}{2x}-2$

[Sol] Translation: **-2 units along the y-axis**

(3) $y=\frac{1}{2x+4}+1=\frac{1}{2(x+\boxed{\mathbf{2}})}+1$

[Sol] Translation: **-2 units along the x-axis,**
and 1 unit along the y-axis

K 129b

3. Given the function $y=\frac{1}{x}$, state the equation of the graph described in each translation.

（1） 3 units along the y-axis

[Sol] $\boldsymbol{y=\frac{1}{x}+3}$

（2） -1 unit along the x-axis

[Sol] $\boldsymbol{y=\frac{1}{x+1}}$

（3） 2 units along the x-axis, and -1 unit along the y-axis

[Sol] $\boldsymbol{y=\frac{1}{x-2}-1}$

（4） -3 units along the x-axis, and 2 units along the y-axis

[Sol] $\boldsymbol{y=\frac{1}{x+3}+2}$

4. Given the function $y=-\frac{2}{x}$, state the equation of the graph described in each translation.

（1） -1 unit along the y-axis

[Sol] $\boldsymbol{y=-\frac{2}{x}-1}$

（2） 3 units along the x-axis

[Sol] $\boldsymbol{y=-\frac{2}{x-3}}$

（3） 1 unit along the x-axis, and -3 units along the y-axis

[Sol] $\boldsymbol{y=-\frac{2}{x-1}-3}$

（4） -2 units along the x-axis, and 1 unit along the y-axis

[Sol] $\boldsymbol{y=-\frac{2}{x+2}+1}$

K130a KUMON

Graphs of Fractional Functions I

1. For each function, write the letter (A)～(F) of the corresponding sketch.

(1) $y=\dfrac{2}{x}+1$ $\cdots\ \boxed{\textbf{(C)}}$

(4) $y=1-\dfrac{2}{x-1}$ $\cdots\ \boxed{\textbf{(E)}}$

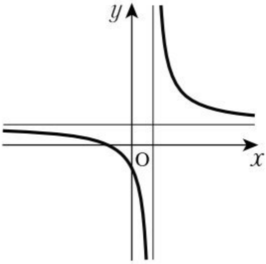
(2) $y=\dfrac{2}{x-1}$ $\cdots\ \boxed{\textbf{(F)}}$

(5) $y=-1-\dfrac{2}{x+1}$ $\cdots\ \boxed{\textbf{(B)}}$

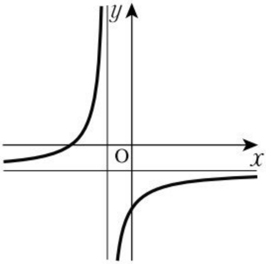
(3) $y=1+\dfrac{2}{x-1}$ $\cdots\ \boxed{\textbf{(A)}}$

(6) $y=-1+\dfrac{2}{x+1}$ $\cdots\ \boxed{\textbf{(D)}}$

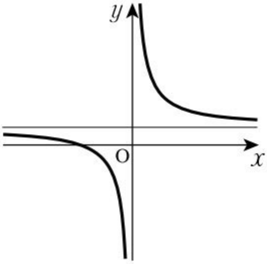
(A)



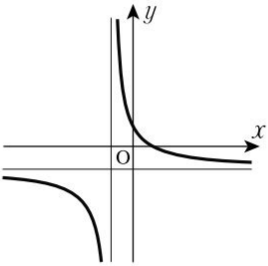
(B)



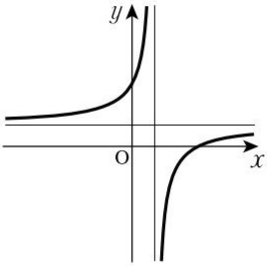
(C)



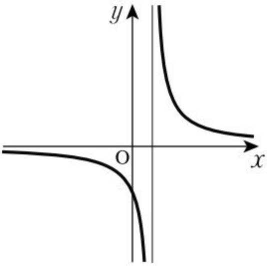
(D)



(E)



(F)



K130b

2. Write the letters (A)～(F) of the functions that satisfy each condition (1)～(3).

(1) The graphs that have been translated from $y=\dfrac{1}{x}$ are ⋯ $\boxed{\textbf{(A)}}$, $\boxed{\textbf{(C)}}$

(2) The graphs that have been translated from $y=-\dfrac{1}{x}$ are ⋯ $\boxed{\textbf{(B)}}$, $\boxed{\textbf{(E)}}$

(3) The graphs that have been translated from $y=\dfrac{1}{2x}$ are ⋯ $\boxed{\textbf{(D)}}$, $\boxed{\textbf{(F)}}$

(A) $y=\dfrac{1}{x-4}$

(B) $y=\dfrac{x-3}{x-2}=\boxed{\mathbf{1}}-\dfrac{\boxed{\mathbf{1}}}{x-2}$

(C) $y=\dfrac{2x+3}{x+1}=\mathbf{2+\dfrac{1}{x+1}}$

(D) $y=\dfrac{1}{2(x-1)}$

(E) $y=\dfrac{x-1}{x}=\mathbf{1-\dfrac{1}{x}}$

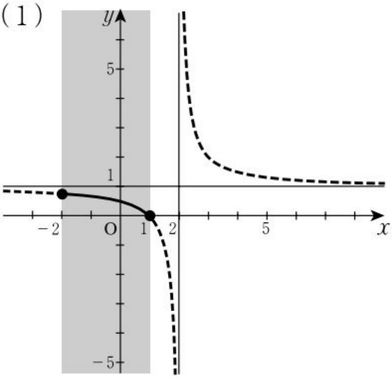
(F) $y=\dfrac{2x+5}{2x+4}=\mathbf{1+\dfrac{1}{2(x+2)}}$

K 131a KUMON

Graphs of Fractional Functions II

Given the fractional function $f(x)=1+\frac{1}{x-2}$, find the maximum and minimum values for each given domain. Also state the range.

(1)



When $-2\leqslant x\leqslant 1$,

$f(-2)=\boxed{\frac{3}{4}}$, $f(1)=\boxed{0}$

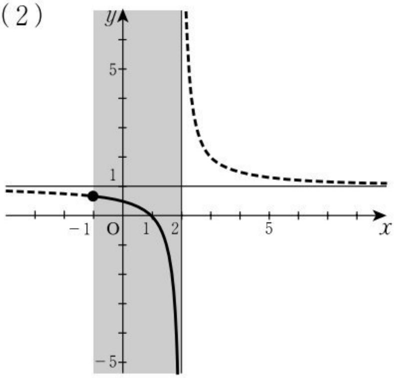
From the graph:

Maximum value: $\boxed{\frac{3}{4}}$ (when $x=\boxed{-2}$)

Minimum value: $\boxed{0}$ (when $x=\boxed{1}$)

Range: $\boxed{0}\leqslant f(x)\leqslant\boxed{\frac{3}{4}}$

(2)



When $-1\leqslant x<2$,

$f(-1)=\boxed{\frac{2}{3}}$, $f(2)$ has no value

Note: When $x=2$, $f(x)$ is undefined.

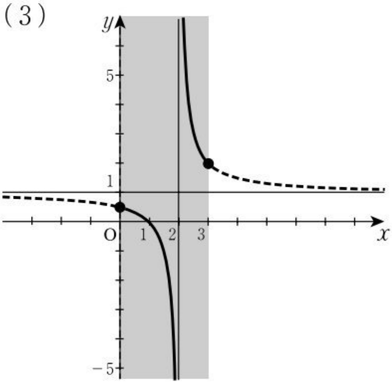
From the graph:

Maximum value: $\boxed{\frac{2}{3}}$ (when $x=\boxed{-1}$)

Minimum value: does not exist

Range: $f(x)\leqslant\boxed{\frac{2}{3}}$

(3)



When $0 \leqslant x \leqslant 3$ (but with $x \neq 2$),

$f(0) = \boxed{\mathbf{\frac{1}{2}}}$, $f(3) = \boxed{\mathbf{2}}$

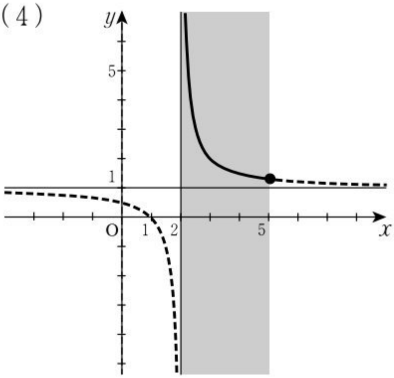
From the graph:

Maximum value: does not exist

Minimum value: $\boxed{\textbf{does not exist}}$

Range: $f(x) \leqslant \boxed{\mathbf{\frac{1}{2}}}$, $f(x) \geqslant \boxed{\mathbf{2}}$

(4)



When $2 < x \leqslant 5$,

$f(2)$ has no value, $f(5) = \boxed{\mathbf{\frac{4}{3}}}$

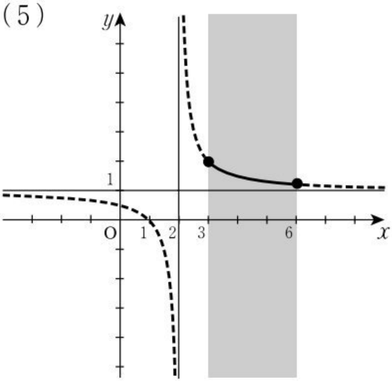
From the graph:

Maximum value: $\boxed{\textbf{does not exist}}$

Minimum value: $\boxed{\mathbf{\frac{4}{3}}}$ (when $x = \boxed{\mathbf{5}}$)

Range: $f(x) \geqslant \boxed{\mathbf{\frac{4}{3}}}$

(5)



When $3 \leqslant x \leqslant 6$,

$f(3) = \boxed{\mathbf{2}}$, $f(6) = \boxed{\mathbf{\frac{5}{4}}}$

From the graph:

Maximum value: $\boxed{\mathbf{2}}$ (when $x = \boxed{\mathbf{3}}$)

Minimum value: $\boxed{\mathbf{\frac{5}{4}}}$ (when $x = \boxed{\mathbf{6}}$)

Range: $\boxed{\mathbf{\frac{5}{4}}} \leqslant f(x) \leqslant \boxed{\mathbf{2}}$

K 132a KUMON

Graphs of Fractional Functions II

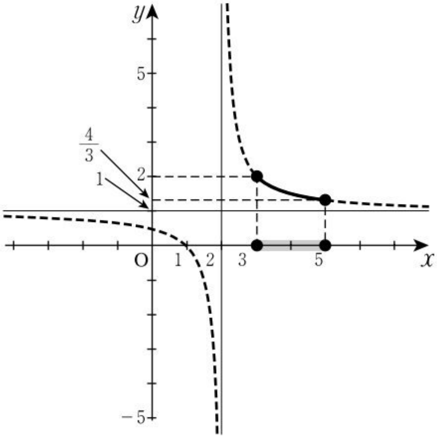
For each fractional function, draw the graph, then state the range that satisfies the given domain.

Ex.

$f(x)=\dfrac{x-1}{x-2}\ \ (3\leqslant x\leqslant 5)$

[Sol] $f(x)=1+\dfrac{1}{x-2}$

- Asymptotes:
  $x=2,\ y=1$
- Intercepts:
  x-axis: $(1,\ 0)$
  y-axis: $\left(0,\ \dfrac{1}{2}\right)$
- Range:
  $f(3)=\boxed{\mathbf{2}},\ f(5)=\boxed{\dfrac{\mathbf{4}}{\mathbf{3}}}$



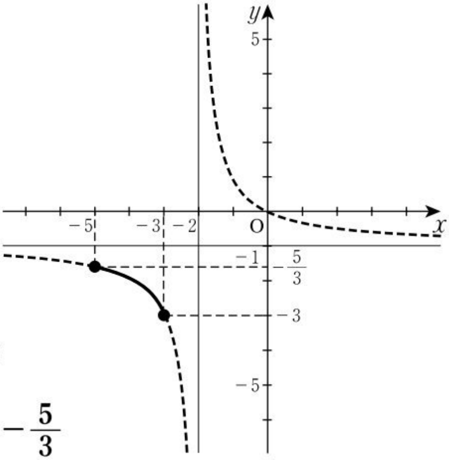
From the graph, $\boxed{\dfrac{\mathbf{4}}{\mathbf{3}}}\leqslant f(x)\leqslant\boxed{\mathbf{2}}$

Answers: in order 2, $\frac{4}{3}$, $\frac{4}{3}$, 2

K132b

（1） $f(x)=\dfrac{-x}{x+2}\ \ (-5\leqslant x\leqslant -3)$

[Sol] $f(x)=-1+\dfrac{2}{x+2}$



- Asymptotes:

  $\boldsymbol{x=-2,\ \ y=-1}$

- Intercepts:

  $\boldsymbol{(0,\ \ 0)}$

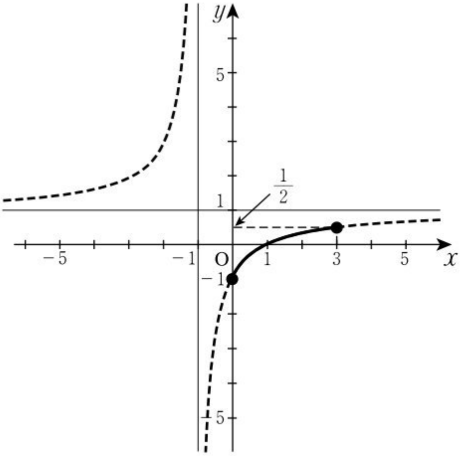
- Range:

  $f(-5)=-\dfrac{5}{3}$, $f(-3)=-3$

  From the graph, $\boldsymbol{-3\leqslant f(x)\leqslant -\dfrac{5}{3}}$

（2） $f(x)=\dfrac{x-1}{x+1}\ \ (0\leqslant x\leqslant 3)$

[Sol] $f(x)=1-\dfrac{2}{x+1}$



- Asymptotes:

  $\boldsymbol{x=-1,\ \ y=1}$

- Intercepts:

  $\boldsymbol{x}$**-axis:** $\boldsymbol{(1,\ \ 0)}$

  $\boldsymbol{y}$**-axis:** $\boldsymbol{(0,\ \ -1)}$

- Range:

  $f(0)=-1,\ \ f(3)=\dfrac{1}{2}$

  From the graph, $\boldsymbol{-1\leqslant f(x)\leqslant \dfrac{1}{2}}$

K 133a KUMON

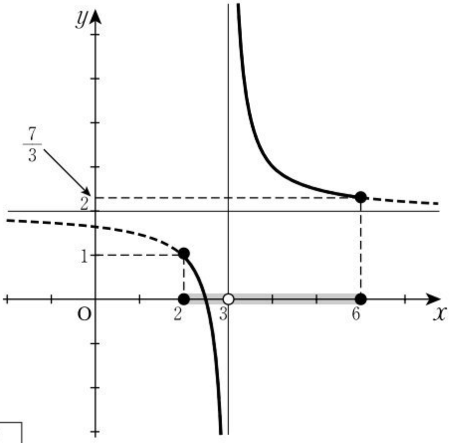
Graphs of Fractional Functions II

For each fractional function, draw the graph and find the range.

Ex.

$f(x)=\dfrac{2x-5}{x-3}$ $(2\leqslant x\leqslant 6$, but with $x\neq 3)$

[Sol] $f(x)=2+\dfrac{1}{x-3}$



- Asymptotes:

  $x=3,\ y=2$

- Intercepts:

  x-axis: $\left(\dfrac{5}{2},\ 0\right)$

  y-axis: $\left(0,\ \dfrac{5}{3}\right)$

- Range:

  $f(2)=\boxed{\mathbf{1}}$, $f(6)=\boxed{\dfrac{\mathbf{7}}{\mathbf{3}}}$

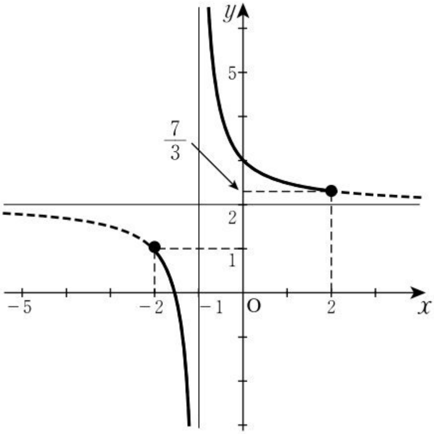
  From the graph, $f(x)\leqslant\boxed{\mathbf{1}}$, $f(x)\geqslant\boxed{\dfrac{\mathbf{7}}{\mathbf{3}}}$

Answers: in order 1, $\frac{7}{3}$, 1, $\frac{7}{3}$

K133b

（1） $f(x)=\dfrac{2x+3}{x+1}$ $(-2\leqslant x\leqslant 2$, but with $x\neq -1)$

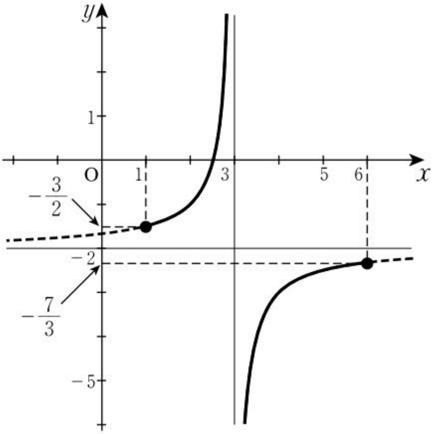
[Sol] $f(x)=2+\dfrac{1}{x+1}$



- Asymptotes:
  $\boldsymbol{x=-1,\ y=2}$
- Intercepts:
  x-axis: $\boldsymbol{\left(-\dfrac{3}{2},\ 0\right)}$
  y-axis: $\boldsymbol{(0,\ 3)}$
- Range:
  $f(-2)=1,\ f(2)=\dfrac{7}{3}$
  From the graph, $\boldsymbol{f(x)\leqslant 1,\ f(x)\geqslant \dfrac{7}{3}}$

（2） $f(x)=\dfrac{-2x+5}{x-3}$ $(1\leqslant x\leqslant 6$, but with $x\neq 3)$

[Sol] $f(x)=-2-\dfrac{1}{x-3}$



- Asymptotes:
  $\boldsymbol{x=3,\ y=-2}$
- Intercepts:
  x-axis: $\boldsymbol{\left(\dfrac{5}{2},\ 0\right)}$
  y-axis: $\boldsymbol{\left(0,\ -\dfrac{5}{3}\right)}$
- Range:
  $f(1)=-\dfrac{3}{2},\ f(6)=-\dfrac{7}{3}$
  From the graph, $\boldsymbol{f(x)\leqslant -\dfrac{7}{3},\ \ f(x)\geqslant -\dfrac{3}{2}}$

Graphs of Fractional Functions II

1. For each fractional function, draw the graph and find the range.

Ex.

$f(x)=\dfrac{2x}{2x-3}$ ($x \geqq 0$, but with $x \neq \dfrac{3}{2}$)

[Sol] $f(x)=1+\dfrac{3}{2x-3}$

- Asymptotes:

  $x=\dfrac{3}{2}$, $y=1$

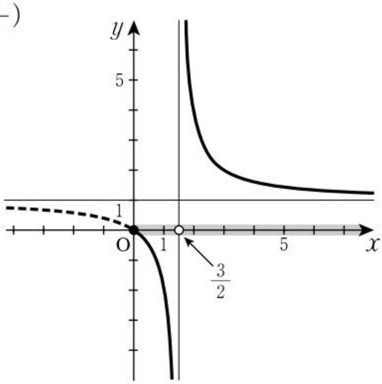
- Intercepts:

  $(0,\ 0)$

- Range:

  $f(0)=0$

  From the graph, $f(x) \leqq 0$, $f(x)>1$



(1) $f(x)=\dfrac{2x-2}{2x+3}$ ($x \geqq -4$, but with $x \neq -\dfrac{3}{2}$)

[Sol] $f(x)=1-\dfrac{5}{2x+3}$

- Asymptotes:

  $\boldsymbol{x=-\dfrac{3}{2},\ \ y=1}$

- Intercepts:

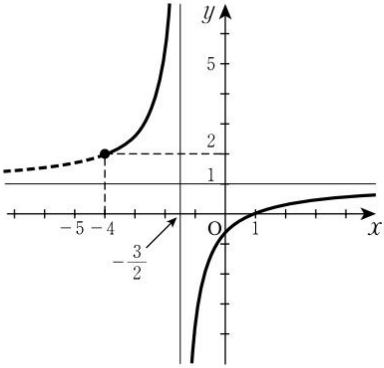
  $\boldsymbol{x}$-axis: $\boldsymbol{(1,\ 0)}$

  $\boldsymbol{y}$-axis: $\boldsymbol{\left(0,\ -\dfrac{2}{3}\right)}$

- Range:

  $f(-4)=2$

  From the graph, $\boldsymbol{f(x)<1,\ f(x) \geqq 2}$



K 134b

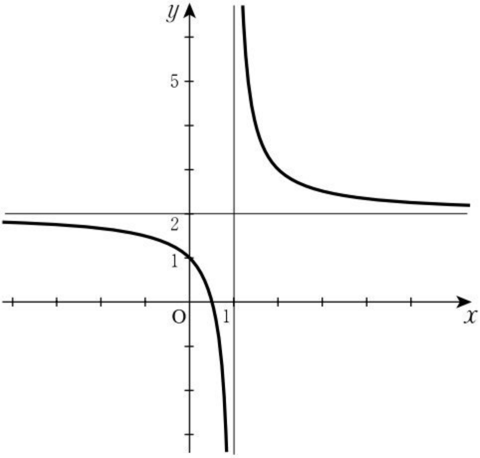
2. Given the fractional function $f(x)=2+\dfrac{1}{x-1}$, find the range that satisfies each given domain.

（1） $2\leqslant x\leqslant 4$

[Sol] $f(2)=3$, $f(4)=\dfrac{7}{3}$

From the graph,

$\boldsymbol{\dfrac{7}{3}\leqslant f(x)\leqslant 3}$



（2） $-2\leqslant x\leqslant 2$ (but with $x\neq 1$)

[Sol] $f(-2)=\dfrac{5}{3}$, $f(2)=3$

From the graph,

$\boldsymbol{f(x)\leqslant \dfrac{5}{3},\ f(x)\geqslant 3}$

（3） $x\geqslant 2$

[Sol] $f(2)=3$

From the graph,

$\boldsymbol{2<f(x)\leqslant 3}$

（4） $x\geqslant -2$ (but with $x\neq 1$)

[Sol] $f(-2)=\dfrac{5}{3}$

From the graph,

$\boldsymbol{f(x)\leqslant \dfrac{5}{3},\ f(x)>2}$

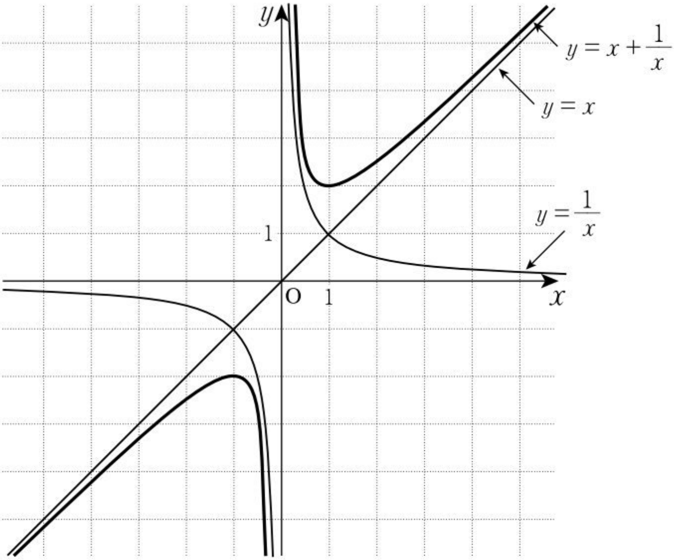
K 135a KUMON

Graphs of Fractional Functions II

1. Graph the following fractional functions.

(1) $y=x+\dfrac{1}{x}$

x	-4	-2	-1	$-\frac{1}{2}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$\frac{1}{x}$	$-\frac{1}{4}$	$-\frac{1}{2}$	-1	-2	-4	☒	$\mathbf{4}$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{\frac{1}{2}}$	$\mathbf{\frac{1}{4}}$
y	$-\frac{17}{4}$	$-\frac{5}{2}$	-2	$-\frac{5}{2}$	$-\frac{17}{4}$	☒	$\mathbf{\frac{17}{4}}$	$\mathbf{\frac{5}{2}}$	$\mathbf{2}$	$\mathbf{\frac{5}{2}}$	$\mathbf{\frac{17}{4}}$



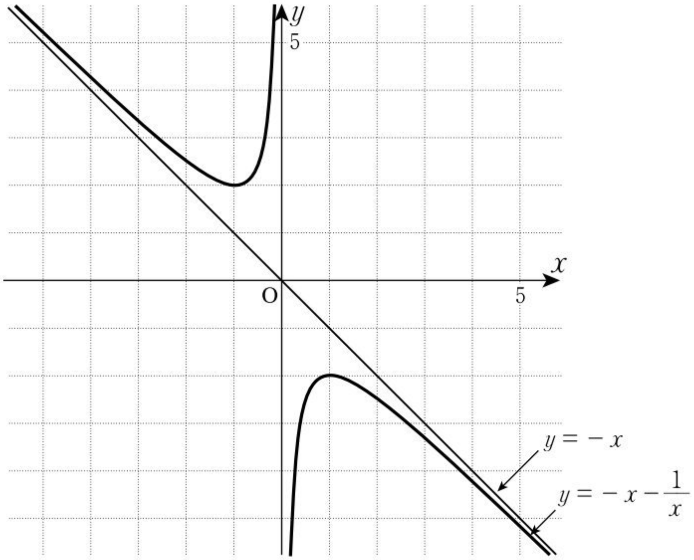
From the graph of $y=x+\dfrac{1}{x}$,

- As x gets further away from zero, the graph gets closer to the line $y=x$. However, it never crosses the line $y=x$.
- As x approaches zero, it gets closer to the y-axis. However, it never crosses the y-axis.

K 135b

(2) $y = -x - \frac{1}{x}$

x	-4	-2	-1	$-\frac{1}{2}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
y	$\frac{17}{4}$	$\frac{5}{2}$	2	$\mathbf{\frac{5}{2}}$	$\mathbf{\frac{17}{4}}$		$\mathbf{-\frac{17}{4}}$	$\mathbf{-\frac{5}{2}}$	$\mathbf{-2}$	$\mathbf{-\frac{5}{2}}$	$\mathbf{-\frac{17}{4}}$



2. Fill in the blank boxes.

The asymptotes of $y = x + \frac{1}{x}$ are: $x = 0$, $y = x$. (Refer to side a.)

The asymptotes of $y = -x - \frac{1}{x}$ are: $x = \boxed{\mathbf{0}}$, $y = \boxed{\boldsymbol{-x}}$.

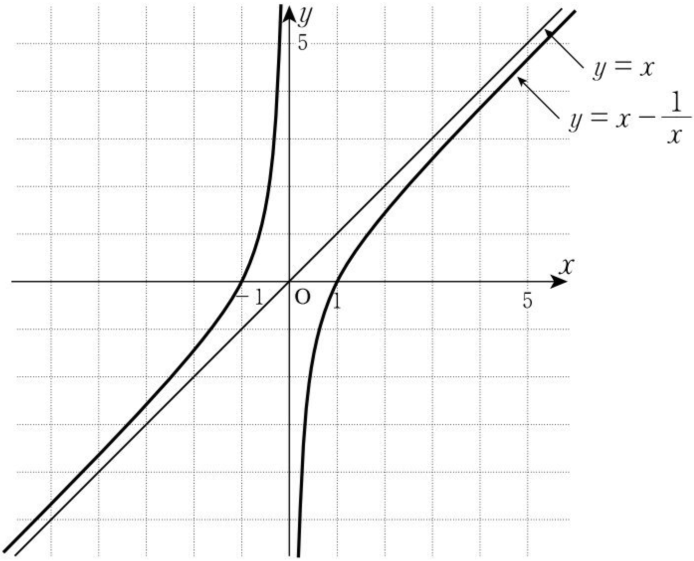
K 136a KUMON

Graphs of Fractional Functions II

1. Graph the following fractional functions.

(1) $y = x - \dfrac{1}{x}$

x	-4	-2	-1	$-\frac{1}{2}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
y	$-\frac{15}{4}$	$-\frac{3}{2}$	0	$\frac{3}{2}$	$\frac{15}{4}$	╳	$\boldsymbol{-\frac{15}{4}}$	$\boldsymbol{-\frac{3}{2}}$	$\boldsymbol{0}$	$\boldsymbol{\frac{3}{2}}$	$\boldsymbol{\frac{15}{4}}$



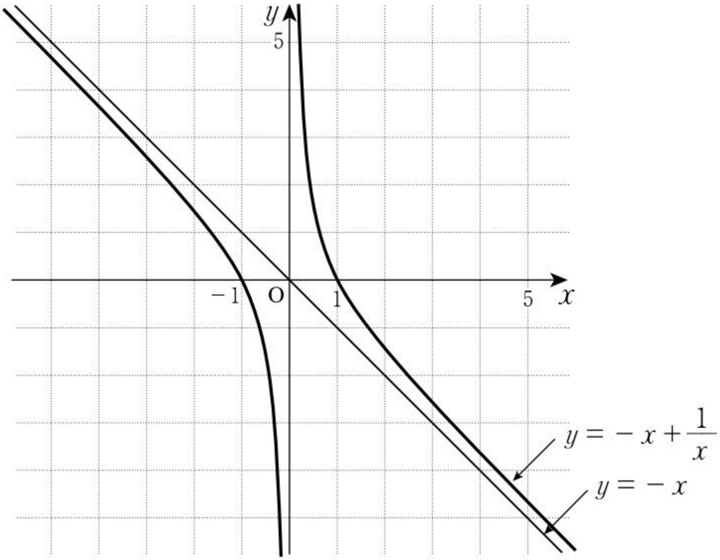
From the graph of $y = x - \dfrac{1}{x}$,

- As x gets further away from zero, the graph gets closer to the line $y = x$.
  However, it never crosses the line $y = x$.
- As x approaches zero, it gets closer to the y-axis.
  However, it never crosses the y-axis.

K136b

(2) $y=-x+\frac{1}{x}$

x	-4	-2	-1	$-\frac{1}{2}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
y	$\frac{15}{4}$	$\frac{3}{2}$	0	$\boldsymbol{-\frac{3}{2}}$	$\boldsymbol{-\frac{15}{4}}$	☒	$\boldsymbol{\frac{15}{4}}$	$\boldsymbol{\frac{3}{2}}$	$\mathbf{0}$	$\boldsymbol{-\frac{3}{2}}$	$\boldsymbol{-\frac{15}{4}}$



2. Fill in the blank boxes.

The asymptotes of $y=x-\frac{1}{x}$ are: $x=0$, $y=x$. (Refer to side a.)

The asymptotes of $y=-x+\frac{1}{x}$ are: $x=\boxed{\mathbf{0}}$, $y=\boxed{\boldsymbol{-x}}$.

The asymptotes of the graph $y=ax+\frac{b}{x}$ are: $x=0$, $y=ax$.

K 137a KUMON

Graphs of Fractional Functions II

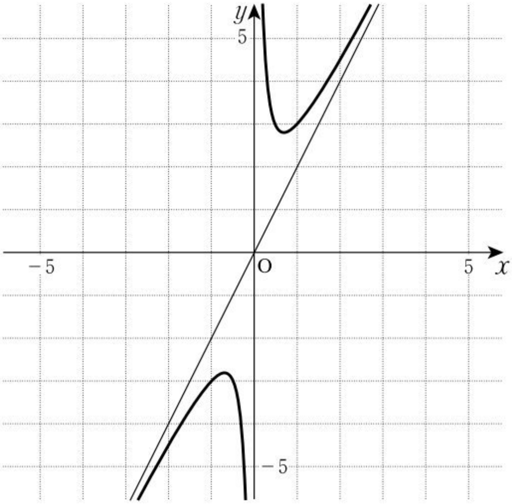
The asymptotes of the graph $y = ax + \frac{b}{x}$ are: $x = 0$, $y = ax$.

Find the asymptotes for each fractional function, and draw the graph.

(1) $y = 2x + \frac{1}{x}$

Asymptotes:

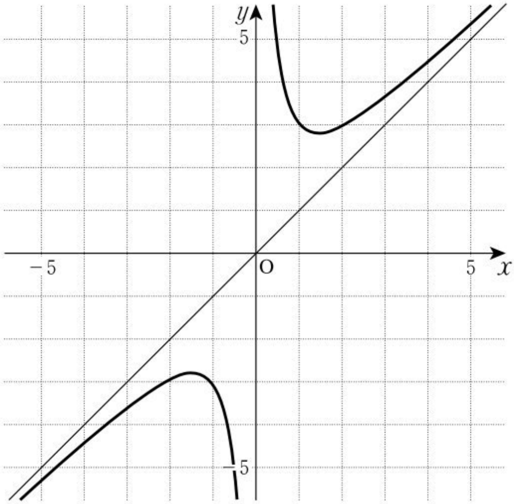
$x = \boxed{\mathbf{0}}$, $y = \boxed{\boldsymbol{2x}}$



(2) $y = x + \frac{2}{x}$

Asymptotes:

$\boldsymbol{x = 0}$, $\boldsymbol{y = x}$

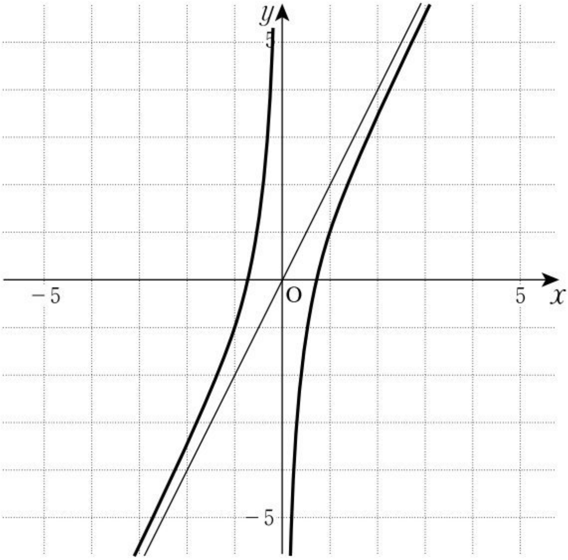


K 137b

（3）　$y = 2x - \dfrac{1}{x}$

Asymptotes:

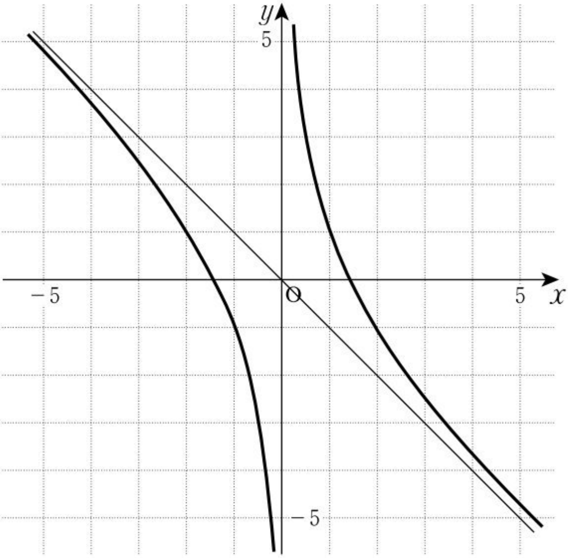
$\boldsymbol{x = 0,\ \ y = 2x}$



（4）　$y = -x + \dfrac{2}{x}$

Asymptotes:

$\boldsymbol{x = 0,\ \ y = -x}$



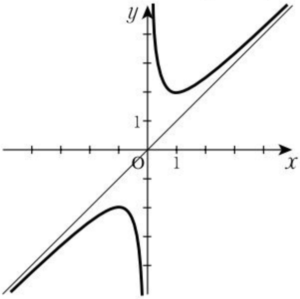
K 138a KUMON

Graphs of Fractional Functions II

1. For each fractional function, find the equations of the asymptotes and draw them on the graph.

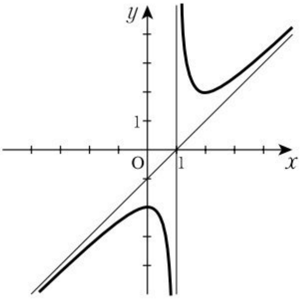
(1) $y = x + \frac{1}{x}$

Asymptotes: $x = \boxed{0}$, $y = \boxed{x}$



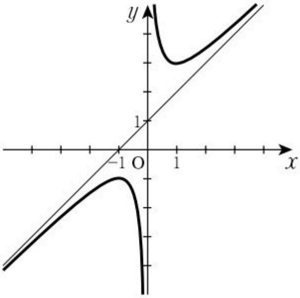
(3) $y = x - 1 + \frac{1}{x-1}$

Asymptotes: $\boldsymbol{x = 1,\ y = x - 1}$



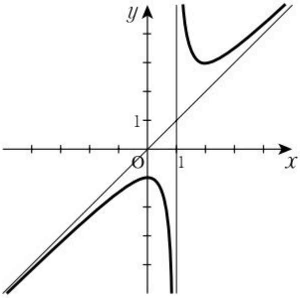
(2) $y = x + 1 + \frac{1}{x} = \left(x + \frac{1}{x}\right) + 1$

Asymptotes: $\boldsymbol{x = 0,\ y = x + 1}$



(4) $y = x + \frac{1}{x-1} = \left(x - 1 + \frac{1}{x-1}\right) + 1$

Asymptotes: $\boldsymbol{x = 1,\ y = x}$



2. Complete the following using the above graphs:

- (2) is a translation of (1), $\boxed{1}$ unit(s) along the y-axis.
- (3) is a translation of (1), $\boxed{1}$ unit(s) along the x-axis.
- (4) is a translation of (1), $\boxed{1}$ unit(s) along the x-axis, and

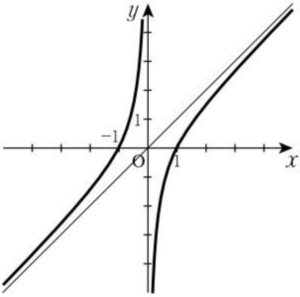
  $\boxed{1}$ unit(s) along the y-axis.

K 138b

3. For each fractional function, find the equations of the asymptotes and draw them on the graph.

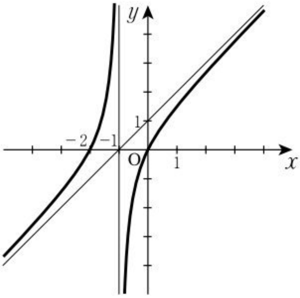
(1) $y=x-\dfrac{1}{x}$

Asymptotes: $\boldsymbol{x=0,\ y=x}$



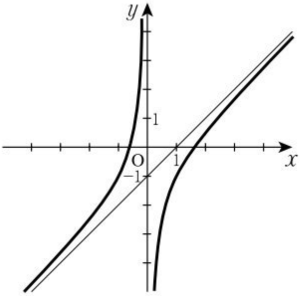
(3) $y=x+1-\dfrac{1}{x+1}$

Asymptotes: $\boldsymbol{x=-1,\ y=x+1}$



(2) $y=x-1-\dfrac{1}{x}=\left(x-\dfrac{1}{x}\right)-1$

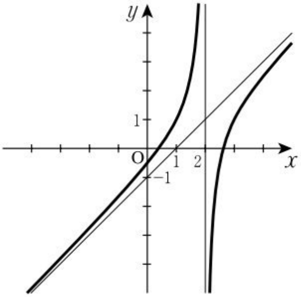
Asymptotes: $\boldsymbol{x=0,\ y=x-1}$



(4) $y=x-1-\dfrac{1}{x-2}$

$=\left(x-2-\dfrac{1}{x-2}\right)+1$

Asymptotes: $\boldsymbol{x=2,\ y=x-1}$



4. Complete the following using the above graphs:

- (2) is a translation of (1), $\boxed{\mathbf{-1}}$ unit(s) along the y-axis.
- (3) is a translation of (1), $\boxed{\mathbf{-1}}$ unit(s) along the x-axis.
- (4) is a translation of (1), $\boxed{\mathbf{2}}$ unit(s) along the x-axis, and

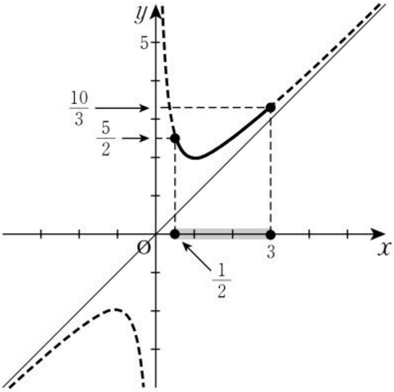
  $\boxed{\mathbf{1}}$ unit(s) along the y-axis.

K 139a KUMON

Graphs of Fractional Functions II

Ex.

Given the fractional function $f(x)=x+\frac{1}{x}$ $(\frac{1}{2}\leqslant x\leqslant 3)$, draw the graph and find the maximum value.



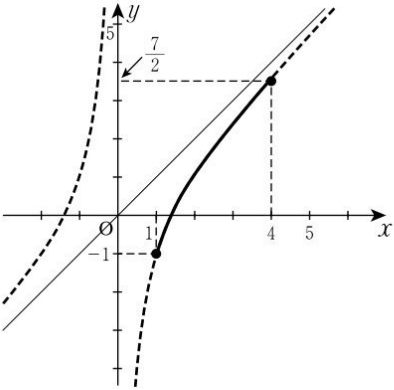
[Sol] $f\left(\frac{1}{2}\right)=\frac{1}{2}+\frac{1}{\frac{1}{2}}=\frac{5}{2}$

$f(3)=3+\frac{1}{3}=\frac{10}{3}$

From the graph:

Maximum value: $f(3)=\frac{10}{3}$

1. Given the function $f(x)=x-\frac{2}{x}$ $(1\leqslant x\leqslant 4)$, draw the graph and find the minimum value.



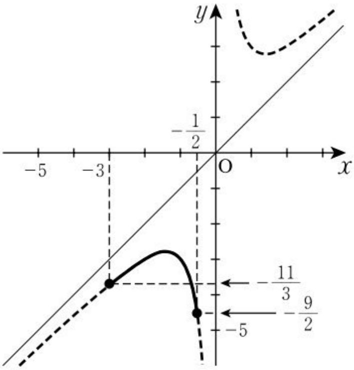
[Sol] $f(1)=1-\frac{2}{1}=-1$

$f(4)=4-\frac{2}{4}=\frac{7}{2}$

From the graph:

Minimum value: $\boldsymbol{f(1)=-1}$

2. Given the function $f(x)=x+\dfrac{2}{x}$ $\left(-3\leqslant x\leqslant -\dfrac{1}{2}\right)$, draw the graph and find the minimum value.



[Sol] $f(-3)=-3+\dfrac{2}{-3}=-\dfrac{11}{3}$

$$f\left(-\frac{1}{2}\right)=-\frac{1}{2}+\frac{2}{-\frac{1}{2}}=-\frac{9}{2}$$

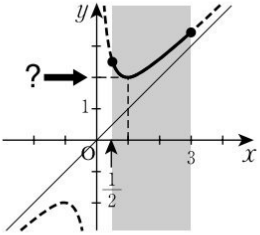
From the graph:

Minimum value: $\boldsymbol{f\left(-\frac{1}{2}\right)=-\frac{9}{2}}$

Consider this!

We can find the minimum value of the function on side a, $f(x)=x+\dfrac{1}{x}$ $\left(\dfrac{1}{2}\leqslant x\leqslant 3\right)$,

using the *arithmetic mean* and the *geometric mean*.



Given two numbers, x and $\dfrac{1}{x}$ (where $x>0$ and $\dfrac{1}{x}>0$),

then from the relationship between the arithmetic mean and the geometric mean,

$$x+\frac{1}{x}\geqslant 2\sqrt{x\cdot\frac{1}{x}}=\boxed{\mathbf{2}}$$

☜ When $x=\dfrac{1}{x}$, i.e. when $x=1$, the inequality sign becomes an equal sign.

Therefore, the minimum value is $f(1)=\boxed{\mathbf{2}}$.

The relationship between the *arithmetic mean* and the *geometric mean* can be found on J197.

When $a>0$, $b>0$, then $\dfrac{a+b}{2}\geqslant\sqrt{ab}$ ☜ When $a=b$, the inequality sign becomes an equal sign.

Answers: 2, 2

K 140a KUMON

Graphs of Fractional Functions II

1. For each function, write the letter (A)～(F) of the corresponding sketch.

(1) $y=1+\frac{1}{x-1}$ $\cdots$ **(C)**

(2) $y=x-1+\frac{1}{x-1}$ $\cdots$ **(F)**

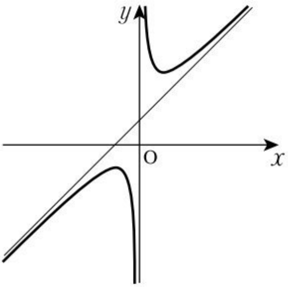
(3) $y=x-\frac{1}{x+1}$ $\cdots$ **(B)**

(4) $y=1-\frac{1}{x+1}$ $\cdots$ **(D)**

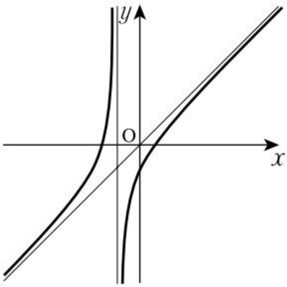
(5) $y=x+1+\frac{1}{x}$ $\cdots$ **(A)**

(6) $y=x-1-\frac{1}{x}$ $\cdots$ **(E)**

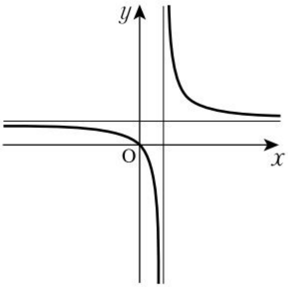
(A)



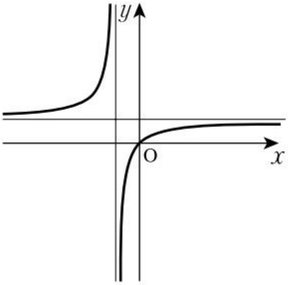
(B)



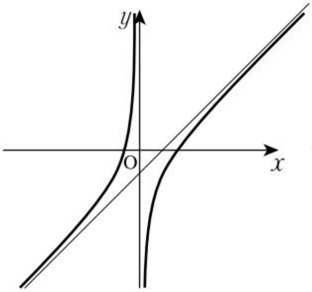
(C)



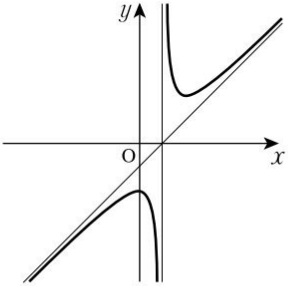
(D)



(E)

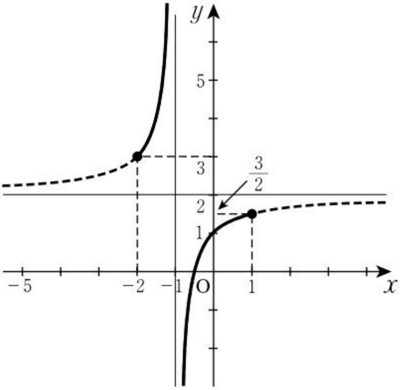


(F)



2. Given the fractional function $f(x)=\dfrac{2x+1}{x+1}$ $(-2\leqslant x\leqslant 1,$ but with $x\neq -1)$, draw the graph and find the range.

[Sol] $f(x)=2-\dfrac{1}{x+1}$



- Asymptotes:

  $\boldsymbol{x=-1,\ y=2}$

- Intercepts:

  x-axis: $\boldsymbol{\left(-\frac{1}{2},\ 0\right)}$

  y-axis: $\boldsymbol{(0,\ 1)}$

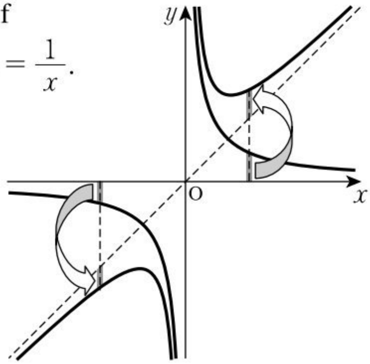
- Range:

  $f(-2)=3,\ f(1)=\dfrac{3}{2}$

  From the graph, $\boldsymbol{f(x)\leqslant\frac{3}{2},\ f(x)\geqslant 3}$

Let's try this!

Find the relationship between the graph of $y=x+\dfrac{1}{x}$ and the graphs of $y=x$ and $y=\dfrac{1}{x}$.



From the graph, we can see that the y-coordinate of $y=x+\dfrac{1}{x}$ is the sum of the y-coordinate of $y=\boxed{\dfrac{1}{x}}$ and the y-coordinate of $y=x$.

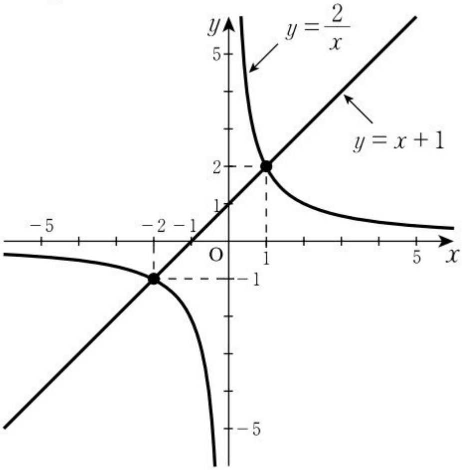
Answer: $\frac{1}{x}$

Fractional Equations and Inequalities

Ex.

Graph the following functions, and find their common points.

$y=\dfrac{2}{x}$, $\quad y=x+1$



[Sol] Let $\dfrac{2}{x}=x+1$,

$x^2+x-2=0$ (where $x\neq 0$)

$(x+2)(x-\boxed{1})=0$

$x=-2, \boxed{1}$

Therefore, the common points are:

$(-2,\ -1), (\boxed{1}, \boxed{2})$

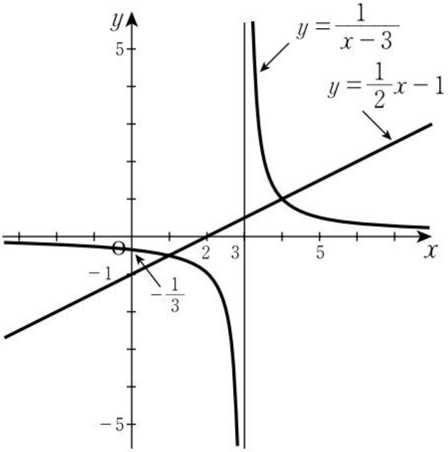
Multiply both sides of the equation by x. However, $x=0$ must be excluded, as the denominator of $\dfrac{2}{x}$ would be zero, and the equation would be undefined.

Answers: in order 1, 1, 1, 2

K141b

Graph the following functions, and find their common points.

(1) $y=\dfrac{1}{x-3}$, $\quad y=\dfrac{1}{2}x-1$



[Sol] Let $\dfrac{1}{x-3}=\dfrac{1}{2}x-1$,

$2=(x-3)x-2(x-3)$

(where $x\neq 3$)

$x^2-5x+4=0$

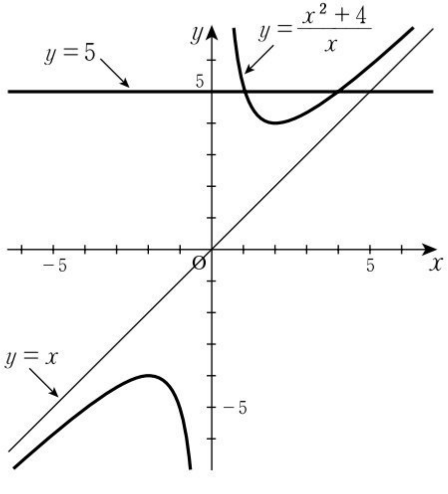
$(x-4)(x-1)=0$

$x=4,\ 1$

Therefore, the common points are:

$\mathbf{(4,\ 1),\ \left(1,\ -\dfrac{1}{2}\right)}$

(2) $y=\dfrac{x^2+4}{x}$, $\quad y=5$



[Sol] $y=\dfrac{x^2+4}{x}=x+\dfrac{4}{x}$

Let $\dfrac{x^2+4}{x}=5$,

$x^2-5x+4=0$ (where $x\neq 0$)

$(x-4)(x-1)=0$

$x=4,\ 1$

Therefore, the common points are:

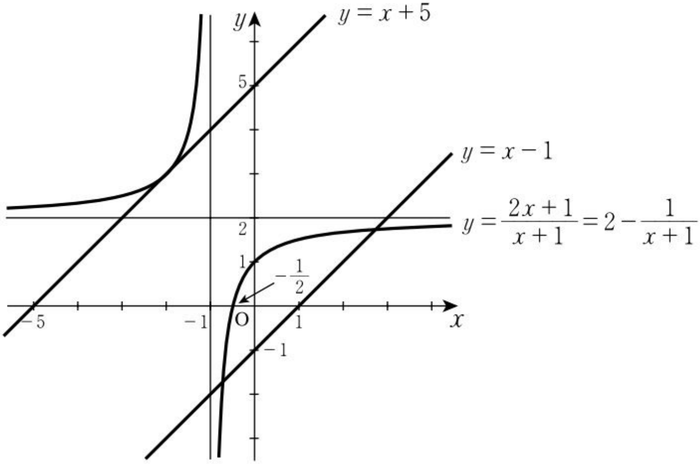
$\mathbf{(4,\ 5),\ (1,\ 5)}$

K 142a KUMON

Fractional Equations and Inequalities

1. For the function $y=\dfrac{2x+1}{x+1}$:

(1) Draw the graph.



(2) Find the common points of the function with each of the following lines.

① $y=x-1$

[Sol] Let $\dfrac{2x+1}{x+1}=x-1$,

$$2x+1=x^2-1$$

(where $x\neq-1$)

$$x^2-2x-2=0$$
$$x=1\pm\sqrt{3}$$

Therefore, the common points are:

$(1+\sqrt{3},\ \sqrt{3})$, $(1-\sqrt{3},\ -\sqrt{3})$

② $y=x+5$

[Sol] Let $\dfrac{2x+1}{x+1}=x+5$,

$$2x+1=(x+5)(x+1)$$

(where $x\neq-1$)

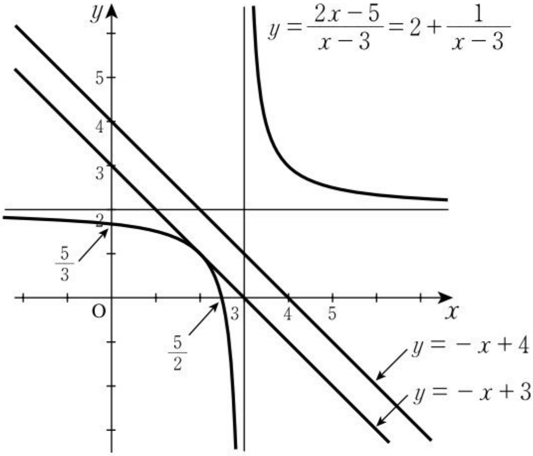
$$x^2+4x+4=0$$
$$(x+2)^2=0$$
$$x=-2$$

Therefore, the common point is:

$(-2,\ 3)$

2. For the function $y=\dfrac{2x-5}{x-3}$:

（1） Draw the graph.



（2） Find the common points of the function with each of the following lines.

① $y=-x+3$

[Sol] Let $\dfrac{2x-5}{x-3}=-x+3$,

$$2x-5=(-x+3)(x-3)$$

(where $x\neq 3$)

$$x^2-4x+4=0$$

$$(x-2)^2=0$$

$$x=2$$

Therefore, the common point is:

(2, 1)

② $y=-x+4$

[Sol] From the graph,

the common point:

$\boxed{\textbf{does not exist}}$

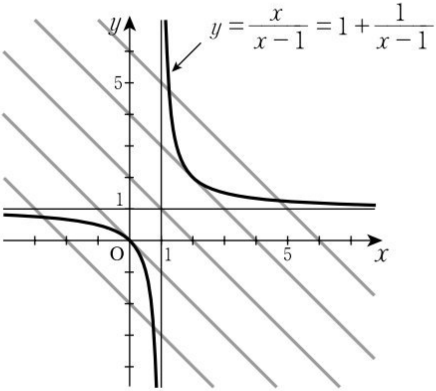
K 143a KUMON

Fractional Equations and Inequalities

Find the number of common points between the given functions.

Ex.

$y=\dfrac{x}{x-1}$, $\quad y=-x+k$



[Sol] Let $\dfrac{x}{x-1}=-x+k$,

$$x=-(x-k)(x-1) \quad (\text{where } x\neq 1)$$

Exclude $x=1$ since at that value the denominator is 0.

$$x^2-kx+k=0$$

Calculating the discriminant, *D*, of this equation,

$$D=k^2-4k=k(k-4)$$

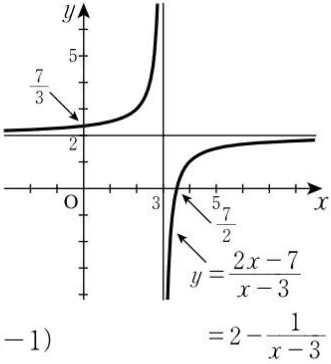
Therefore:

- When $D>0$, i.e. when $k<0$, $k>4$,
  the number of common points is $\boxed{2}$.
- When $D=0$, i.e. when $k=0$, 4,
  the number of common points is $\boxed{1}$.
- When $D<0$, i.e. when $0<k<4$,
  the number of common points is $\boxed{0}$.

Answers: in order 2, 1, 0

K 143b

（1） $y=\dfrac{2x-7}{x-3}$, $\quad y=x+k$



[Sol] Let $\dfrac{2x-7}{x-3}=x+k$,

$$2x-7=(x+k)(x-3)$$

(where $x \neq 3$)

$$x^2+(k-5)x-3k+7=0$$

Calculating the discriminant, D, of this equation,

$$D=(k-5)^2-4(-3k+7)=(k+3)(k-1)$$

Therefore:

- When $D>0$, i.e. **when $k<-3$, $k>1$, the number of common points is 2**.
- When $D=0$, i.e. **when $k=-3$, 1, the number of common points is 1**.
- When $D<0$, i.e. **when $-3<k<1$, the number of common points is 0**.

K 144a KUMON

Fractional Equations and Inequalities

1. Solve the following fractional equations.

Ex.

$$\frac{x+4}{x+2}=\frac{x}{3}$$

[Sol] Multiplying both sides of the equation by $3(x+2)$,

$3(x+4)=x(x+2)$ (where $x\neq -2$)

$x^2-x-12=0$

$(x-4)(x+3)=0$

$x=4,\ -3$

(1) $\dfrac{x+5}{x-1}=\dfrac{x}{2}$

[Sol] Multiplying both sides of the equation by $2(x-1)$,

$2(x+5)=x(x-1)$ (where $x\neq 1$)

$x^2-3x-10=0$

$(x-5)(x+2)=0$

$\boldsymbol{x=5,\ -2}$

(2) $\dfrac{2}{x-1}=\dfrac{x}{x+2}$

[Sol] Multiplying both sides of the equation by $(x-1)(x+2)$,

$2(x+2)=x(x-1)$ (where $x\neq 1,\ x\neq -2$)

$x^2-3x-4=0$

$(x-4)(x+1)=0$

$\boldsymbol{x=4,\ -1}$

2. Solve the following fractional equation.

Ex.

$$\frac{1}{x-1}-\frac{2}{x+1}=\frac{x}{x-1}$$

[Sol] Multiplying both sides of the equation by $(x-1)(x+1)$,

☜ $(x-1)(x+1)$ is the least common multiple, LCM, of the denominator.

$x+1-2(x-1)=x(x+1)$ (where $x\neq 1,\ x\neq -1$)

$x^2+2x-3=0$

$(x+3)(x-1)=0$

$x=-3,\ 1$

$x=1$ is invalid. ☜ Since the denominator is 0 when $x=1$.

Therefore, $x=-3$

(1) $\dfrac{x+1}{x-1}+\dfrac{1}{x+3}=\dfrac{2}{x-1}$

[Sol] Multiplying both sides of the equation by $(x-1)(x+3)$,

☜ $(x-1)(x+3)$ is the least common multiple, LCM, of the denominator.

$(x+1)(x+3)+x-1=2(x+3)$ (where $x\neq 1,\ x\neq -3$)

$x^2+3x-4=0$

$(x+4)(x-1)=0$

$x=-4,\ 1$

$x=1$ is invalid.

Therefore, $\boldsymbol{x=-4}$

Note: When solving fractional equations, we must always check each solution and exclude any that make a denominator in the original equation equal to zero.

K 145a KUMON

Fractional Equations and Inequalities

Solve the following fractional equations.

(1) $\dfrac{3}{x-2}-\dfrac{x}{x+1}=\dfrac{x+1}{x-2}$

[Sol] Multiplying both sides of the equation by $(x-2)(x+1)$,

$3(x+1)-x(x-2)=(x+1)^2$ (where $x\neq 2,\ x\neq -1$)

$2x^2-3x-2=0$

$(2x+1)(x-2)=0$

$x=-\dfrac{1}{2},\ 2$

$x=2$ is invalid.

Therefore, $\boldsymbol{x=-\dfrac{1}{2}}$

(2) $\dfrac{x-3}{x+1}+\dfrac{x}{x-2}=\dfrac{3}{x-2}$

[Sol] Multiplying both sides of the equation by $(x+1)(x-2)$,

$(x-3)(x-2)+x(x+1)=3(x+1)$ (where $x\neq -1,\ x\neq 2$)

$2x^2-7x+3=0$

$(2x-1)(x-3)=0$

$\boldsymbol{x=\dfrac{1}{2},\ 3}$

K 145b

(3) $\dfrac{x}{x-3}-\dfrac{2}{x+1}=\dfrac{12}{(x-3)(x+1)}$

[Sol] Multiplying both sides of the equation by $(x-3)(x+1)$,

$x(x+1)-2(x-3)=12$ (where $x\neq 3$, $x\neq -1$)

$x^2-x-6=0$

$(x-3)(x+2)=0$

$x=3,\ -2$

$x=3$ is invalid.

Therefore, $\boldsymbol{x=-2}$

(4) $\dfrac{x}{x^2-3x+2}-\dfrac{2}{x-2}=1$

[Sol] $\dfrac{x}{(x-2)(x-1)}-\dfrac{2}{x-2}=1$

Multiplying both sides of the equation by $(x-2)(x-1)$,

$x-2(x-1)=(x-2)(x-1)$ (where $x\neq 2$, $x\neq 1$)

$x^2-2x=0$

$x(x-2)=0$

$x=0,\ 2$

$x=2$ is invalid.

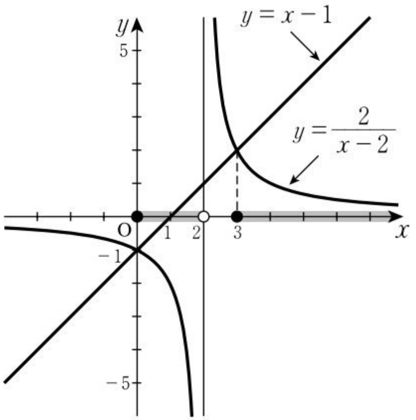
Therefore, $\boldsymbol{x=0}$

Fractional Equations and Inequalities

Solve the following inequalities by drawing the graphs.

Ex.

$$\frac{2}{x-2} \leqslant x-1$$



☜ The values of x satisfying the inequality $\frac{2}{x-2} \leqslant x-1$, are shaded on the graph.

[Sol] Solving $\frac{2}{x-2} = x-1$ ☜ Solving the original inequality as an equation.

$$2 = (x-1)(x-2) \quad (\text{where } x \neq 2)$$

$$x^2 - 3x = 0$$

$$x(x-3) = 0$$

$$x = 0,\ 3$$

From the graph,

$$\boxed{0} \leqslant x < \boxed{2},\quad x \geqslant \boxed{3}$$

Answers: in order 0, 2, 3

K 146b

（1）　$\dfrac{2}{x+1} < \dfrac{1}{2}x - 1$

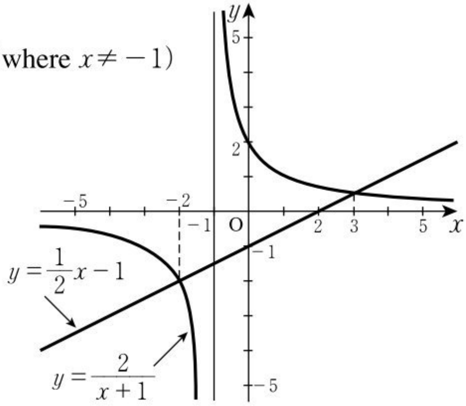
[Sol] Solving $\dfrac{2}{x+1} = \dfrac{1}{2}x - 1$

$4 = x(x+1) - 2(x+1)$　(where $x \neq -1$)

$x^2 - x - 6 = 0$

$(x-3)(x+2) = 0$

$x = 3,\ -2$



From the graph,

$\boldsymbol{-2 < x < -1,\ \ x > 3}$

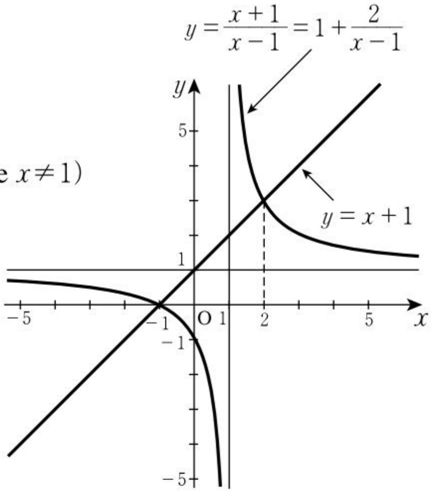
（2）　$\dfrac{x+1}{x-1} \geqslant x + 1$

[Sol] Solving $\dfrac{x+1}{x-1} = x + 1$

$x + 1 = (x+1)(x-1)$　(where $x \neq 1$)

$(x-2)(x+1) = 0$

$x = 2,\ -1$



From the graph,

$\boldsymbol{x \leqslant -1,\ \ 1 < x \leqslant 2}$

Note: It is also possible to solve the inequalities above by rearranging them as

（1）$\dfrac{1}{2}x - 1 - \dfrac{2}{x+1} > 0$

（2）$x + 1 - \dfrac{x+1}{x-1} \leqslant 0$

then sketching the LHS of each inequality.

K 147a KUMON

Fractional Equations and Inequalities

Solve the following inequalities by drawing the graphs.

(1) $x+1 \leqslant \dfrac{1}{x-1}$

[Sol] Solving $x+1=\dfrac{1}{x-1}$

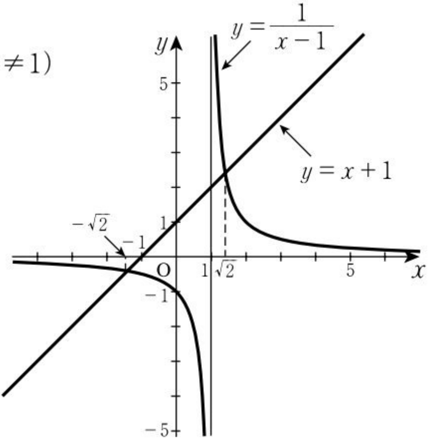
$(x+1)(x-1)=1$ (where $x \neq 1$)

$x^2-2=0$

$x=\pm\sqrt{2}$

From the graph,

$\boldsymbol{x \leqslant -\sqrt{2},\ 1<x \leqslant \sqrt{2}}$



(2) $x>\dfrac{1}{x-2}$

[Sol] Solving $x=\dfrac{1}{x-2}$

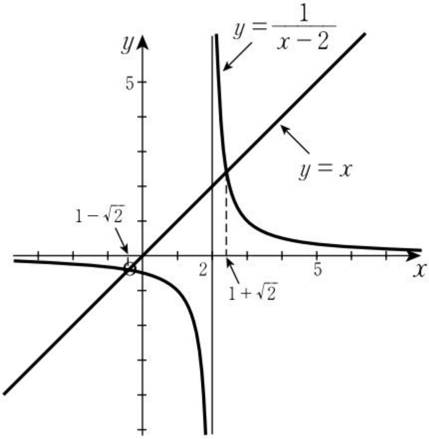
$x(x-2)=1$ (where $x \neq 2$)

$x^2-2x-1=0$

$x=1\pm\sqrt{2}$

From the graph,

$\boldsymbol{1-\sqrt{2}<x<2,\ x>1+\sqrt{2}}$

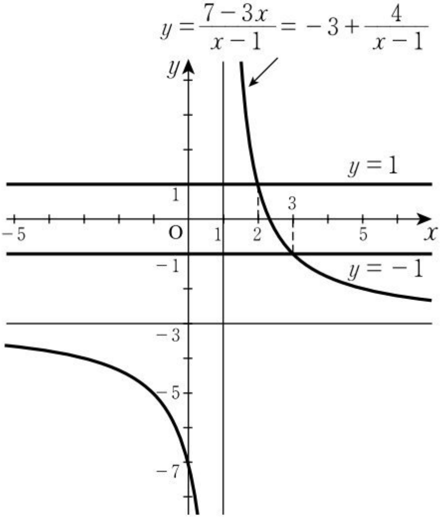


K 147b

(3) $-1 < \dfrac{7-3x}{x-1} \leqq 1$ …①

[Sol]

$$\begin{cases} -1 < \dfrac{7-3x}{x-1} & \cdots ② \\ \dfrac{7-3x}{x-1} \leqq 1 & \cdots ③ \end{cases}$$



For ②,

solving $-1 = \dfrac{7-3x}{x-1}$,

$-x+1 = 7-3x$ (where $x \neq 1$)

$x = 3$

From the graph,

$1 < x < 3$ …④

For ③,

solving $\dfrac{7-3x}{x-1} = 1$,

$7-3x = x-1$ (where $x \neq 1$)

$x = 2$

From the graph,

$x < 1,\ x \geqq 2$ …⑤

From ④ and ⑤, the solution to ① is:

$\boldsymbol{2 \leqq x < 3}$

K 148a KUMON

Fractional Equations and Inequalities

1. Solve the following inequality, using the method shown in the example.

Ex.

$$\frac{(x+1)(x-2)}{x} \leqslant 0$$

[Sol] Rewriting, $x-1-\frac{2}{x} \leqslant 0$,

$x-1 \leqslant \frac{2}{x}$

Solving $x-1=\frac{2}{x}$,

$x(x-1)=2$ (where $x \neq 0$)

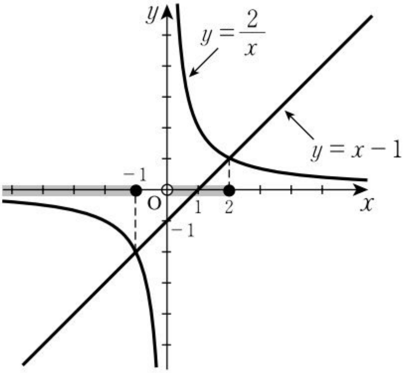
$x^2-x-2=0$

$(x-2)(x+1)=0$

$x=2, -1$

From the graph,

$x \leqslant -1,\ 0<x \leqslant 2$



(1) $\frac{(x+3)(x-1)}{x} \geqslant 0$

[Sol] Rewriting, $x+2-\frac{3}{x} \geqslant 0$,

$x+2 \geqslant \frac{3}{x}$

Solving $x+2=\frac{3}{x}$,

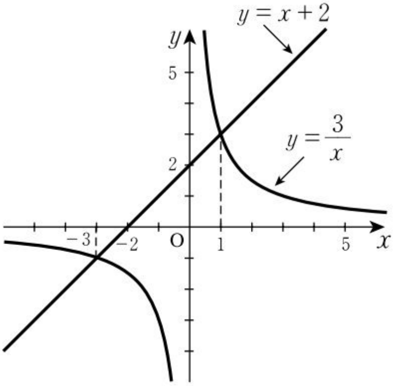
$x^2+2x-3=0$ (where $x \neq 0$)

$(x+3)(x-1)=0$

$x=-3, 1$

From the graph,

$\boldsymbol{-3 \leqslant x < 0,\ x \geqslant 1}$



2. Solve the following inequality, using the method shown in the example.

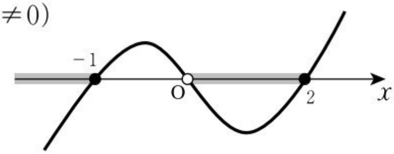
Ex.

$$\frac{(x+1)(x-2)}{x} \leqslant 0$$

[Sol] Multiply both sides of the inequality by x^2,

☜ Here, we multiply by x^2 rather than x, as x^2 is positive for all x (except $x=0$).

$$x(x+1)(x-2) \leqslant 0 \quad (\text{where } x \neq 0)$$



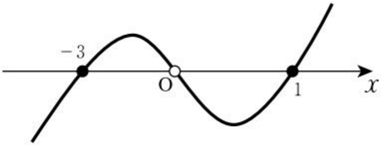
From the graph,

$$x \leqslant -1,\ 0 < x \leqslant 2$$

(1) $\dfrac{(x+3)(x-1)}{x} \geqslant 0$

[Sol] Multiply both sides of the inequality by x^2,

$$x(x+3)(x-1) \geqslant 0 \quad (\text{where } x \neq 0)$$



From the graph,

$$\boldsymbol{-3 \leqslant x < 0,\ x \geqslant 1}$$

K 149a KUMON

Fractional Equations and Inequalities

Solve the following inequalities.

Ex.

$$\frac{5}{x+2} \geqslant \frac{2}{x-1}$$

[Sol] $\dfrac{5}{x+2} - \dfrac{2}{x-1} \geqslant 0$

$$\frac{5(x-1)-2(x+2)}{(x+2)(x-1)} \geqslant 0$$

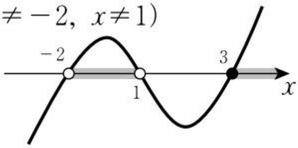
$$\frac{3(x-3)}{(x+2)(x-1)} \geqslant 0$$

Multiplying both sides by $(x+2)^2(x-1)^2$, ☜ The square of the denominator is $(x+2)^2(x-1)^2$.

$$3(x-3)(x+2)(x-1) \geqslant 0 \quad (\text{where } x \neq -2,\ x \neq 1)$$

From the graph,

$$-2 < x < 1,\ x \geqslant 3$$



(1) $\dfrac{2}{x-2} \leqslant \dfrac{1}{x+1}$

[Sol] $\dfrac{2}{x-2} - \dfrac{1}{x+1} \leqslant 0$

$$\frac{2(x+1)-(x-2)}{(x-2)(x+1)} \leqslant 0$$

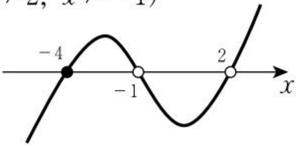
$$\frac{x+4}{(x-2)(x+1)} \leqslant 0$$

Multiplying both sides by $(x-2)^2(x+1)^2$,

$$(x+4)(x-2)(x+1) \leqslant 0 \quad (\text{where } x \neq 2,\ x \neq -1)$$

From the graph,

$$\boldsymbol{x \leqslant -4,\ -1 < x < 2}$$



K 149b

（2） $\frac{1}{x+2}+\frac{2}{x-4} \geqq 0$

[Sol] $\frac{x-4+2(x+2)}{(x+2)(x-4)} \geqq 0$

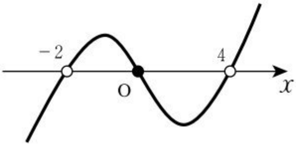
$\frac{3x}{(x+2)(x-4)} \geqq 0$

Multiplying both sides by $(x+2)^2(x-4)^2$,

$3x(x+2)(x-4) \geqq 0$ （where $x \neq -2,\ x \neq 4$）

From the graph,

$\boldsymbol{-2 < x \leqq 0,\ x > 4}$



（3） $\frac{1}{x+1}+\frac{1}{x-5} \leqq \frac{2}{(x+1)(x-5)}$

[Sol] $\frac{1}{x+1}+\frac{1}{x-5}-\frac{2}{(x+1)(x-5)} \leqq 0$

$\frac{x-5+x+1-2}{(x+1)(x-5)} \leqq 0$

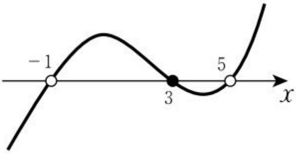
$\frac{2(x-3)}{(x+1)(x-5)} \leqq 0$

Multiplying both sides by $(x+1)^2(x-5)^2$,

$2(x-3)(x+1)(x-5) \leqq 0$ （where $x \neq -1,\ x \neq 5$）

From the graph,

$\boldsymbol{x < -1,\ 3 \leqq x < 5}$



K 150a KUMON

Fractional Equations and Inequalities

1. Find the value of the constant k for which the graph of $y=\dfrac{2x+1}{x-1}$ touches the line $y=-x+k$ (i.e. so that the two graphs have exactly one common point).

[Sol] Let $\dfrac{2x+1}{x-1}=-x+k$,

$$2x+1=(-x+k)(x-1) \quad (\text{where } x \neq 1)$$

$$x^2+(1-k)x+k+1=0$$

Calculating the discriminant, D, of this equation,

$$D=(1-k)^2-4(k+1)=k^2-6k-3=0$$

Therefore, $\boldsymbol{k=3\pm2\sqrt{3}}$

2. Solve the fractional equation.

$$\frac{x}{x-1}-\frac{2}{x+3}=\frac{4}{x^2+2x-3}$$

[Sol] $\dfrac{x}{x-1}-\dfrac{2}{x+3}=\dfrac{4}{(x+3)(x-1)}$

Multiplying both sides by $(x+3)(x-1)$,

$$x(x+3)-2(x-1)=4 \quad (\text{where } x \neq -3,\ x \neq 1)$$

$$x^2+x-2=0$$

$$(x+2)(x-1)=0$$

$$x=-2,1$$

$x=1$ is invalid.

Therefore, $\boldsymbol{x=-2}$

K150b

3. Solve the fractional inequality.

$$x \leqq \frac{2}{x-1}$$

[Sol] $x - \dfrac{2}{x-1} \leqq 0$

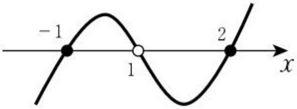
$$\frac{x(x-1)-2}{x-1} \leqq 0$$

$$\frac{(x+1)(x-2)}{x-1} \leqq 0$$

$(x+1)(x-2)(x-1) \leqq 0$ (where $x \neq 1$)

From the graph,

$$\boldsymbol{x \leqq -1,\ 1 < x \leqq 2}$$



[From the graph of $y = x$ and $y = \dfrac{2}{x-1}$, $\boldsymbol{x \leqq -1,\ 1 < x \leqq 2}$]

Let's try this!

Looking at the inequality on K148, $\dfrac{(x+1)(x-2)}{x} \leqq 0$, we can solve it using an alternative method.

[Sol] (i) When $x < 0$

Multiplying both sides by x, ☜

$(x+1)(x-2) \geqq 0$

$x \leqq -1,\ x \geqq 2$

Since $x < 0$, $x \leqq \boxed{\mathbf{-1}}$

Note that when multiplying both sides by a negative number, the direction of the inequality sign is reversed.

(ii) When $x > 0$

Multiplying both sides by x, ☜

$(x+1)(x-2) \leqq 0$

$-1 \leqq x \leqq 2$

Since $x > 0$, $\boxed{\mathbf{0}} < x \leqq 2$

Note that when multiplying both sides by a positive number, the direction of the inequality sign does not change.

From (i) and (ii), $x \leqq \boxed{\mathbf{-1}}$, $\boxed{\mathbf{0}} < x \leqq 2$

Answers: in order −1, 0, −1, 0

K 151 a KUMON

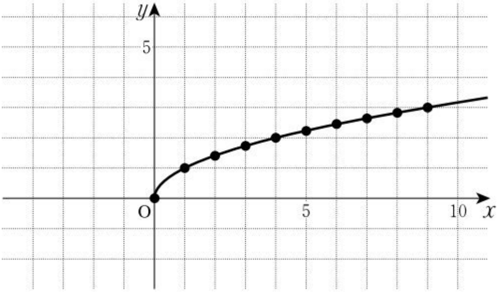
Graphs of Irrational Functions

Graph the following irrational functions.

Ex.

$y=\sqrt{x}$

x	-2	-1	0	1	2	3	4	5	6	7	8	9
y	☒	☒	0	1	$\sqrt{2}$	$\sqrt{3}$	2	$\sqrt{5}$	$\sqrt{6}$	$\sqrt{7}$	$2\sqrt{2}$	3



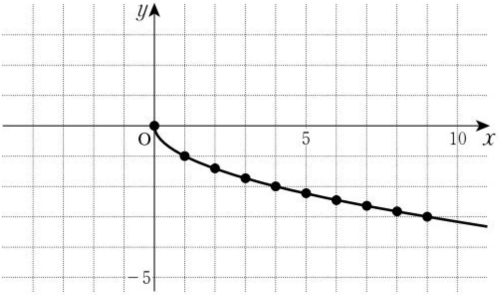
Domain: $x \geqq 0$

Range: $y \geqq 0$

Note: When $x<0$, y is undefined.

(1) $y=-\sqrt{x}$

x	-2	-1	0	1	2	3	4	5	6	7	8	9
y	☒	☒	**0**	**−1**	$-\sqrt{2}$	$-\sqrt{3}$	**−2**	$-\sqrt{5}$	$-\sqrt{6}$	$-\sqrt{7}$	$-2\sqrt{2}$	**−3**



Domain: $x \geqq \boxed{0}$

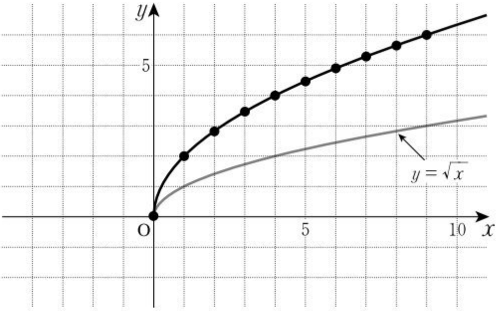
Range: $y \leqq 0$

[Reference] $\sqrt{2}=1.41\ldots$, $\sqrt{3}=1.73\ldots$, $\sqrt{5}=2.23\ldots$, $\sqrt{7}=2.64\ldots$

K151b

（2）　$y=2\sqrt{x}$

x	-2	-1	0	1	2	3	4	5	6	7	8	9
y			0	$\mathbf{2}$	$2\sqrt{2}$	$2\sqrt{3}$	$\mathbf{4}$	$2\sqrt{5}$	$2\sqrt{6}$	$2\sqrt{7}$	$4\sqrt{2}$	$\mathbf{6}$

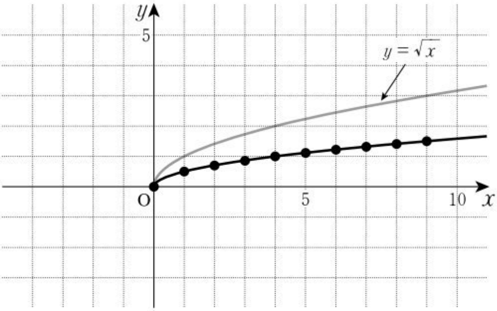


Domain: $x \geqq \boxed{\mathbf{0}}$

Range: $y \geqq \boxed{\mathbf{0}}$

（3）　$y=\dfrac{\sqrt{x}}{2}$

x	-2	-1	0	1	2	3	4	5	6	7	8	9
y			$\mathbf{0}$	$\mathbf{\frac{1}{2}}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	$\mathbf{1}$	$\frac{\sqrt{5}}{2}$	$\frac{\sqrt{6}}{2}$	$\frac{\sqrt{7}}{2}$	$\sqrt{2}$	$\mathbf{\frac{3}{2}}$



Domain: $x \geqq \boxed{\mathbf{0}}$

Range: $y \geqq \boxed{\mathbf{0}}$

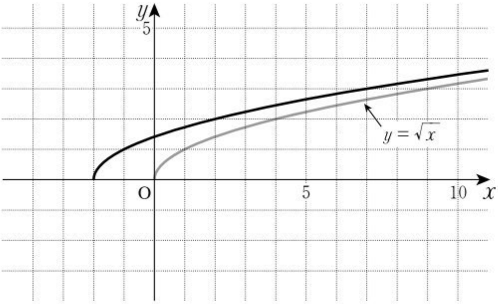
K 152a KUMON

Graphs of Irrational Functions

Graph the following irrational functions.

(1) $y=\sqrt{x+2}$

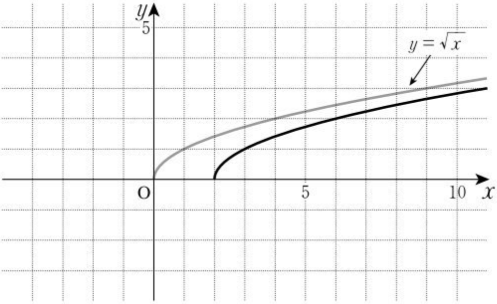
x	-4	-3	-2	-1	0	1	2	3	4	5	6	7
y	☒	☒	$\mathbf{0}$	$\mathbf{1}$	$\mathbf{\sqrt{2}}$	$\mathbf{\sqrt{3}}$	$\mathbf{2}$	$\mathbf{\sqrt{5}}$	$\mathbf{\sqrt{6}}$	$\mathbf{\sqrt{7}}$	$\mathbf{2\sqrt{2}}$	$\mathbf{3}$



Domain: $x \geqq -2$

Range: $y \geqq \boxed{\mathbf{0}}$

(2) $y=\sqrt{x-2}$



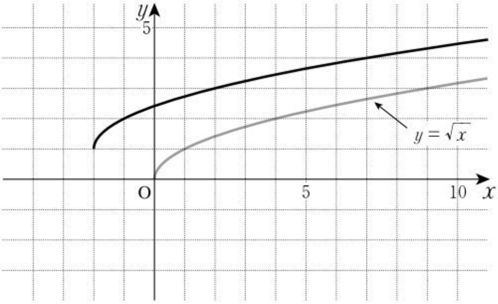
Domain: $\boxed{\boldsymbol{x \geqq 2}}$

Range: $\boxed{\boldsymbol{y \geqq 0}}$

[Reference] $\sqrt{2}=1.41\ldots$, $\sqrt{3}=1.73\ldots$, $\sqrt{5}=2.23\ldots$, $\sqrt{7}=2.64\ldots$

K 152b

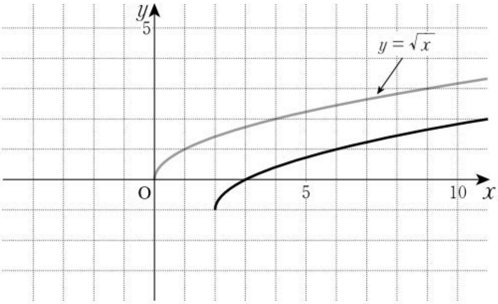
（3） $y=\sqrt{x+2}+1$



Domain: $\boxed{\boldsymbol{x \geqq -2}}$

Range: $\boxed{\boldsymbol{y \geqq 1}}$

（4） $y=\sqrt{x-2}-1$



Domain: $\boxed{\boldsymbol{x \geqq 2}}$

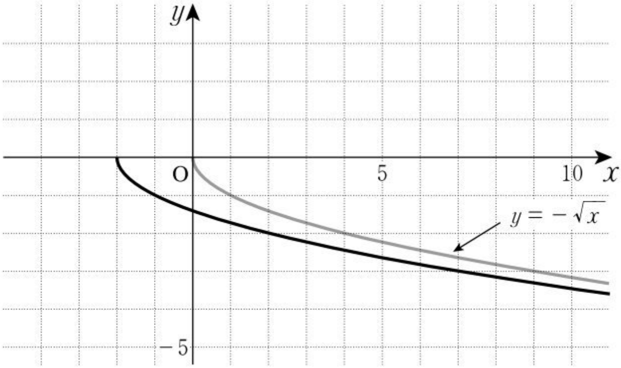
Range: $\boxed{\boldsymbol{y \geqq -1}}$

K 153a KUMON

Graphs of Irrational Functions

Graph the following irrational functions.

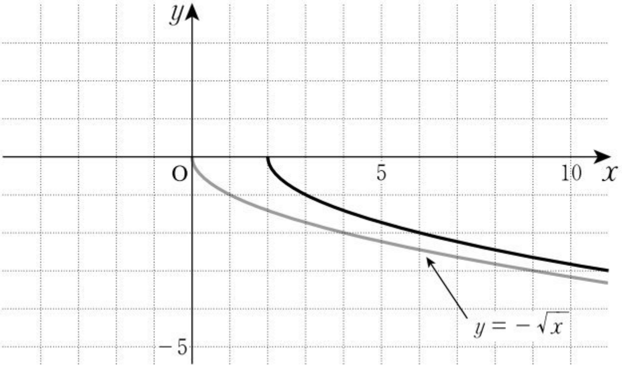
(1) $y=-\sqrt{x+2}$



Domain: $x \geqslant \boxed{-2}$

Range: $y \leqslant 0$

(2) $y=-\sqrt{x-2}$

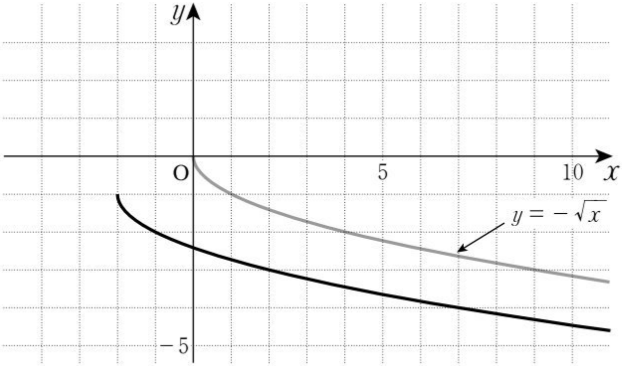


Domain: $\boxed{x \geqslant 2}$

Range: $\boxed{y \leqslant 0}$

K 153b

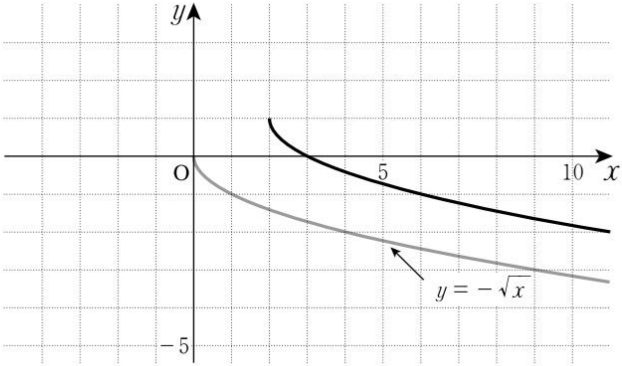
(3)　$y=-\sqrt{x+2}-1$



Domain: $\boxed{\boldsymbol{x \geqslant -2}}$

Range:　$y \leqslant -1$

(4)　$y=-\sqrt{x-2}+1$



Domain: $\boxed{\boldsymbol{x \geqslant 2}}$

Range: $\boxed{\boldsymbol{y \leqslant 1}}$

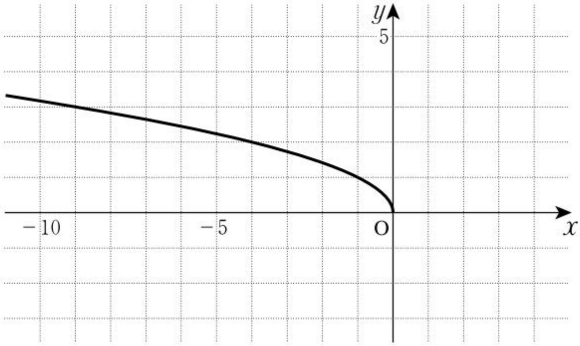
K 154a KUMON

Graphs of Irrational Functions

Graph the following irrational functions.

(1) $y=\sqrt{-x}$

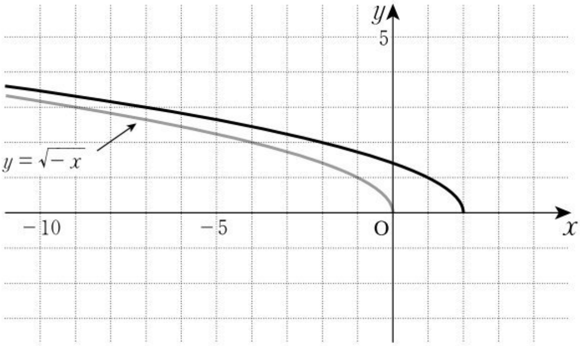
x	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2
y	3	$2\sqrt{2}$	$\sqrt{7}$	$\sqrt{6}$	$\sqrt{5}$	2	$\sqrt{3}$	$\sqrt{2}$	1	0	☒	☒



Domain: $x \leqslant 0$

Range: $y \geqslant 0$

(2) $y=\sqrt{-x+2}$

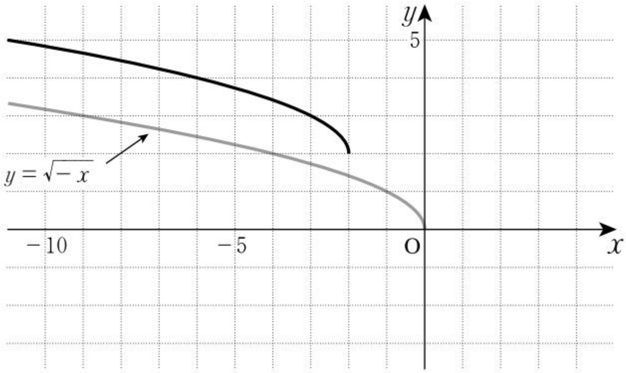


Domain: $x \leqslant 2$

Range: $\boxed{y \geqslant 0}$

K 154b

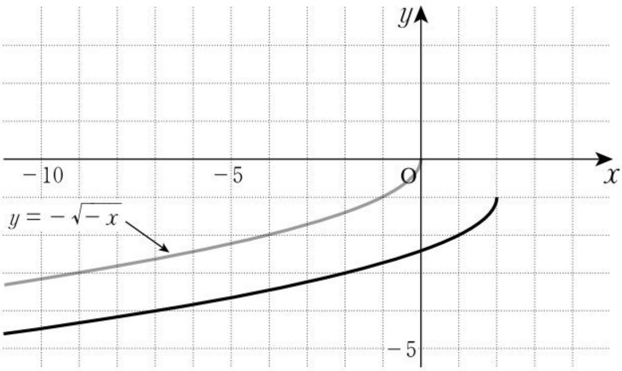
(3)　$y=\sqrt{-x-2}+2$



Domain: $\boxed{\boldsymbol{x \leqq -2}}$

Range: $\boxed{\boldsymbol{y \geqq 2}}$

(4)　$y=-\sqrt{-x+2}-1$



Domain: $\boxed{\boldsymbol{x \leqq 2}}$

Range: $\boxed{\boldsymbol{y \leqq -1}}$

K 155a KUMON

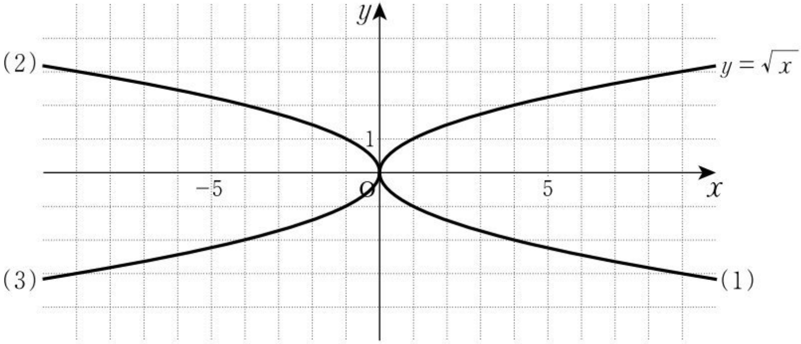
Graphs of Irrational Functions

1. Graph the following irrational functions.

(1) $y = -\sqrt{x}$

(2) $y = \sqrt{-x}$

(3) $y = -\sqrt{-x}$



2. Complete the following using the above graphs.

Ex. $y = \sqrt{x}$ and $y = -\sqrt{x}$ are symmetric with respect to the $\boxed{x\text{-axis}}$.

(1) $y = \sqrt{x}$ and $y = \sqrt{-x}$ are symmetric with respect to the $\boxed{\boldsymbol{y}\textbf{-axis}}$.

(2) $y = \sqrt{-x}$ and $y = -\sqrt{-x}$ are symmetric with respect to the $\boxed{\boldsymbol{x}\textbf{-axis}}$.

(3) $y = -\sqrt{x}$ and $y = -\sqrt{-x}$ are symmetric with respect to the $\boxed{\boldsymbol{y}\textbf{-axis}}$.

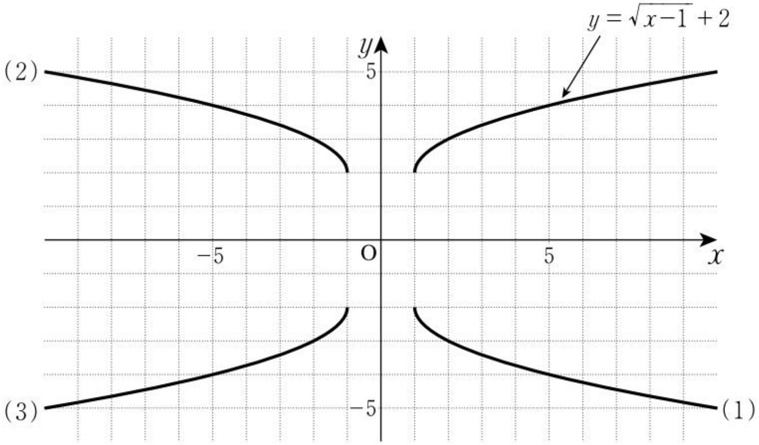
K 155b

3. Graph the following irrational functions.

（1） $y=-\sqrt{x-1}-2$

（2） $y=\sqrt{-(x+1)}+2$

（3） $y=-\sqrt{-(x+1)}-2$



4. Complete the following using the above graphs.

（1） $y=\sqrt{x-1}+2$ and $y=-\sqrt{x-1}-2$ are symmetric with respect to the **x-axis**.

（2） $y=\sqrt{x-1}+2$ and $y=\sqrt{-(x+1)}+2$ are symmetric with respect to the **y-axis**.

（3） $y=\sqrt{-(x+1)}+2$ and $y=-\sqrt{-(x+1)}-2$ are symmetric with respect to the **x-axis**.

（4） $y=-\sqrt{x-1}-2$ and $y=-\sqrt{-(x+1)}-2$ are symmetric with respect to the **y-axis**.

K 156a KUMON

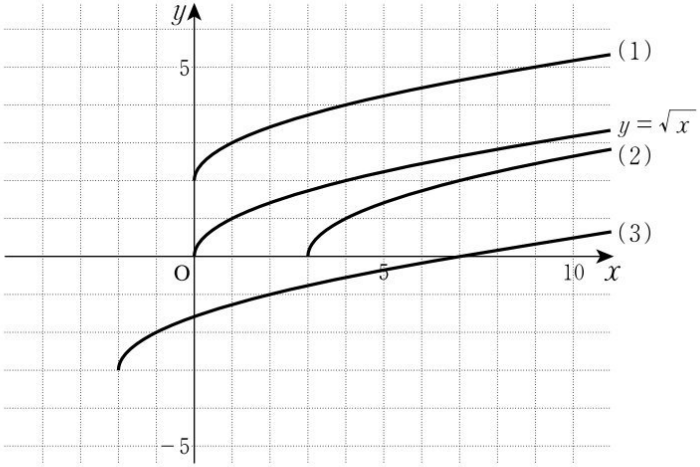
Graphs of Irrational Functions

1. Graph the following irrational functions.

(1) $y=\sqrt{x}+2$

(2) $y=\sqrt{x-3}$

(3) $y=\sqrt{x+2}-3$



2. Complete the following using the above graphs.

(1) $y=\sqrt{x}+2$ is a translation of $y=\sqrt{x}$, $\boxed{\mathbf{2}}$ unit(s) along the y-axis.

(2) $y=\sqrt{x-3}$ is a translation of $y=\sqrt{x}$, $\boxed{\mathbf{3}}$ unit(s) along the x-axis.

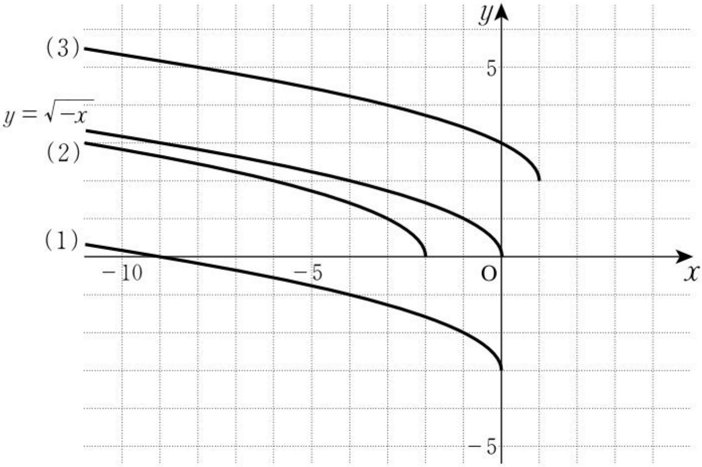
(3) $y=\sqrt{x+2}-3$ is a translation of $y=\sqrt{x}$, $\boxed{\mathbf{-2}}$ unit(s) along the x-axis, and $\boxed{\mathbf{-3}}$ unit(s) along the y-axis.

3. Graph the following irrational functions.

(1) $y=\sqrt{-x}-3$

(2) $y=\sqrt{-(x+2)}$

(3) $y=\sqrt{-(x-1)}+2$



4. Complete the following using the above graphs.

(1) $y=\sqrt{-x}-3$ is a translation of $y=\sqrt{-x}$, $\boxed{\mathbf{-3}}$ unit(s) along the y-axis.

(2) $y=\sqrt{-(x+2)}$ is a translation of $y=\sqrt{-x}$, $\boxed{\mathbf{-2}}$ unit(s) along the x-axis.

(3) $y=\sqrt{-(x-1)}+2$ is a translation of $y=\sqrt{-x}$, $\boxed{\mathbf{1}}$ unit(s) along the x-axis, and $\boxed{\mathbf{2}}$ unit(s) along the y-axis.

K 157a KUMON

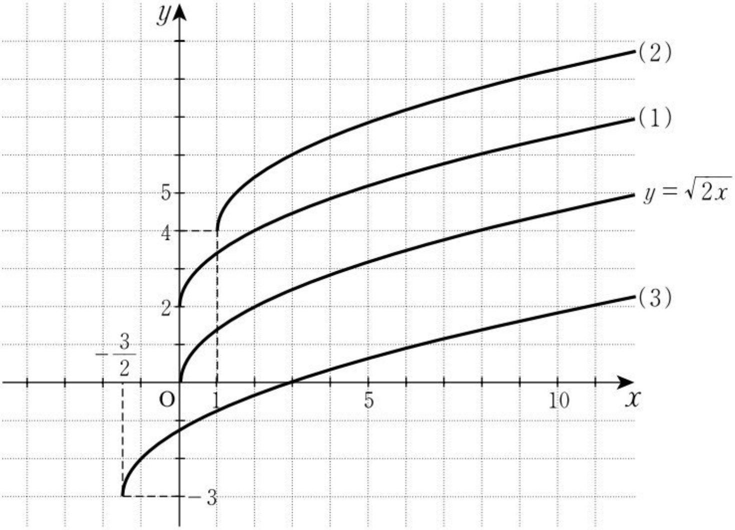
Graphs of Irrational Functions

1. Graph the following irrational functions.

(1) $y=\sqrt{2x}+2$

(2) $y=\sqrt{2x-2}+4=\sqrt{2(x-1)}+4$

(3) $y=\sqrt{2x+3}-3=\sqrt{2\left(x+\boxed{\mathbf{\frac{3}{2}}}\right)}-3$



2. Complete the following using the above graphs.

(1) $y=\sqrt{2x}+2$ is a translation of $y=\sqrt{2x}$, $\boxed{\mathbf{2}}$ unit(s) along the y-axis.

(2) $y=\sqrt{2x-2}+4$ is a translation of $y=\sqrt{2x}$, 1 unit along the x-axis, and $\boxed{\mathbf{4}}$ unit(s) along the y-axis.

(3) $y=\sqrt{2x+3}-3$ is a translation of $y=\sqrt{2x}$, $\boxed{\mathbf{-\frac{3}{2}}}$ unit(s) along the x-axis, and $\boxed{\mathbf{-3}}$ unit(s) along the y-axis.

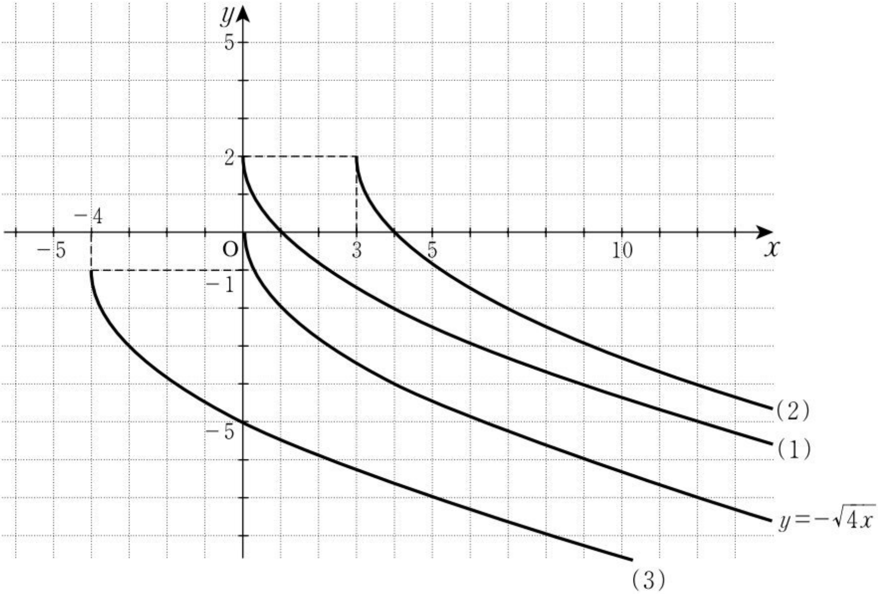
K 157b

3. Graph the following irrational functions.

(1) $y=-\sqrt{4x}+2$

(2) $y=-\sqrt{4x-12}+2=-\sqrt{4(x-\boxed{\mathbf{3}})}+2$

(3) $y=-\sqrt{4x+16}-1=\mathbf{-\sqrt{4(x+4)}-1}$



4. Complete the following using the above graphs.

(1) $y=-\sqrt{4x}+2$ is a translation of $y=-\sqrt{4x}$, $\boxed{\mathbf{2}}$ unit(s) along the y-axis.

(2) $y=-\sqrt{4x-12}+2$ is a translation of $y=-\sqrt{4x}$, $\boxed{\mathbf{3}}$ unit(s) along the x-axis, and $\boxed{\mathbf{2}}$ unit(s) along the y-axis.

(3) $y=-\sqrt{4x+16}-1$ is a translation of $y=-\sqrt{4x}$, $\boxed{\mathbf{-4}}$ unit(s) along the x-axis, and $\boxed{\mathbf{-1}}$ unit(s) along the y-axis.

Note: An irrational function of the form $y=\sqrt{k(x-p)}+q$ is a translation of $y=\sqrt{kx}$, p units along the x-axis, and q units along the y-axis.

An irrational function of the form $y=-\sqrt{k(x-p)}+q$ is a translation of $y=-\sqrt{kx}$, p units along the x-axis, and q units along the y-axis.

K 158a KUMON

Graphs of Irrational Functions

1. Describe how the following functions have been translated from $y=\sqrt{-x}$.

(1) $y=\sqrt{-x+1}-2=\sqrt{-(x-1)}-2$

1 unit along the x-axis, and $\boxed{\mathbf{-2}}$ unit(s) along the y-axis

(2) $y=\sqrt{-x-3}=\sqrt{-(x+3)}$

-3 units along the x-axis

(3) $y=\sqrt{-x+2}+3=\sqrt{-(x-2)}+3$

2 units along the x-axis, and 3 units along the y-axis

2. Describe how the following functions have been translated from $y=\sqrt{5x}$.

(1) $y=\sqrt{5x-10}+5=\sqrt{5(x-2)}+5$

2 units along the x-axis, and 5 units along the y-axis

(2) $y=\sqrt{5x}-10$

-10 units along the y-axis

(3) $y=\sqrt{5x+2}-3=\sqrt{5\left(x+\frac{2}{5}\right)}-3$

$-\frac{2}{5}$ units along the x-axis, and -3 units along the y-axis

K 158b

3. Find the equations of the curves obtained when $y=\sqrt{x}$ is translated as follows:

（1） 3 units along the y-axis

$$\boldsymbol{y=\sqrt{x}+3}$$

（2） -2 units along the x-axis

$$\boldsymbol{y=\sqrt{x+2}}$$

（3） 2 units along the x-axis, and -3 units along the y-axis

$$\boldsymbol{y=\sqrt{x-2}-3}$$

4. Find the equations of the curves obtained when $y=\sqrt{-3x}$ is translated as follows:

（1） -1 unit along the y-axis

$$\boldsymbol{y=\sqrt{-3x}-1}$$

（2） 1 unit along the x-axis

$$\boldsymbol{y}=\sqrt{-3(x-1)}\boldsymbol{=\sqrt{-3x+3}}$$

（3） $-\frac{1}{3}$ units along the x-axis, and 5 units along the y-axis

$$\boldsymbol{y}=\sqrt{-3\left(x+\frac{1}{3}\right)}+5\boldsymbol{=\sqrt{-3x-1}+5}$$

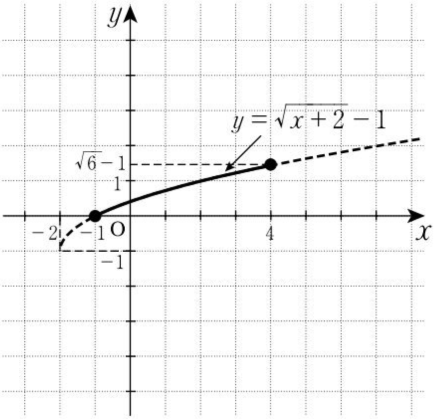
K 159a KUMON

Graphs of Irrational Functions

Graph each irrational function, and find the maximum and minimum values within the given domain.

Ex.

$f(x)=\sqrt{x+2}-1\ \ (-1\leqq x\leqq 4)$



[Sol]

$f(-1)=\sqrt{1}-1=0$

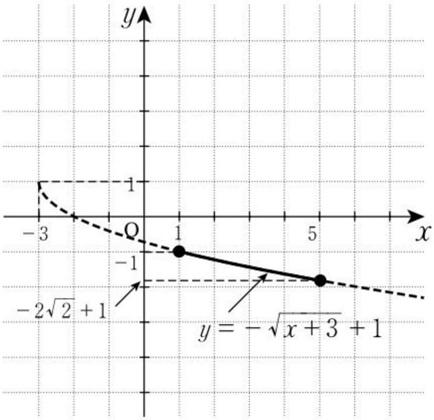
$f(4)=\sqrt{6}-1$

From the graph:

Maximum value: $f(4)=\sqrt{6}-1$

Minimum value: $f(-1)=0$

(1) $f(x)=-\sqrt{x+3}+1\ \ (1\leqq x\leqq 5)$



[Sol]

$f(1)=-\sqrt{4}+1=-1$

$f(5)=-\sqrt{8}+1=-2\sqrt{2}+1$

From the graph:

Maximum value:

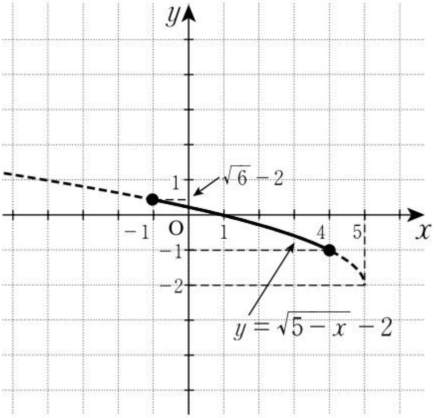
$f(1)=-1$

Minimum value:

$f(5)=-2\sqrt{2}+1$

K 159b

(2)　$f(x)=\sqrt{5-x}-2$ $(-1 \leqq x \leqq 4)$



[Sol]

$f(-1)=\sqrt{6}-2$

$f(4)=\sqrt{1}-2=-1$

From the graph:

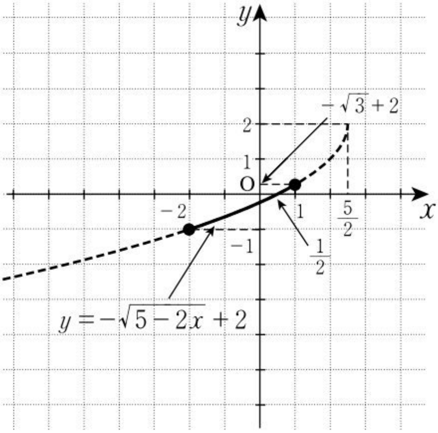
Maximum value:

$f(-1)=\sqrt{6}-2$

Minimum value:

$f(4)=-1$

(3)　$f(x)=-\sqrt{5-2x}+2$ $(-2 \leqq x \leqq 1)$



[Sol]

$f(-2)=-\sqrt{9}+2=-1$

$f(1)=-\sqrt{3}+2$

From the graph:

Maximum value:

$f(1)=-\sqrt{3}+2$

Minimum value:

$f(-2)=-1$

K 160a KUMON

Graphs of Irrational Functions

1. For each function, write the letter (A)～(F) of the corresponding sketch.

(1) $y=\sqrt{x-1}-2$ … $\boxed{\textbf{(D)}}$

(4) $y=-\sqrt{-x+1}+2$ … $\boxed{\textbf{(B)}}$

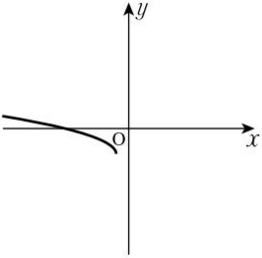
(2) $y=-\sqrt{x-1}+2$ … $\boxed{\textbf{(C)}}$

(5) $y=-\sqrt{2x-2}-1$ … $\boxed{\textbf{(E)}}$

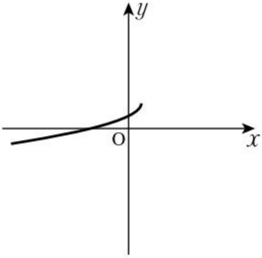
(3) $y=\sqrt{2x+2}+1$ … $\boxed{\textbf{(F)}}$

(6) $y=\sqrt{-x-1}-2$ … $\boxed{\textbf{(A)}}$

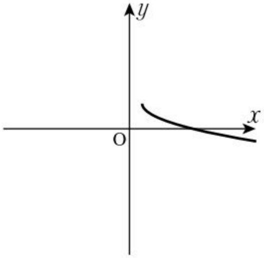
(A)



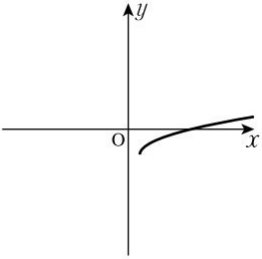
(B)



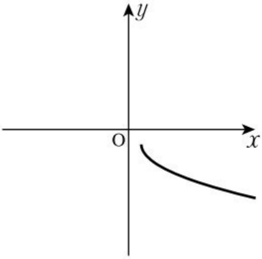
(C)



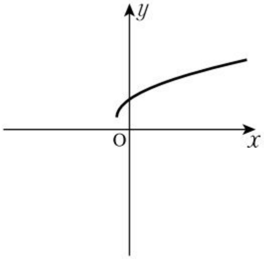
(D)



(E)



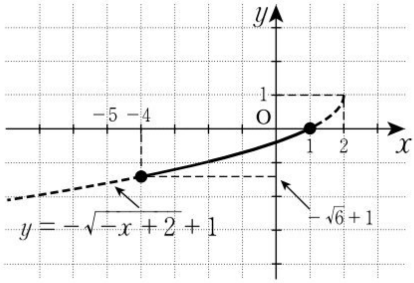
(F)



2. Find the equation of the function obtained when $y=\sqrt{2x}$ is translated 1 unit along the x-axis, and -2 units along the y-axis.

[Sol] $\boldsymbol{y}=\sqrt{2(x-1)}-2\boldsymbol{=\sqrt{2x-2}-2}$

3. Graph the irrational function $f(x)=-\sqrt{-x+2}+1\ (-4\leqslant x\leqslant 1)$, then find the maximum and minimum values within the given domain.



[Sol]

$f(-4)=-\sqrt{6}+1$

$f(1)=0$

From the graph:

Maximum value:

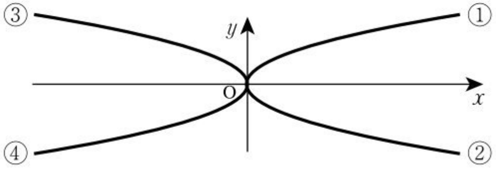
$\boldsymbol{f(1)=0}$

Minimum value:

$\boldsymbol{f(-4)=-\sqrt{6}+1}$

Note Summary

- Given an irrational function of the form $y=\sqrt{k(x-p)}+q$, the domain and range of the function are as follows:
  When $k>0$, Domain: $x\geqslant p$, Range: $y\geqslant q$
  When $k<0$, Domain: $x\leqslant p$, Range: $y\geqslant q$
- An irrational function of the form $y=\sqrt{k(x-p)}+q$ is a translation of $y=\sqrt{kx}$, p units along the x-axis, and q units along the y-axis.
- The graphs of some common irrational functions are shown below.
  ① $y=\sqrt{x}$, ② $y=-\sqrt{x}$, ③ $y=\sqrt{-x}$, ④ $y=-\sqrt{-x}$

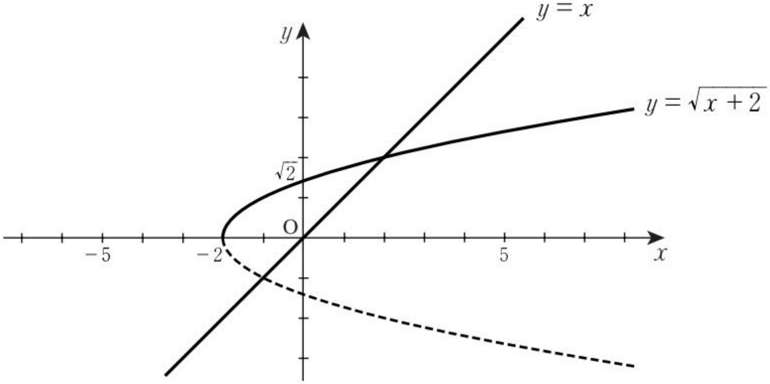


K 161a KUMON

Irrational Equations and Inequalities

Ex.

Graph $y=\sqrt{x+2}$ and $y=x$, and find the common points.



[Sol] Let $\sqrt{x+2}=x$

Squaring both sides,

$$x+2=x^2$$
$$x^2-x-2=0$$
$$(x-2)(x+1)=0$$
$$x=2,\ -1$$

From the graph, $x=\boxed{-1}$ is an extraneous solution.

☜ The graphs of $y=x$ and $y=\sqrt{x+2}$ do not have a common point at $x=-1$. (See the footnote.)

Therefore, the common point is ($\boxed{2}$, $\boxed{2}$).

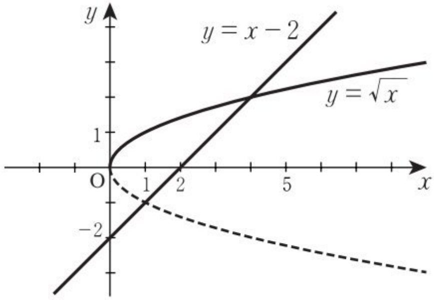
Answers: in order -1, 2, 2

The x-coordinate of the point of intersection between $y=x$ and $y=-\sqrt{x+2}$ (shown on the dashed curve of the graph) is $x=-1$.

K161b

Graph each pair of functions, then find the common points.

(1)　$y=\sqrt{x}$,　$y=x-2$



[Sol]

Let $\sqrt{x}=x-2$

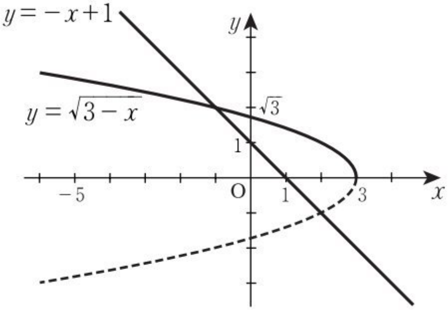
Squaring both sides,

$$x=(x-2)^2$$
$$x^2-5x+4=0$$
$$(x-4)(x-1)=0$$
$$x=4,\ 1$$

From the graph, $x=1$ is an extraneous solution.

Therefore, the common point is **(4, 2)**.

(2)　$y=\sqrt{3-x}$,　$y=-x+1$



[Sol]

Let $\sqrt{3-x}=-x+1$

Squaring both sides,

$$3-x=(-x+1)^2$$
$$x^2-x-2=0$$
$$(x-2)(x+1)=0$$
$$x=2,\ -1$$

From the graph, $x=2$ is an extraneous solution.

Therefore, the common point is **(−1, 2)**.

K 162a KUMON

Irrational Equations and Inequalities

In each question, draw the graph and use it to solve the irrational equation.

Ex.

$\sqrt{4-x}=x-2$

[Sol] Squaring both sides,

$4-x=(x-2)^2$

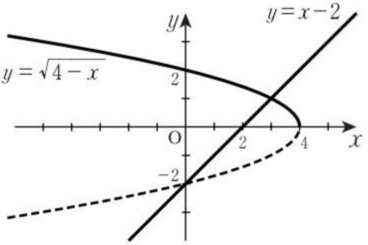
$x^2-3x=0$

$x(x-3)=0$

$x=0,\ 3$

From the graph, $x=0$ is an extraneous solution.

Therefore, $x=3$



Note: When drawing the graph, first find the x and y-intercepts.

(1) $-\sqrt{x+3}=x+1$

[Sol] Squaring both sides,

$x+3=(x+1)^2$

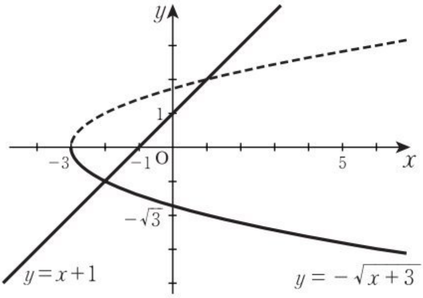
$x^2+x-2=0$

$(x+2)(x-1)=0$

$x=-2,\ 1$

From the graph, $x=1$ is an extraneous solution.

Therefore, $\boldsymbol{x=-2}$



K 162b

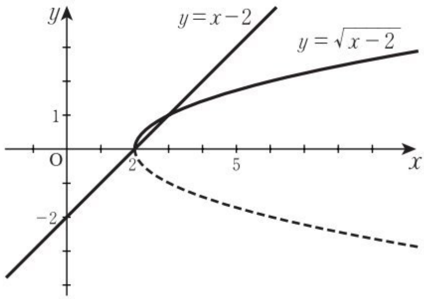
(2) $\sqrt{x-2}=x-2$ (There are two solutions.)

[Sol] Squaring both sides,

$$x-2=(x-2)^2$$
$$x^2-5x+6=0$$
$$(x-3)(x-2)=0$$
$$x=3,\ 2$$

From the graph, both solutions apply.

Therefore, **$x=3,\ 2$**



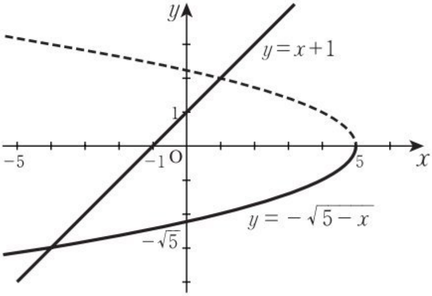
(3) $-\sqrt{5-x}=x+1$

[Sol] Squaring both sides,

$$5-x=(x+1)^2$$
$$x^2+3x-4=0$$
$$(x+4)(x-1)=0$$
$$x=-4,\ 1$$

From the graph, $x=1$ is an extraneous solution.

Therefore, **$x=-4$**



K 163a KUMON

Irrational Equations and Inequalities

In each question, draw the graph and use it to solve the irrational equation.

(1) $\sqrt{2x+5}=x+1$

[Sol] Squaring both sides,

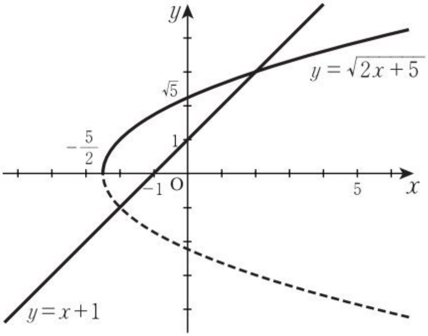
$$2x+5=(x+1)^2$$

$$x^2=4$$

$$x=\pm 2$$

From the graph, $x=-2$ is an extraneous solution.

Therefore, $\boldsymbol{x=2}$



(2) $\sqrt{2x+3}=-2x+3$

[Sol] Squaring both sides,

$$2x+3=(-2x+3)^2$$

$$4x^2-14x+6=0$$

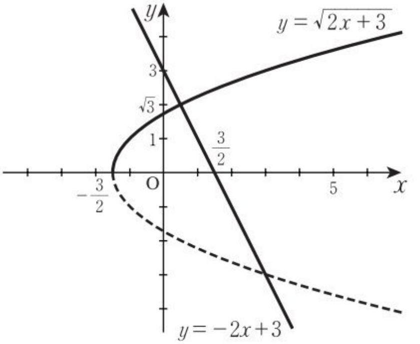
$$2x^2-7x+3=0$$

$$(2x-1)(x-3)=0$$

$$x=\frac{1}{2},\ 3$$

From the graph, $x=3$ is an extraneous solution.

Therefore, $\boldsymbol{x=\frac{1}{2}}$



K 163b

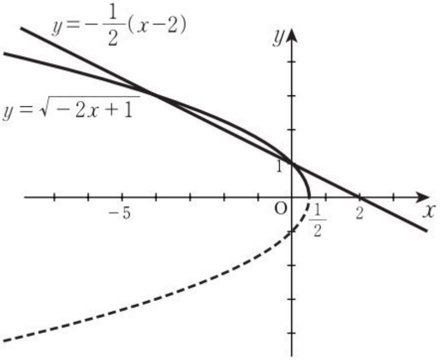
(3) $\sqrt{-2x+1}=-\dfrac{1}{2}(x-2)$

[Sol] Squaring both sides,

$$-2x+1=\frac{(x-2)^2}{4}$$
$$x^2+4x=0$$
$$x(x+4)=0$$
$$x=0,\ -4$$

From the graph, both solutions apply.

Therefore, $\boldsymbol{x=0,\ -4}$



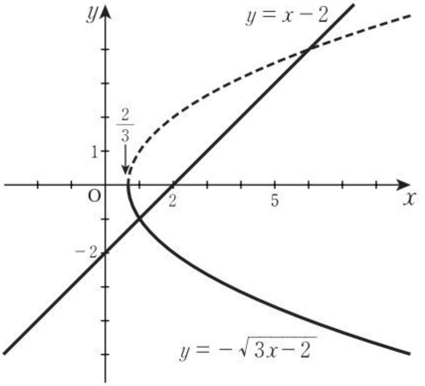
(4) $-\sqrt{3x-2}=x-2$

[Sol] Squaring both sides,

$$3x-2=(x-2)^2$$
$$x^2-7x+6=0$$
$$(x-6)(x-1)=0$$
$$x=6,\ 1$$

From the graph, $x=6$ is an extraneous solution.

Therefore, $\boldsymbol{x=1}$

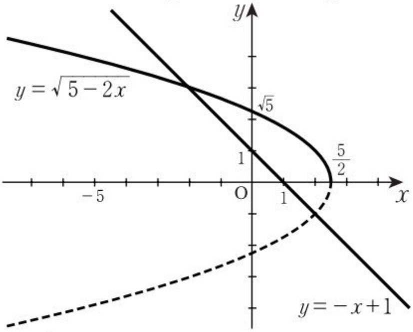


K 164a KUMON

Irrational Equations and Inequalities

Ex.

Solve the irrational equation $\sqrt{5-2x}=-x+1$, using the following methods.



[Sol] Squaring both sides,

$$5-2x=(-x+1)^2$$
$$x^2=4$$
$$x=\pm 2$$

[Sol 1] From the graph, $x=2$ is an extraneous solution.

Therefore, $x=-2$

[Sol 2] Substituting into the original equation,

(i) When $x=2$, the left hand side, LHS $=1$,
the right hand side, RHS $=-1$

Therefore, $x=2$ is an extraneous solution. ☜ Since the LHS $\neq$ RHS

(ii) When $x=-2$, the left hand side, LHS $=3$,
the right hand side, RHS $=3$

From (i) and (ii), $x=-2$

Solve the following irrational equations by using the second method, [Sol 2], shown in the example.

(1) $x+1=\sqrt{5-x}$

[Sol] Squaring both sides,

$$(x+1)^2=5-x$$
$$x^2+3x-4=0$$
$$(x+4)(x-1)=0$$
$$x=-4,\ 1$$

Substituting into the original equation,

(i) When $x=-4$, LHS $=-3$, RHS $=3$
Therefore, $x=-4$ is an extraneous solution.

(ii) When $x=1$, LHS $=2$, RHS $=2$

From (i) and (ii), $\boldsymbol{x=1}$

K 164b

(2) $x-1=\sqrt{2x+6}$

[Sol] Squaring both sides,

$$(x-1)^2=2x+6$$

$$x^2-4x-5=0$$

$$(x-5)(x+1)=0$$

$$x=5,\ -1$$

Substituting into the original equation,

(i) When $x=5$, LHS $=4$, RHS $=4$

(ii) When $x=-1$, LHS $=-2$, RHS $=2$

Therefore, $x=-1$ is an extraneous solution.

From (i) and (ii), $\boldsymbol{x=5}$

(3) $\sqrt{x+3}=\dfrac{x+5}{3}$

[Sol] Squaring both sides,

$$x+3=\frac{(x+5)^2}{9}$$

$$x^2+x-2=0$$

$$(x+2)(x-1)=0$$

$$x=-2,\ 1$$

Substituting into the original equation,

(i) When $x=-2$, LHS $=1$, RHS $=1$

(ii) When $x=1$, LHS $=2$, RHS $=2$

From (i) and (ii), $\boldsymbol{x=-2,\ 1}$

Note: When solving irrational equations we must always check our solutions using either method ① or ②.

① The solution must correspond to the common point on the graph.

② When substituting into the original equation, we must show that LHS = RHS.

K 165a KUMON

Irrational Equations and Inequalities

Solve the following irrational inequalities using the graphs.

Ex.

$\sqrt{2x+1} > x-1$

[Sol] Let $\sqrt{2x+1} = x-1$ ☜ First, solve the equation.

Squaring both sides,

$2x+1=(x-1)^2$

$x^2-4x=0$

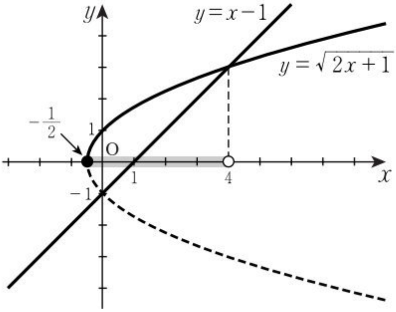
$x(x-4)=0$

$x=0,\ 4$

From the graph, $x=0$ is an extraneous solution.

Thus, $x=4$

Therefore, $-\dfrac{1}{2} \leqq x < 4$ ☜ For these values of x, the graph of $y=\sqrt{2x+1}$ lies above the graph of $y=x-1$.



(1) $\sqrt{x+3} < -x+3$

[Sol] Let $\sqrt{x+3} = -x+3$

Squaring both sides,

$x+3=(-x+3)^2$

$x^2-7x+6=0$

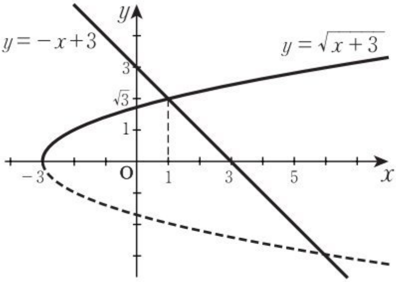
$(x-6)(x-1)=0$

$x=6,\ 1$

From the graph, $x=6$ is an extraneous solution.

Thus, $x=1$

Therefore, $\mathbf{-3 \leqq x < 1}$



K 165b

(2) $\sqrt{x-1}<x-3$

[Sol] Let $\sqrt{x-1}=x-3$

Squaring both sides,

$x-1=(x-3)^2$

$x^2-7x+10=0$

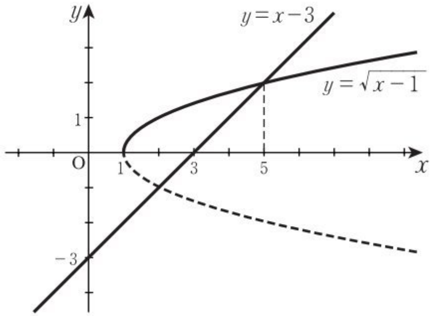
$(x-5)(x-2)=0$

$x=5,\ 2$

From the graph, $x=2$ is an extraneous solution.

Thus, $x=5$

Therefore, $\boldsymbol{x>5}$



(3) $\sqrt{5-2x}>-x+1$

[Sol] Let $\sqrt{5-2x}=-x+1$

Squaring both sides,

$5-2x=(-x+1)^2$

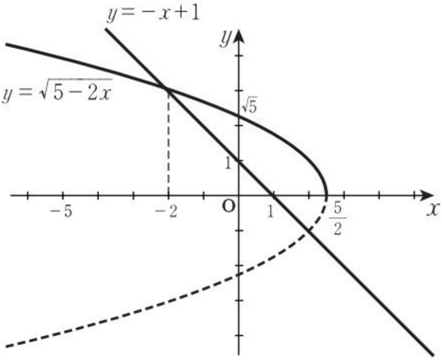
$x^2=4$

$x=\pm 2$

From the graph, $x=2$ is an extraneous solution.

Thus, $x=-2$

Therefore, $\boldsymbol{-2<x\leqq\frac{5}{2}}$



K 166a KUMON

Irrational Equations and Inequalities

Solve the following irrational inequalities using the graphs.

(1) $\sqrt{5-x} < 3-x$

[Sol] Let $\sqrt{5-x} = 3-x$

Squaring both sides,

$$5-x=(3-x)^2$$

$$x^2-5x+4=0$$

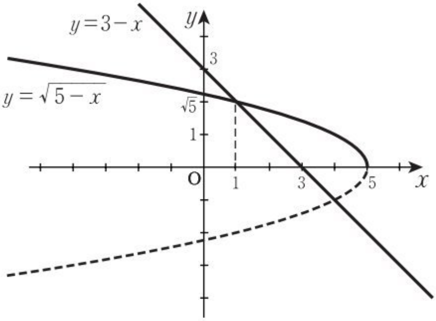
$$(x-4)(x-1)=0$$

$$x=4,\ 1$$

From the graph, $x=4$ is an extraneous solution.

Thus, $x=1$

Therefore, $\boldsymbol{x<1}$



(2) $\sqrt{x+3} > \dfrac{x+3}{2}$

[Sol] Let $\sqrt{x+3} = \dfrac{x+3}{2}$

Squaring both sides,

$$x+3=\frac{(x+3)^2}{4}$$

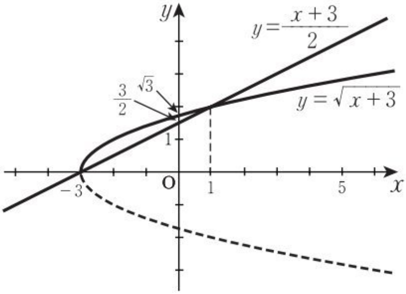
$$x^2+2x-3=0$$

$$(x+3)(x-1)=0$$

$$x=-3,\ 1$$

From the graph,

$$\boldsymbol{-3<x<1}$$



K 166b

（3） $\sqrt{2x+1} > \dfrac{x+2}{2}$

[Sol] Let $\sqrt{2x+1} = \dfrac{x+2}{2}$

Squaring both sides,

$$2x+1 = \frac{(x+2)^2}{4}$$

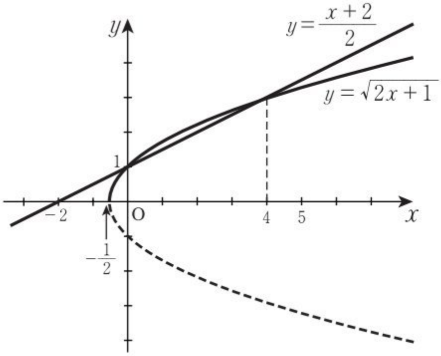
$$x^2-4x=0$$

$$x(x-4)=0$$

$$x=0,\ 4$$

From the graph,

$$\mathbf{0 < x < 4}$$



（4） $\sqrt{2x+1} < \dfrac{x+2}{2}$

[Sol] From the graph in question（3）,

$$\mathbf{-\frac{1}{2} \leqslant x < 0,\ \ x > 4}$$

K 167a KUMON

Irrational Equations and Inequalities

Solve the following irrational inequalities using the graphs.

Ex.

$\sqrt{2x+1} < \sqrt{x} + 1$

[Sol] Let $\sqrt{2x+1} = \sqrt{x} + 1$

Squaring both sides,

$2x+1 = x+2\sqrt{x}+1$

$x = 2\sqrt{x}$

Squaring both sides again,

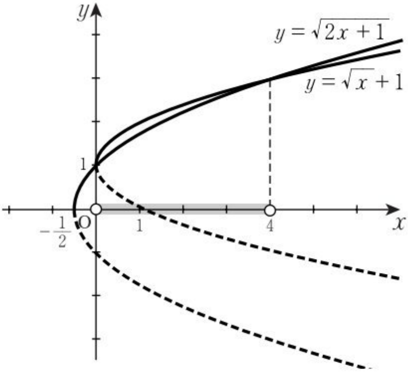
$x^2 = 4x$

$x(x-4) = 0$

$x = 0,\ 4$

From the graph,

$0 < x < 4$



(1) $\sqrt{2x+1} > \sqrt{x} + 1$

[Sol] From the graph in the example,

$x > 4$

K 167b

(2)　$\sqrt{x}-1<\sqrt{5-x}$

[Sol] Let $\sqrt{x}-1=\sqrt{5-x}$

Squaring both sides,

$$x-2\sqrt{x}+1=5-x$$

$$2x-4=2\sqrt{x}$$

$$x-2=\sqrt{x}$$

Squaring both sides again,

$$x^2-4x+4=x$$

$$x^2-5x+4=0$$

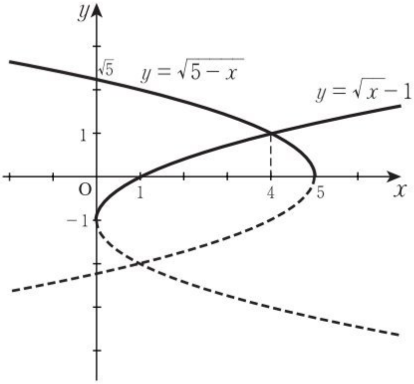
$$(x-4)(x-1)=0$$

$$x=4,\ 1$$

From the graph, $x=1$ is an extraneous solution.

Thus, $x=4$

Therefore, $\boldsymbol{0\leqq x<4}$



(3)　$\sqrt{x}-1>\sqrt{5-x}$

[Sol] From the graph in question (2),

$$\boldsymbol{4<x\leqq 5}$$

K 168a KUMON

Irrational Equations and Inequalities

1. Solve the irrational inequality $\sqrt{x} < x < \sqrt{2x+8}$ using the following method.

(1) Solve $\sqrt{x} < x$ using the graph.

[Sol] Let $\sqrt{x} = x$

Squaring both sides,

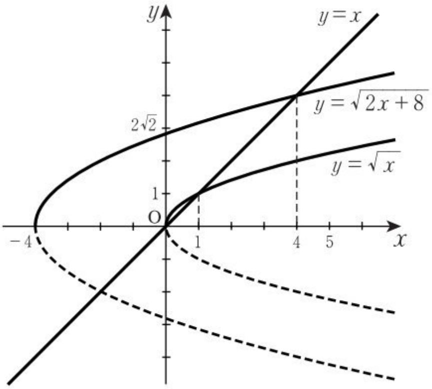
$x = x^2$

$x(x-1) = 0$

$x = 0,\ 1$

From the graph,

$\boldsymbol{x > 1}$



(2) Solve $x < \sqrt{2x+8}$ using the graph.

[Sol] Let $x = \sqrt{2x+8}$

Squaring both sides,

$x^2 = 2x + 8$

$x^2 - 2x - 8 = 0$

$(x-4)(x+2) = 0$

$x = 4,\ -2$

From the graph,

$\boldsymbol{-4 \leqq x < 4}$

(3) Use the results of (1) and (2) to find the solution of the original inequality.

[Sol] $\boldsymbol{1 < x < 4}$

K 168b

2. Solve the irrational inequality $\sqrt{6-x}<x<\sqrt{2x+8}$ using the method followed on side a.

[Sol]

(i) $\sqrt{6-x}<x$

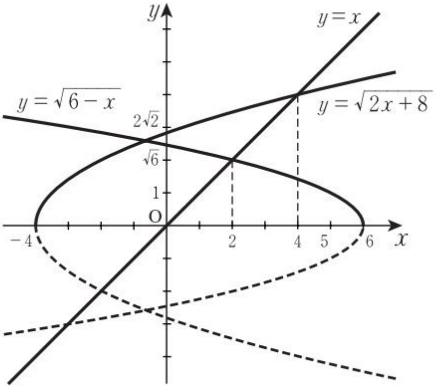
Let $\sqrt{6-x}=x$

Squaring both sides,

$$6-x=x^2$$
$$(x+3)(x-2)=0$$
$$x=-3,\ 2$$

From the graph,

$$2<x\leqslant 6$$



(ii) $x<\sqrt{2x+8}$

Let $x=\sqrt{2x+8}$

Squaring both sides,

$$x^2=2x+8$$
$$x^2-2x-8=0$$
$$(x-4)(x+2)=0$$
$$x=4,\ -2$$

From the graph,

$$-4\leqslant x<4$$

From (i) and (ii), the solution of the original inequality is:

$$\mathbf{2<x<4}$$

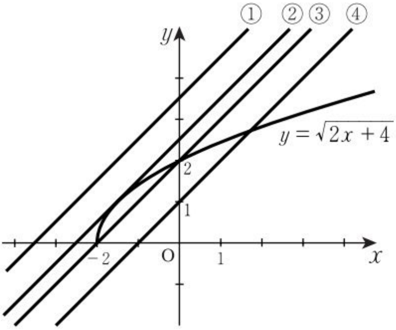
K 169a KUMON

Irrational Equations and Inequalities

Ex.

Using the graph, find the number of real solutions of the irrational equation:

$\sqrt{2x+4}=x+k$



[Sol] Squaring both sides,

$$2x+4=x^2+2kx+k^2$$

$$x^2+2(k-1)x+k^2-4=0$$

When the line is at ①, there are no common points.

When the line is at ②, it touches the curve.

From $\frac{D}{4}=(k-1)^2-(k^2-4)=-2k+5=0$,

$$k=\frac{5}{2}$$

When the line is at ③, $y=x+k$ passes through point $(-2,\ 0)$.

From $0=-2+k$, ☜ Substitute $x=-2$ and $y=0$ into $y=x+k$.

$$k=2$$

When the line is at ④, there is 1 common point.

Therefore:

When $k>\frac{5}{2}$, there is/are $\boxed{0}$ real solution(s).

When $k=\frac{5}{2}$, there is/are $\boxed{1}$ real solution(s).

When $2\leqslant k<\frac{5}{2}$, there is/are $\boxed{2}$ real solution(s).

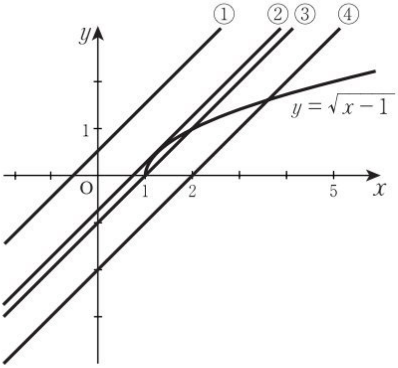
When $k<2$, there is/are $\boxed{1}$ real solution(s).

Answers: in order 0, 1, 2, 1

K 169b

1. Using the graph and the method shown on side a, find the number of real solutions of the irrational equation:

   $\sqrt{x-1}=x+k$



[Sol] Squaring both sides,

$$x-1=x^2+2kx+k^2$$
$$x^2+(2k-1)x+k^2+1=0$$

When the line is at ①,
there are no common points.

When the line is at ②,
it touches the curve.

From $D=(2k-1)^2-4(k^2+1)$
$=-4k-3=0,$

$$k=-\frac{3}{4}$$

When the line is at ③, $y=x+k$ passes through point $(1,\ 0)$.

From $0=1+k$,

$$k=-1$$

When the line is at ④, there is 1 common point.

Therefore:

When $k>-\frac{3}{4}$, there are 0 real solutions.

When $k=-\frac{3}{4}$, there is 1 real solution.

When $-1\leqq k<-\frac{3}{4}$, there are 2 real solutions.

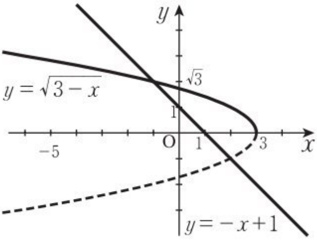
When $k<-1$, there is 1 real solution.

K 170a KUMON

Irrational Equations and Inequalities

1. Solve the following irrational equations.

(1) $\sqrt{3-x}=-x+1$



[Sol] Squaring both sides,

$$3-x=(-x+1)^2$$
$$x^2-x-2=0$$
$$(x-2)(x+1)=0$$
$$x=2,\ -1$$

From the graph, $x=2$ is an extraneous solution.

Therefore, $\boldsymbol{x=-1}$

Alternative Solution

Squaring both sides, $3-x=(-x+1)^2$

Therefore, $x=2,\ -1$

Substituting into the original equation:

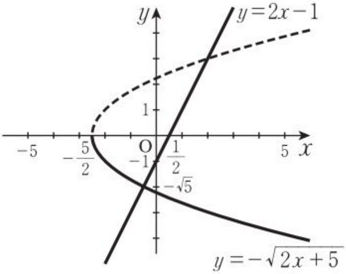
(i) When $x=2$, LHS $=1$, RHS $=-1$

Therefore, $x=2$ is an extraneous solution.

(ii) When $x=-1$, LHS $=2$, RHS $=2$

From (i) and (ii), $\boldsymbol{x=-1}$

(2) $-\sqrt{2x+5}=2x-1$



[Sol] Squaring both sides,

$$2x+5=(2x-1)^2$$
$$2x^2-3x-2=0$$
$$(2x+1)(x-2)=0$$
$$x=-\frac{1}{2},\ 2$$

From the graph, $x=2$ is an extraneous solution.

Therefore, $\boldsymbol{x=-\frac{1}{2}}$

Alternative Solution

Squaring both sides, $2x+5=(2x-1)^2$

Therefore, $x=-\frac{1}{2},\ 2$

Substituting into the original equation:

(i) When $x=-\frac{1}{2}$, LHS $=-2$, RHS $=-2$

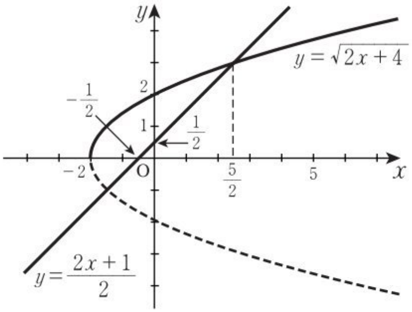
(ii) When $x=2$, LHS $=-3$, RHS $=3$

Therefore, $x=2$ is an extraneous solution.

From (i) and (ii), $\boldsymbol{x=-\frac{1}{2}}$

2. Solve the irrational inequality $\sqrt{2x+4} > \dfrac{2x+1}{2}$.

[Sol] Let $\sqrt{2x+4} = \dfrac{2x+1}{2}$



Squaring both sides,

$$2x+4=\frac{(2x+1)^2}{4}$$

$$4x^2-4x-15=0$$

$$(2x-5)(2x+3)=0$$

$$x=\frac{5}{2},\ -\frac{3}{2}$$

From the graph, $x=-\dfrac{3}{2}$ is an extraneous solution.

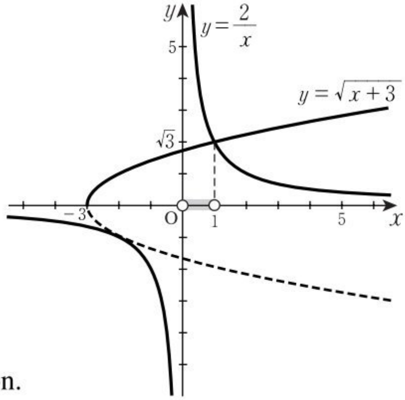
Thus, $x=\dfrac{5}{2}$

Therefore, $\mathbf{-2 \leqslant x < \dfrac{5}{2}}$

Consider this!

Solve the irrational inequality $\sqrt{x+3} < \dfrac{2}{x}$.

[Sol] Let $\sqrt{x+3} = \dfrac{2}{x}$



Squaring both sides,

$$x+3=\frac{4}{x^2}$$

$$x^3+3x^2-4=0$$

$$(x-1)(x^2+4x+4)=0$$

$$(x-1)(x+2)^2=0$$

$$x=1,\ -2$$

From the graph,

$x=-2$ is an extraneous solution.

Therefore, $\boxed{\mathbf{0}} < x < \boxed{\mathbf{1}}$

Answers: in order 0, 1

K 171 a KUMON

Exponential Functions

1. Simplify the following expressions.

Ex.

$a^3\times a^4=a^7,\ \ a^7\div a^4=a^3,\ \ (a^3)^4=a^{12}$

(1) $a^2\times a^7=\boldsymbol{a^9}$

(2) $a^7\div a^2=\boldsymbol{a^5}$

(3) $a^2\div a^7=\boldsymbol{\dfrac{1}{a^5}}$

(4) $a^3\div a^3=\mathbf{1}$

(5) $(a^3)^5=\boldsymbol{a^{15}}$

(6) $(ab)^3=\boldsymbol{a^3b^3}$

(7) $(a^2b^3)^5=\boldsymbol{a^{10}b^{15}}$

(8) $(-3a^3b^2)^3=\boldsymbol{-27a^9b^6}$

(9) $(-2a^4b^5)^4=\boldsymbol{16a^{16}b^{20}}$

(10) $\left(\dfrac{a}{b}\right)^3=\boldsymbol{\dfrac{a^3}{b^3}}$

(11) $\left(\dfrac{bc^2}{a^3}\right)^4=\boldsymbol{\dfrac{b^4c^8}{a^{12}}}$

(12) $\left(-\dfrac{2bc^2}{3a^3}\right)^5=\boldsymbol{-\dfrac{32b^5c^{10}}{243a^{15}}}$

Definition of $\boldsymbol{a^0}$ and $\boldsymbol{a^{-n}}$

1. Using $a^m\times a^n=a^{m+n}$, then when $m=0$,
   we find $a^0\times a^n=a^n$. Thus, $a^0=1$.
2. Using $a^m\div a^n=a^{m-n}$, then when $m=0$,
   we find $1\div a^n=a^{-n}$. Thus, $a^{-n}=\dfrac{1}{a^n}$.

Laws of Exponents I

When $a \neq 0$, $b \neq 0$,

$a^0 = 1$　　$a^{-n} = \dfrac{1}{a^n}$

$a^m \times a^n = a^{m+n}$　　$a^m \div a^n = a^{m-n}$

$(a^m)^n = a^{mn}$　　$(ab)^m = a^m b^m$

2. Evaluate the following expressions.

(1) $2^{-2} = \dfrac{1}{2^2} = \mathbf{\dfrac{1}{4}}$

(2) $3^{-4} = \mathbf{\dfrac{1}{81}}$

(3) $10^0 = \mathbf{1}$

(4) $5^{-1} = \mathbf{\dfrac{1}{5}}$

(5) $2^{-5} = \mathbf{\dfrac{1}{32}}$

(6) $2^0 = \mathbf{1}$

3. Calculate the following expressions.

(1) $(3^4)^0 = \mathbf{1}$

(2) $4^3 = 2^{\boxed{6}} = \mathbf{64}$

(3) $4^{-3} = 2^{\boxed{-6}} = \mathbf{\dfrac{1}{64}}$

(4) $2^4 \times 2^3 = 2^7 = \mathbf{128}$

(5) $2^3 \times 2^{-6} = 2^{-3} = \mathbf{\dfrac{1}{8}}$

(6) $3^5 \div 3^2 = 3^3 = \mathbf{27}$

(7) $2^{-1} \div 2^{-4} = 2^3 = \mathbf{8}$

(8) $2^2 \div 2^{-3} = 2^5 = \mathbf{32}$

Note: 'Exponent' is another word for 'power' or 'index'. Functions containing exponents are usually called exponential functions, so the word 'exponent' is used here.

K 172a KUMON

Exponential Functions

Laws of Exponents I

When $a \neq 0$, $b \neq 0$,

$a^0 = 1$ $\qquad a^{-n} = \dfrac{1}{a^n}$

$a^m \times a^n = a^{m+n}$ $\qquad a^m \div a^n = a^{m-n}$

$(a^m)^n = a^{mn}$ $\qquad (ab)^m = a^m b^m$

1. Use the *laws of exponents* I to simplify the following expressions.

(1) $a^0 \div a^5 = a^{\boxed{-5}} = \boldsymbol{\dfrac{1}{a^5}}$

(2) $a^{13} \times a^{-3} \div a = \boldsymbol{a^9}$

(3) $a^{-10} \div a^{-3} \times a^4 = a^{-3} = \boldsymbol{\dfrac{1}{a^3}}$

(4) $a^5 \times a^{-7} \times (a^2)^3 = \boldsymbol{a^4}$

(5) $a^0 \times (a^2)^4 \times a^{-12} = a^{-4} = \boldsymbol{\dfrac{1}{a^4}}$

(6) $(a^2)^3 \times a^{-4} \times a^{-2} = \mathbf{1}$

(7) $(a^4)^3 \div (a^3)^5 = a^{-3} = \boldsymbol{\dfrac{1}{a^3}}$

2. Using the *laws of exponents* I on side a, calculate the following expressions.

(1) $3^{-1} \div 3^{2} = 3^{-3} = \mathbf{\frac{1}{27}}$

(2) $2^{3} \times 2^{-2} \div 2^{-1} = 2^{2} = \mathbf{4}$

(3) $\frac{1}{2^{-2}} = 2^{2} = \mathbf{4}$

(4) $(2^{3})^{-1} = 2^{-3} = \mathbf{\frac{1}{8}}$

(5) $4^{2} \times 2^{-6} = 2^{4} \times 2^{-6} = 2^{-2} = \mathbf{\frac{1}{4}}$

(6) $8^{4} \times 16^{-2} \div 64 = 2^{12} \times 2^{-8} \div 2^{6} = 2^{-2} = \mathbf{\frac{1}{4}}$

(7) $(3 \times 2^{2})^{-2} = 3^{-2} \times 2^{-4} = \mathbf{\frac{1}{144}}$

(8) $(-3)^{2} \times (2^{-3} \times 3^{2})^{-1} = 2^{3} = \mathbf{8}$

K 173a KUMON

Exponential Functions

1. Evaluate the following expressions.

(1) $2^3 = \mathbf{8}$

(2) $(-2)^4 = \mathbf{16}$

(3) $2^5 = \mathbf{32}$

(4) $(-2)^6 = \mathbf{64}$

(5) $2^7 = \mathbf{128}$

(6) $(-2)^8 = \mathbf{256}$

(7) $(-3)^3 = \mathbf{-27}$

(8) $3^4 = \mathbf{81}$

(9) $(-3)^5 = \mathbf{-243}$

(10) $3^6 = \mathbf{729}$

(11) $(-4)^3 = \mathbf{-64}$

(12) $4^4 = \mathbf{256}$

(13) $5^3 = \mathbf{125}$

(14) $(-5)^4 = \mathbf{625}$

(15) $6^3 = \mathbf{216}$

(16) $(-7)^3 = \mathbf{-343}$

K 173b

2. Evaluate the following expressions.

Ex.

$\sqrt[3]{8}=2$ $\sqrt[3]{-8}=-2$

(1) $\sqrt{25}=\mathbf{5}$

(2) $\sqrt[3]{27}=\mathbf{3}$

(3) $\sqrt[3]{-27}=\mathbf{-3}$

(4) $\sqrt[3]{64}=\mathbf{4}$

(5) $\sqrt[3]{125}=\mathbf{5}$

(6) $\sqrt[3]{216}=\mathbf{6}$

(7) $\sqrt[4]{16}=\mathbf{2}$

(8) $\sqrt[4]{81}=\mathbf{3}$

(9) $\sqrt[4]{625}=\mathbf{5}$

(10) $\sqrt[5]{32}=\mathbf{2}$

(11) $\sqrt[6]{64}=\mathbf{2}$

(12) $\sqrt[7]{-128}=\mathbf{-2}$

(13) $\sqrt[5]{-243}=\mathbf{-3}$

(14) $\sqrt[4]{256}=\mathbf{4}$

(15) $\sqrt[3]{-64}=\mathbf{-4}$

(16) $\sqrt[3]{-125}=\mathbf{-5}$

(17) $\sqrt[5]{-32}=\mathbf{-2}$

(18) $-\sqrt[5]{32}=\mathbf{-2}$

Note: The number that, **when raised to the power of n, is equal to a** is called the n^{th} **root** of a.

For example, $\sqrt[3]{8}$ is the 3rd root of 8, $\sqrt[3]{-8}$ is the 3rd root of -8, $\sqrt[n]{a}$ is the n^{th} root of a.

K 174a KUMON

Exponential Functions

1. Complete the following questions.

(1) $\sqrt{16}=\mathbf{4}$

(2) The square roots of 16 are

$\mathbf{\pm 4}$.

(3) $\sqrt{25}=\mathbf{5}$

(4) The square roots of 25 are

$\mathbf{\pm 5}$.

(5) $\sqrt[4]{16}=\mathbf{2}$

(6) The 4th roots of 16 are

$\mathbf{\pm 2}$.

(7) $\sqrt[3]{8}=\mathbf{2}$

(8) The cube root of 8 is

$\mathbf{2}$.

(9) $\sqrt[3]{-8}=\mathbf{-2}$

(10) The cube root of -8 is

$\mathbf{-2}$.

(11) The 5th root of 32 is

$\mathbf{2}$.

(12) The 5th root of -32 is

$\mathbf{-2}$.

The number of n^{th} roots

(i) When n is an even number (square roots, 4th roots...):
When $a>0$, there are two real n^{th} roots $\sqrt[n]{a}$ and $-\sqrt[n]{a}$.
When $a<0$, there are no real n^{th} roots.

(ii) When n is an odd number (cube roots, 5th roots...):
There is only one real n^{th} root.

However, if we include complex numbers, there are *three* cube roots, *four* fourth roots, etc. (See K180.)

K 174b

Laws of Exponents II

When $a>0$, $b>0$ and m, n and p are positive integers:

$\sqrt[n]{a^m}=(\sqrt[n]{a})^m$ $\qquad \sqrt[mp]{a^{np}}=\sqrt[m]{a^n}$ $\qquad \sqrt[m]{\sqrt[n]{a}}=\sqrt[mn]{a}$

$\sqrt[n]{a}\sqrt[n]{b}=\sqrt[n]{ab}$ $\qquad \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}=\sqrt[n]{\dfrac{a}{b}}$

2. Evaluate the following expressions.

(1) $\sqrt[3]{1000}=\mathbf{10}$

(2) $\sqrt[3]{8000}=\mathbf{20}$

(3) $\sqrt{0.01}=\mathbf{0.1}$

(4) $\sqrt[4]{0.0001}=\mathbf{0.1}$

(5) $\sqrt[3]{2^{12}}=2^4=\mathbf{16}$

(6) $\sqrt[3]{2^6}=2^2=\mathbf{4}$

(7) $\sqrt[4]{5^{12}}=5^3=\mathbf{125}$

(8) $\sqrt[3]{8^4}=2^4=\mathbf{16}$

K 175a KUMON

Exponential Functions

Laws of Exponents II

When $a > 0$, $b > 0$ and m, n and p are positive integers:

$\sqrt[n]{a^m} = (\sqrt[n]{a})^m$ $\qquad \sqrt[mp]{a^{np}} = \sqrt[m]{a^n}$ $\qquad \sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$

$\sqrt[n]{a}\sqrt[n]{b} = \sqrt[n]{ab}$ $\qquad \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}$

Evaluate the following expressions.

(1) $\sqrt[4]{25} = \sqrt{\mathbf{5}}$

(2) $\sqrt{16^3} = \mathbf{64}$

(3) $\sqrt[5]{-\dfrac{1}{32}} = \mathbf{-\dfrac{1}{2}}$

(4) $\sqrt[3]{\dfrac{64}{125}} = \mathbf{\dfrac{4}{5}}$

(5) $(\sqrt[3]{-2})^3 = \mathbf{-2}$

(6) $\sqrt[3]{(-2)^3} = \mathbf{-2}$

(7) $\sqrt[4]{(-2)^4} = \mathbf{2}$

(8) $(-\sqrt[3]{3})^9 = \mathbf{-27}$

K175b

(9) $\sqrt[5]{\sqrt{2^{20}}} = \sqrt[10]{2^{20}} = 2^2 = \mathbf{4}$

(10) $\sqrt{\sqrt[3]{2^6}} = \sqrt[6]{2^6} = \mathbf{2}$

(11) $\sqrt{\sqrt[3]{729}} = \sqrt[6]{729} = \mathbf{3}$

(12) $\sqrt[3]{\sqrt{64}} = \sqrt[6]{64} = \mathbf{2}$

(13) $\dfrac{\sqrt{75}}{\sqrt{3}} = \sqrt{\dfrac{75}{3}} = \sqrt{25} = \mathbf{5}$

(14) $\dfrac{\sqrt[3]{2}}{\sqrt[3]{54}} = \sqrt[3]{\dfrac{1}{27}} = \mathbf{\dfrac{1}{3}}$

(15) $\dfrac{\sqrt[4]{6}}{\sqrt[4]{96}} = \sqrt[4]{\dfrac{1}{16}} = \mathbf{\dfrac{1}{2}}$

(16) $\sqrt[3]{\sqrt{8}} = \sqrt[6]{8} = \sqrt{\sqrt[3]{8}} = \sqrt{\boxed{\mathbf{2}}}$

(17) $\sqrt{\sqrt[3]{16}} = \sqrt[6]{16} = \sqrt[3]{\sqrt{16}} = \sqrt[3]{\mathbf{4}}$

K 176a KUMON

Exponential Functions

1. Evaluate the following expressions.

(1) $\sqrt{8}\times\sqrt{2}=\sqrt{4^2}=\mathbf{4}$

(2) $\sqrt[3]{2}\times\sqrt[3]{4}=\sqrt[3]{8}=\boxed{\mathbf{2}}$

(3) $\sqrt[3]{12}\times\sqrt[3]{18}=\sqrt[3]{2^3\cdot3^3}=\mathbf{6}$

(4) $\sqrt[5]{4}\times\sqrt[5]{8}=\sqrt[5]{2^5}=\mathbf{2}$

(5) $\sqrt[3]{2^5}\times\sqrt[3]{16}=\sqrt[3]{2^5\cdot2^4}=\mathbf{8}$

(6) $\dfrac{\sqrt[4]{400}}{\sqrt{10}}=\dfrac{\sqrt[4]{20^2}}{\sqrt{10}}=\dfrac{\sqrt{20}}{\sqrt{10}}=\sqrt{\mathbf{2}}$

(7) $\sqrt[3]{2}\div\sqrt[3]{16}=\dfrac{\sqrt[3]{2}}{\sqrt[3]{16}}=\sqrt[3]{\dfrac{1}{8}}=\mathbf{\dfrac{1}{2}}$

(8) $\sqrt[3]{2}\div\sqrt[3]{-16}=\dfrac{\sqrt[3]{2}}{-\sqrt[3]{16}}=-\sqrt[3]{\dfrac{1}{8}}=\mathbf{-\dfrac{1}{2}}$

Note: $\sqrt[3]{-16}=-\sqrt[3]{16}$, $\sqrt[5]{-32}=-\sqrt[5]{32}=-2$
When n is an odd number, $\sqrt[n]{-a}=-\sqrt[n]{a}$ (where $a>0$).

K 176b

2. Simplify the following expressions. Assume $a > 0$.

(1) $\sqrt[3]{a^2} \times \sqrt[3]{a^4} = \sqrt[3]{a^6} = \boldsymbol{a}^2$

(2) $\sqrt[3]{a^7} \times \sqrt[3]{-a^2} = \sqrt[3]{a^7} \times (-\sqrt[3]{a^2}) = -\sqrt[3]{a^9} = \boldsymbol{-a}^3$

(3) $\sqrt[3]{a^7} \div \sqrt[3]{-a} = \dfrac{\sqrt[3]{a^7}}{-\sqrt[3]{a}} = -\sqrt[3]{\dfrac{a^7}{a}} = -\sqrt[3]{a^6} = \boldsymbol{-a}^2$

(4) $\sqrt[3]{2a^2} \times \sqrt[3]{4a} = \sqrt[3]{8a^3} = \boldsymbol{2a}$

(5) $\sqrt[3]{-2a^4} \div \sqrt[3]{-16a} = \dfrac{-\sqrt[3]{2a^4}}{-\sqrt[3]{16a}} = \sqrt[3]{\dfrac{2a^4}{16a}} = \sqrt[3]{\dfrac{a^3}{8}} = \boldsymbol{\dfrac{a}{2}}$

(6) $\sqrt[3]{a^{12}} = \boldsymbol{a}^4$

(7) $\sqrt[6]{a^2} = \sqrt[3]{\boldsymbol{a}}$

(8) $\sqrt{\sqrt[3]{16a^2}} = \sqrt[3]{\boldsymbol{4a}}$

K 177a KUMON

Exponential Functions

If $(a^m)^n = a^{mn}$, then when $m = \dfrac{2}{3}$ and $n = 3$, we have $(a^{\frac{2}{3}})^3 = a^2$.

This shows us that $a^{\frac{2}{3}}$ is the cube root of a^2. Therefore, $a^{\frac{2}{3}} = \sqrt[3]{a^2}$.

Laws of Exponents III

When $a > 0$, m is an integer, and n is a positive integer,

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} \qquad (a^{\frac{1}{n}} = \sqrt[n]{a}\ )$$

1. Rewrite the following expressions.

Ex.

$$a^{\frac{2}{5}} = \sqrt[5]{a^2}\ ,\ a^{-\frac{1}{4}} = \frac{1}{\sqrt[4]{a}}$$

Ex.

$$\sqrt[5]{a^3} = a^{\frac{3}{5}}\ ,\ \frac{1}{\sqrt[3]{a}} = a^{-\frac{1}{3}}$$

(1) $a^{\frac{1}{2}} = \sqrt{\boldsymbol{a}}$

(2) $a^{\frac{1}{4}} = \sqrt[4]{\boldsymbol{a}}$

(3) $a^{\frac{2}{3}} = \sqrt[3]{\boldsymbol{a}^2}$

(4) $a^{-\frac{1}{5}} = \dfrac{\mathbf{1}}{\sqrt[5]{\boldsymbol{a}}}$

(5) $a^{-\frac{3}{4}} = \dfrac{\mathbf{1}}{\sqrt[4]{\boldsymbol{a}^3}}$

(6) $\sqrt{a} = \boldsymbol{a}^{\frac{1}{2}}$

(7) $\sqrt[5]{a} = \boldsymbol{a}^{\frac{1}{5}}$

(8) $\sqrt[4]{a^3} = \boldsymbol{a}^{\frac{3}{4}}$

(9) $\dfrac{1}{\sqrt[4]{x}} = \boldsymbol{x}^{-\frac{1}{4}}$

(10) $\dfrac{1}{\sqrt[3]{x^2}} = \boldsymbol{x}^{-\frac{2}{3}}$

K177b

2. Evaluate the following expressions.

(1) $9^{\frac{1}{2}}=(3^2)^{\frac{1}{2}}=\mathbf{3}$

(2) $4^{\frac{3}{2}}=(2^2)^{\frac{3}{2}}=2^3=\mathbf{8}$

(3) $32^{\frac{4}{5}}=(2^5)^{\frac{4}{5}}=2^4=\mathbf{16}$

(4) $8^{-\frac{2}{3}}=(2^3)^{-\frac{2}{3}}=2^{-2}=\mathbf{\frac{1}{4}}$

(5) $9^{-\frac{3}{2}}=(3^2)^{-\frac{3}{2}}=3^{-3}=\mathbf{\frac{1}{27}}$

(6) $64^{-\frac{2}{3}}=(2^6)^{-\frac{2}{3}}=2^{-4}=\mathbf{\frac{1}{16}}$

(7)* $(-64)^{\frac{2}{3}}$

$=(-2^6)^{\frac{2}{3}}=[(-2^6)^2]^{\frac{1}{3}}$

$=(2^{12})^{\frac{1}{3}}=2^4=\mathbf{16}$

(8) $\left(-\frac{8}{27}\right)^{\frac{1}{3}}$

$=\left[\left(-\frac{2}{3}\right)^3\right]^{\frac{1}{3}}=\mathbf{-\frac{2}{3}}$

(9) $(8^{\frac{3}{2}})^{-\frac{4}{9}}$

$=[(2^3)^{\frac{3}{2}}]^{-\frac{4}{9}}=2^{-2}=\mathbf{\frac{1}{4}}$

(10) $(27^{-\frac{3}{2}})^{\frac{4}{9}}$

$=[(3^3)^{-\frac{3}{2}}]^{\frac{4}{9}}=3^{-2}=\mathbf{\frac{1}{9}}$

K 178a KUMON

Exponential Functions

Evaluate the following expressions.

Ex.

$2^{\frac{1}{2}} \times 2^{\frac{3}{2}} = 2^{\frac{1}{2}+\frac{3}{2}} = 2^2 = 4$ $\qquad$ $2^{\frac{1}{2}} \div 2^{\frac{3}{2}} = 2^{\frac{1}{2}-\frac{3}{2}} = 2^{-1} = \frac{1}{2}$

(1) $2^{-3} \times 2^4 = 2^1 = \mathbf{2}$

(2) $3^{-4} \div 3^{-5} = 3^1 = \mathbf{3}$

(3) $8^{-\frac{1}{3}} \times 9^0 = 8^{-\frac{1}{3}} = \mathbf{\frac{1}{2}}$

(4) $(-3)^0 \times 64^{-\frac{1}{3}} = (2^6)^{-\frac{1}{3}} = 2^{-2} = \mathbf{\frac{1}{4}}$

(5) $\left(\frac{8}{27}\right)^{-\frac{1}{3}} \times 9^0 = \left[\left(\frac{2}{3}\right)^3\right]^{-\frac{1}{3}} = \left(\frac{2}{3}\right)^{-1} = \mathbf{\frac{3}{2}}$

(6) $4^{\frac{1}{2}} \times 4^{\frac{1}{3}} \times 4^{\frac{1}{6}} = 4^{\frac{1}{2}+\frac{1}{3}+\frac{1}{6}} = 4^1 = \mathbf{4}$

(7) $8^{\frac{1}{2}} \times 8^{-\frac{1}{3}} \times 8^{-\frac{3}{2}} = 8^{\frac{1}{2}-\frac{1}{3}-\frac{3}{2}} = 8^{-\frac{4}{3}} = (2^3)^{-\frac{4}{3}} = 2^{-4} = \mathbf{\frac{1}{16}}$

K178b

(8) $8^{\frac{3}{5}}\times16^{\frac{2}{5}}\times4^{-\frac{1}{5}}=2^{\frac{9}{5}}\cdot2^{\frac{8}{5}}\cdot2^{-\frac{2}{5}}=2^3=\mathbf{8}$

(9) $\left(\frac{9}{25}\right)^{\frac{1}{2}}\times\left(\frac{27}{125}\right)^{-\frac{1}{3}}=\frac{3}{5}\times\left(\frac{3}{5}\right)^{-1}=\mathbf{1}$

(10) $\frac{14^5\times15^6\times6^2}{10^7\times21^5}=\frac{2^5\cdot7^5\cdot5^6\cdot3^6\cdot2^2\cdot3^2}{2^7\cdot5^7\cdot3^5\cdot7^5}=\mathbf{\frac{27}{5}}$

(11) $(-8)^{\frac{1}{3}}\times81^{-\frac{1}{4}}=-2\times\frac{1}{3}=\mathbf{-\frac{2}{3}}$

(12) $\left(\frac{9}{4}\right)^{\frac{1}{2}}\times216^{\frac{2}{3}}=\frac{3}{2}\times6^2=\mathbf{54}$

(13) $(2^{\frac{1}{2}}+2^{-\frac{1}{2}})^2=(2^{\frac{1}{2}})^{\boxed{\mathbf{2}}}+2\cdot2^{\frac{1}{2}}\cdot2^{-\frac{1}{2}}+(2^{-\frac{1}{2}})^{\boxed{\mathbf{2}}}=2+2+\frac{1}{2}=\mathbf{\frac{9}{2}}$

(14) $(3^{\frac{1}{2}}-3^{-\frac{1}{2}})^2=(3^{\frac{1}{2}})^2-2\cdot3^{\frac{1}{2}}\cdot3^{-\frac{1}{2}}+(3^{-\frac{1}{2}})^2=3-2+\frac{1}{3}=\mathbf{\frac{4}{3}}$

(15) $(5^{\frac{1}{2}}+5^{-\frac{1}{2}})(5^{\frac{1}{2}}-5^{-\frac{1}{2}})=(5^{\frac{1}{2}})^2-(5^{-\frac{1}{2}})^2=5-\frac{1}{5}=\mathbf{\frac{24}{5}}$

K 179a KUMON

Exponential Functions

Simplify each expression, writing the answer in exponential form.

(1) $\frac{1}{a^4}=a^{\boxed{-4}}$

(2) $\sqrt[3]{a}=a^{\boxed{\frac{1}{3}}}$

(3) $a^{\frac{1}{3}}\cdot a^{-2}=a^{\frac{1}{3}-2}=\boldsymbol{a}^{-\frac{5}{3}}$

(4) $a\sqrt{a}=a\cdot a^{\frac{1}{2}}=\boldsymbol{a}^{\frac{3}{2}}$

(5) $\sqrt{a}\cdot\sqrt[3]{a}=a^{\frac{1}{2}}\cdot a^{\frac{1}{3}}=\boldsymbol{a}^{\frac{5}{6}}$

(6) $\sqrt[3]{a}\cdot\sqrt[4]{a^3}=a^{\frac{1}{3}}\cdot a^{\frac{3}{4}}=\boldsymbol{a}^{\frac{13}{12}}$

(7) $\sqrt[3]{\sqrt{a}}=(a^{\frac{1}{2}})^{\frac{1}{3}}=\boldsymbol{a}^{\frac{1}{6}}$

(8) $(\sqrt[3]{a^2})^6=(a^{\frac{2}{3}})^6=\boldsymbol{a}^4$

K 179b

（9） $\sqrt{a\cdot\sqrt[3]{a}}=(a\cdot a^{\frac{1}{3}})^{\frac{1}{2}}=(a^{\frac{4}{3}})^{\frac{1}{2}}=\boldsymbol{a^{\frac{2}{3}}}$

（10） $\sqrt{a\sqrt{a}}=(a\cdot a^{\frac{1}{2}})^{\frac{1}{2}}=(a^{\frac{3}{2}})^{\frac{1}{2}}=\boldsymbol{a^{\frac{3}{4}}}$

（11） $\dfrac{\sqrt{a}}{a}=a^{\frac{1}{2}}\cdot a^{-1}=\boldsymbol{a^{-\frac{1}{2}}}$

（12） $\dfrac{\sqrt[3]{a}}{a^2}=a^{\frac{1}{3}}\cdot a^{-2}=\boldsymbol{a^{-\frac{5}{3}}}$

（13） $\dfrac{a\sqrt{a}}{\sqrt[3]{a}}=a\cdot a^{\frac{1}{2}}\cdot a^{-\frac{1}{3}}=\boldsymbol{a^{\frac{7}{6}}}$

（14） $\dfrac{x^2}{y^3}=x^{\boxed{\mathbf{2}}}y^{\boxed{\mathbf{-3}}}$

（15） $\sqrt[3]{x}\cdot\sqrt[4]{y^3}=\boldsymbol{x^{\frac{1}{3}}y^{\frac{3}{4}}}$

（16） $\dfrac{\sqrt[3]{x^2}}{y\sqrt{y}}=\boldsymbol{x^{\frac{2}{3}}y^{-\frac{3}{2}}}$

K 180a KUMON

Exponential Functions

Simplify the following expressions.

(1) $\sqrt[3]{36}\times\sqrt[3]{48}=\sqrt[3]{2^6\cdot 3^3}=\mathbf{12}$

(2) $\sqrt[3]{16}\div\sqrt[3]{2^{10}}=\sqrt[3]{2^{-6}}=2^{-2}=\mathbf{\dfrac{1}{4}}$

(3) $(\sqrt[3]{\sqrt{2}})^{12}=(\sqrt[6]{2})^{12}=\mathbf{4}$

(4) $32^{-\frac{3}{5}}=(2^5)^{-\frac{3}{5}}=2^{-3}=\mathbf{\dfrac{1}{8}}$

(5) $(64^{-\frac{3}{4}})^{\frac{2}{3}}=[(2^6)^{-\frac{3}{4}}]^{\frac{2}{3}}=2^{-3}=\mathbf{\dfrac{1}{8}}$

(6) $8^{\frac{1}{2}}\times 8^{-\frac{1}{3}}\times 8^{-\frac{3}{2}}=8^{\frac{1}{2}-\frac{1}{3}-\frac{3}{2}}=8^{-\frac{4}{3}}=(2^3)^{-\frac{4}{3}}=2^{-4}=\mathbf{\dfrac{1}{16}}$

(7) $2^{\frac{1}{3}}\times 3^{\frac{1}{2}}\div 6^{\frac{1}{6}}\div\left(\dfrac{3}{2}\right)^{\frac{1}{3}}=2^{\frac{1}{3}}\cdot 3^{\frac{1}{2}}\cdot 2^{-\frac{1}{6}}\cdot 3^{-\frac{1}{6}}\cdot 3^{-\frac{1}{3}}\cdot 2^{\frac{1}{3}}=2^{\frac{1}{2}}\cdot 3^{0}=\mathbf{\sqrt{2}}$

(8) $\sqrt{a}\cdot\sqrt[3]{a^2}=a^{\frac{1}{2}}\cdot a^{\frac{2}{3}}=\boldsymbol{a^{\frac{7}{6}}}$ $\left(=\sqrt[6]{\boldsymbol{a}^7}\right)$

(9) $\sqrt{a^3\cdot\sqrt{a^2}}=(a^3\cdot a)^{\frac{1}{2}}=\boldsymbol{a^2}$

(10) $\dfrac{\sqrt{a}}{\sqrt[5]{a^4}}=a^{\frac{1}{2}}\cdot a^{-\frac{4}{5}}=\boldsymbol{a^{-\frac{3}{10}}}$ $\left[=\dfrac{\mathbf{1}}{\sqrt[10]{\boldsymbol{a}^3}}\right]$

(11) $\dfrac{a\sqrt{a}}{\sqrt[3]{a}}=a\cdot a^{\frac{1}{2}}\cdot a^{-\frac{1}{3}}=\boldsymbol{a^{\frac{7}{6}}}$ $\left(=\sqrt[6]{\boldsymbol{a}^7}\right)$

Consider this!

Let's find all the cube roots of 8 (including complex numbers).

If we let x be the cube root of 8, then $x^3=8$.

Therefore, $x^3-8=0$

$$(x-2)(x^2+2x+4)=0$$

Thus, $x-2=0$ or $x^2+2x+4=0$

From $x-2=0$, $x=2$

From $x^2+2x+4=0$, $x=-1\pm\sqrt{3}\,i$ ☜ From the ***Quadratic Formula***

Therefore, the cube roots of 8 are: 2, $-1+\sqrt{3}\,i$, $-1-\sqrt{3}\,i$

Challenge!

Find the cube roots of -1 as above:

$x^3+1=0$

$(x+1)(x^2-x+1)=0$

The cube roots of -1 are: $\boxed{\mathbf{-1}}$, $\boxed{\dfrac{\mathbf{1+\sqrt{3}\,\boldsymbol{i}}}{\mathbf{2}}}$, $\boxed{\dfrac{\mathbf{1-\sqrt{3}\,\boldsymbol{i}}}{\mathbf{2}}}$

Answers: -1, $\dfrac{1+\sqrt{3}i}{2}$, $\dfrac{1-\sqrt{3}i}{2}$

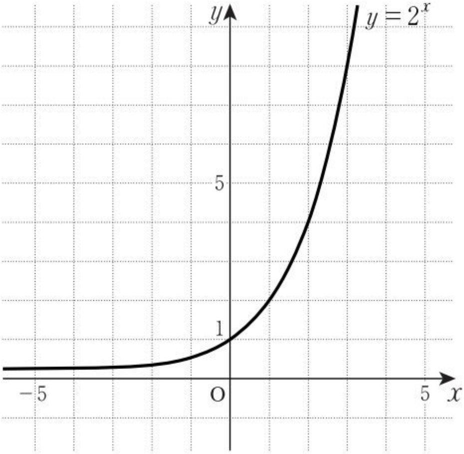
Graphs of Exponential Functions

Graph each of the following exponential functions.

Ex.

$y = 2^x$

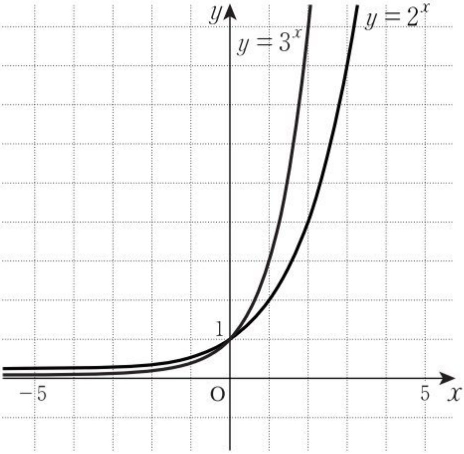
x	y
-3	$\frac{1}{8}$
-2	$\frac{1}{4}$
$-\frac{3}{2}$	$\frac{\sqrt{2}}{4}$
-1	$\frac{1}{2}$
$-\frac{1}{2}$	$\frac{\sqrt{2}}{2}$
0	1
$\frac{1}{2}$	$\sqrt{2}$
1	2
$\frac{3}{2}$	$2\sqrt{2}$
2	4
3	8



Note: The asymptote of $y = 2^x$ is the x-axis, since for all values of x, $y > 0$, and the graph approaches (but does not cross) the x-axis.

(1) $y = 3^x$

x	y
-2	$\mathbf{\frac{1}{9}}$
$-\frac{3}{2}$	$\mathbf{\frac{\sqrt{3}}{9}}$
-1	$\mathbf{\frac{1}{3}}$
$-\frac{1}{2}$	$\mathbf{\frac{\sqrt{3}}{3}}$
0	$\mathbf{1}$
$\frac{1}{2}$	$\mathbf{\sqrt{3}}$
1	$\mathbf{3}$
$\frac{3}{2}$	$\mathbf{3\sqrt{3}}$
2	$\mathbf{9}$

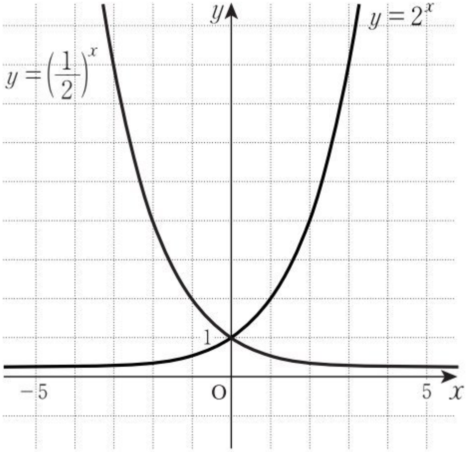


The asymptote of the graph of $y = 3^x$ is the $\boxed{\mathbf{x}}$-axis.

K181b

(2) $y=\left(\frac{1}{2}\right)^x$

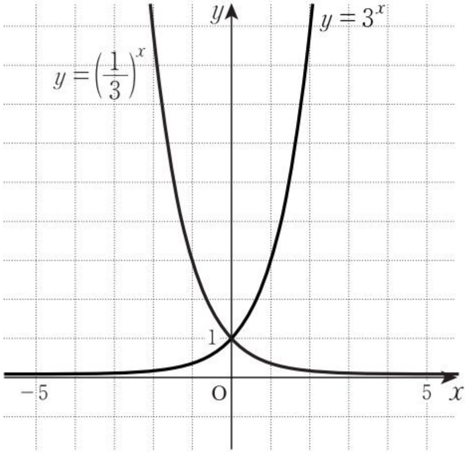
x	y
-3	$\mathbf{8}$
-2	$\mathbf{4}$
$-\frac{3}{2}$	$\mathbf{2\sqrt{2}}$
-1	$\mathbf{2}$
$-\frac{1}{2}$	$\mathbf{\sqrt{2}}$
0	$\mathbf{1}$
$\frac{1}{2}$	$\mathbf{\frac{\sqrt{2}}{2}}$
1	$\mathbf{\frac{1}{2}}$
$\frac{3}{2}$	$\mathbf{\frac{\sqrt{2}}{4}}$
2	$\mathbf{\frac{1}{4}}$
3	$\mathbf{\frac{1}{8}}$



The asymptote of the graph of $y=\left(\frac{1}{2}\right)^x$ is the $\boxed{\boldsymbol{x}}$-axis.

(3) $y=\left(\frac{1}{3}\right)^x$

x	y
-2	$\mathbf{9}$
$-\frac{3}{2}$	$\mathbf{3\sqrt{3}}$
-1	$\mathbf{3}$
$-\frac{1}{2}$	$\mathbf{\sqrt{3}}$
0	$\mathbf{1}$
$\frac{1}{2}$	$\mathbf{\frac{\sqrt{3}}{3}}$
1	$\mathbf{\frac{1}{3}}$
$\frac{3}{2}$	$\mathbf{\frac{\sqrt{3}}{9}}$
2	$\mathbf{\frac{1}{9}}$



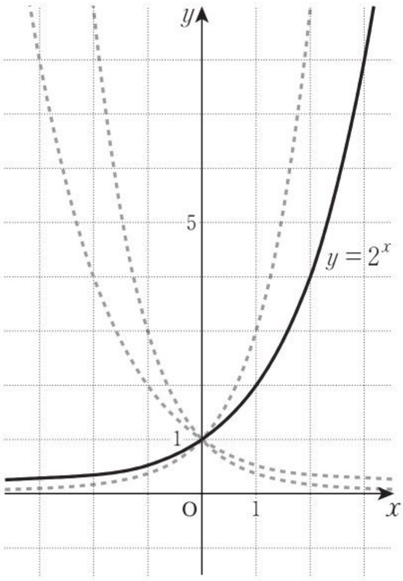
The asymptote of the graph of $y=\left(\frac{1}{3}\right)^x$ is the $\boxed{\boldsymbol{x}}$-axis.

K 182a KUMON

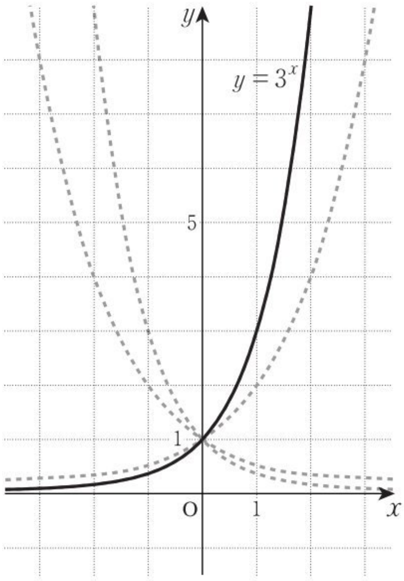
Graphs of Exponential Functions

1. Trace the graph of each exponential function.

(1) $y = 2^x$



(2) $y = 3^x$



2. Complete the following using the above graphs.

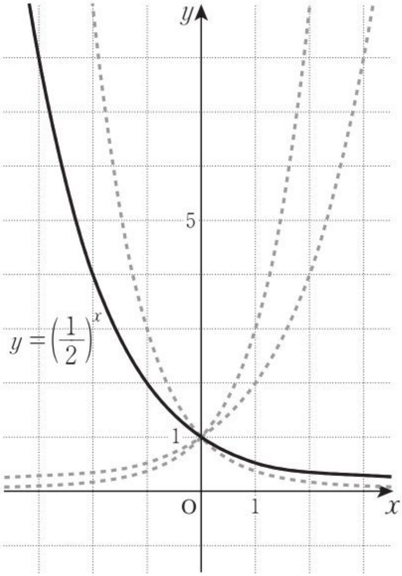
When $x < 0$, the graph of $y = 2^x$ is above $y = 3^x$.

(1) When $x = 0$, the graphs of $y = 2^x$ and $y = 3^x$ both pass through point (0, $\boxed{1}$). ☞ $2^0 = 3^0 = 1$

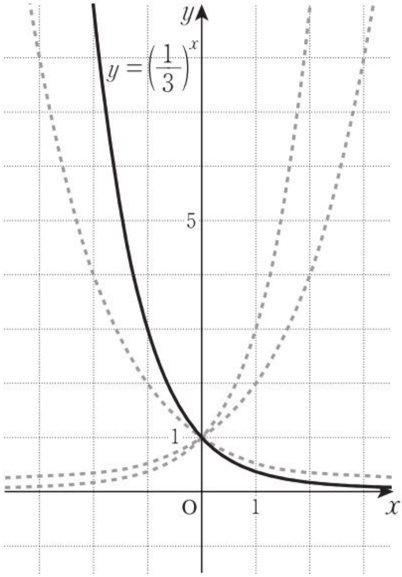
(2) When $x > 0$, the graph of $y = 2^x$ is $\boxed{\textbf{below}}$ $y = 3^x$.

3. Trace the graph of each exponential function.

(1) $y=\left(\frac{1}{2}\right)^x$



(2) $y=\left(\frac{1}{3}\right)^x$



4. Complete the following using the above graphs.

(1) When $x<0$, the graph of $y=\left(\frac{1}{2}\right)^x$ is $\boxed{\textbf{below}}$ $y=\left(\frac{1}{3}\right)^x$.

(2) When $x=0$, the graphs of $y=\left(\frac{1}{2}\right)^x$ and $y=\left(\frac{1}{3}\right)^x$ both pass through point $(0, \boxed{\textbf{1}})$. $\left(\frac{1}{2}\right)^0=\left(\frac{1}{3}\right)^0=1$

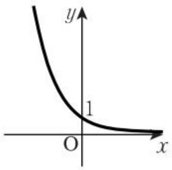
(3) When $x>0$, the graph of $y=\left(\frac{1}{2}\right)^x$ is $\boxed{\textbf{above}}$ $y=\left(\frac{1}{3}\right)^x$.

Note Summary

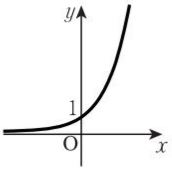
When $a>0$ and $a\neq 1$, a function of the form $y=a^x$ is called an exponential function of x, where a is the base.

The graph of $y=a^x$ passes through point $(0, 1)$, and the x-axis is the asymptote.

When $0<a<1$



When $a>1$



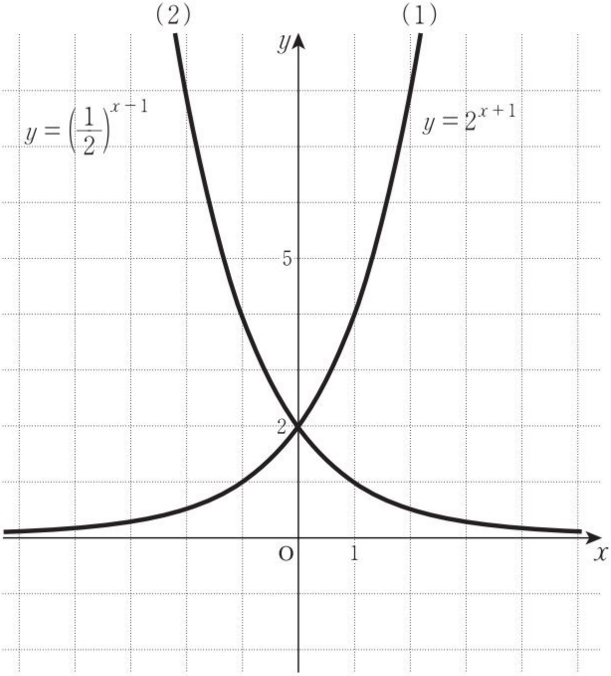
K 183a

Graphs of Exponential Functions

Graph the following exponential functions.

(1) $y = 2^{x+1}$

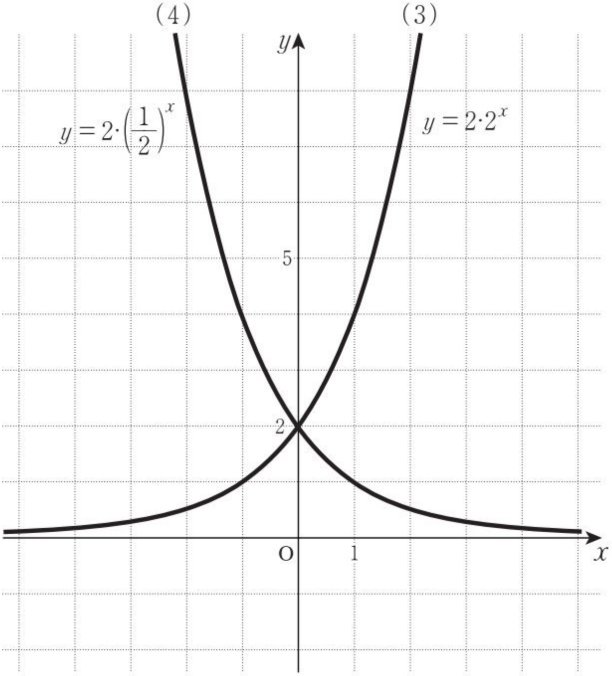
(2) $y = \left(\frac{1}{2}\right)^{x-1}$



K 183b

（3） $y = 2 \cdot 2^x$

（4） $y = 2 \cdot \left(\frac{1}{2}\right)^x$



➢ The graphs of $y = 2^{x+1}$ and $y = 2 \cdot 2^x$ are the same. ☜ See graphs （1） and （3）.

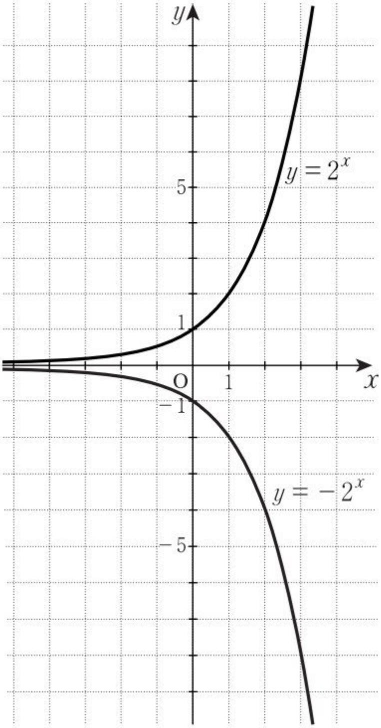
➢ The graphs of $y = \left(\frac{1}{2}\right)^{x-1}$ and $y = 2 \cdot \left(\frac{1}{2}\right)^x$ are the same. ☜ See graphs （2） and （4）.

K 184a KUMON

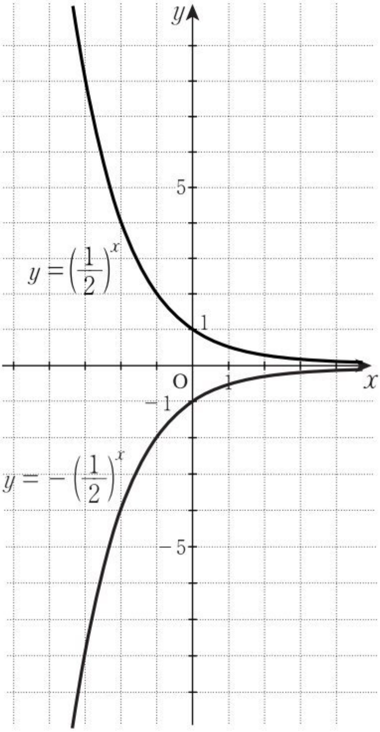
Graphs of Exponential Functions

1. Graph the following exponential functions.

(1) $y = -2^x$



(2) $y = -\left(\frac{1}{2}\right)^x$



2. Complete the following using the above graphs.

The graphs of $y = 2^x$ and $y = -2^x$ are symmetric with respect to the x-axis.

(1) The graphs of $y = \left(\frac{1}{2}\right)^x$ and $y = -\left(\frac{1}{2}\right)^x$ are symmetric with respect to the $\boxed{x}$-axis.

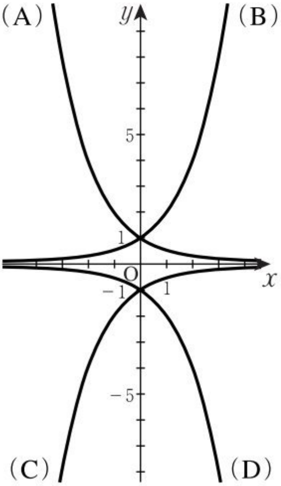
3. For each function, write the letter (A)～(D) of the corresponding graph.

(1) $y=2^x$ ··· $\boxed{\textbf{(B)}}$

(2) $y=\left(\frac{1}{2}\right)^x$ ··· $\boxed{\textbf{(A)}}$

(3) $y=-2^x$ ··· $\boxed{\textbf{(D)}}$

(4) $y=-\left(\frac{1}{2}\right)^x$ ··· $\boxed{\textbf{(C)}}$



4. Complete the following using the above graphs.

The graphs of $y=\left(\frac{1}{2}\right)^x$ and $y=2^x$ are symmetric with respect to the y-axis.

(1) The graphs of $y=-\left(\frac{1}{2}\right)^x$ and $y=-2^x$ are symmetric with respect to the $\boxed{\boldsymbol{y}}$-axis.

(2) The graphs of $y=-2^x$ and $y=2^x$ are symmetric with respect to the $\boxed{\boldsymbol{x}}$-axis.

Consider this!

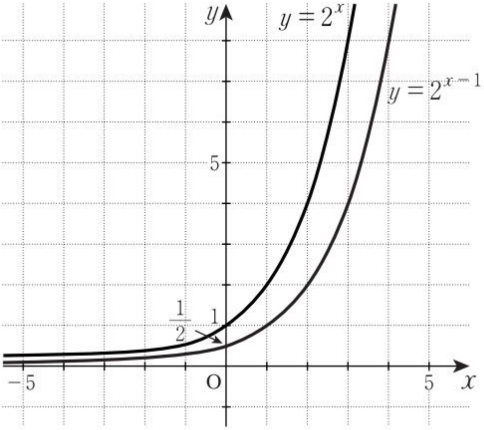
Since $y=\left(\frac{1}{2}\right)^x$ can be transformed as follows: $y=\left(\frac{1}{2}\right)^x=(2^{-1})^x=2^{-x}$, we see that it is symmetric to $y=2^x$ with respect to the $\boxed{\boldsymbol{y}}$-axis.

Answer: y

Graphs of Exponential Functions

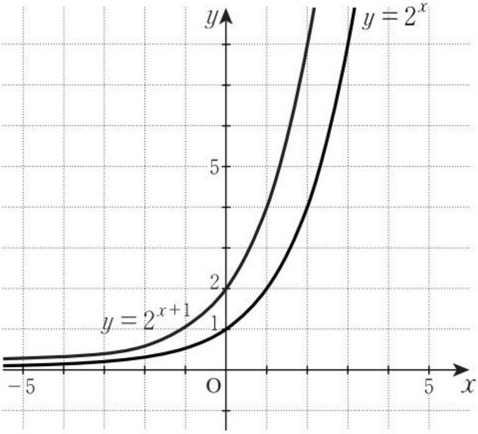
Graph the following exponential functions.

(1) $y = 2^{x-1}$



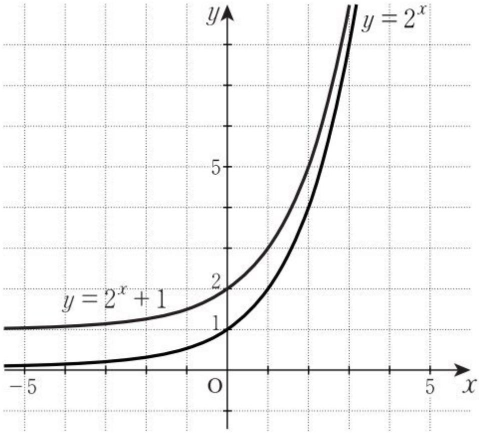
The graph of $y = 2^{x-1}$
is a translation of $y = 2^x$,
1 unit along the x-axis.

(2) $y = 2^{x+1}$



The graph of $y = 2^{x+1}$
is a translation of $y = 2^x$,
$\boxed{-1}$ unit(s) along the x-axis.

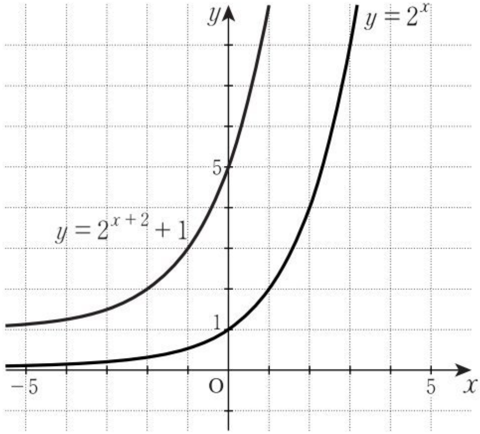
(3) $y = 2^x + 1$



The graph of $y = 2^x + 1$ is a translation of $y = 2^x$,

$\boxed{\mathbf{1}}$ unit(s) along the y-axis.

(4) $y = 2^{x+2} + 1$



The graph of $y = 2^{x+2} + 1$ is a translation of $y = 2^x$,

$\boxed{\mathbf{-2}}$ unit(s) along the x-axis,

and $\boxed{\mathbf{1}}$ unit(s) along the y-axis.

Note: The graph of $y = a^{x-p} + q$ (where $a > 0$ and $a \neq 1$) is a translation of $y = a^x$, p units along the x-axis, and q units along the y-axis.

K 186a KUMON

Graphs of Exponential Functions

1. In each question, complete the statement describing how the graph of the given exponential function has been translated from $y = 3^x$.

(1) $y = 3^{x-2}$ [Sol] $\boxed{\mathbf{2}}$ unit(s) along the x-axis

(2) $y = 3 \cdot 3^x$ [Sol] Since $y = 3 \cdot 3^x = 3^{\boxed{\boldsymbol{x+1}}}$,

$\boxed{\mathbf{-1}}$ unit(s) along the x-axis

(3) $y = \dfrac{3^x}{3}$ [Sol] Since $y = \dfrac{3^x}{3} = 3^{\boxed{\boldsymbol{x-1}}}$,

$\boxed{\mathbf{1}}$ unit(s) along the x-axis

(4) $y = 3^{x+2} + 1$ [Sol] $\boxed{\mathbf{-2}}$ unit(s) along the x-axis, and

$\boxed{\mathbf{1}}$ unit(s) along the y-axis

2. In each question, state how the graph of the given exponential function has been translated from $y = \left(\dfrac{1}{2}\right)^x$.

(1) $y = \left(\dfrac{1}{2}\right)^{x-2}$ [Sol] **2 units along the x-axis**

(2) $y = \left(\dfrac{1}{2}\right)^x - 2$ [Sol] **-2 units along the y-axis**

(3) $y = 2 \cdot \left(\dfrac{1}{2}\right)^x$ [Sol] Since $y = 2 \cdot \left(\dfrac{1}{2}\right)^x = \left(\dfrac{1}{2}\right)^{x-1}$,

1 unit along the x-axis

(4) $y = \left(\dfrac{1}{2}\right)^{x+2} + 2$ [Sol] **-2 units along the x-axis, and**
2 units along the y-axis

K 186b

3. Find the functions whose graphs have been translated from $y = 2^x$ in the following ways.

(1) 1 unit along the x-axis	[Sol] $y = 2^{\boxed{x-1}}$
(2) -2 units along the y-axis	[Sol] $\boldsymbol{y = 2^x - 2}$
(3) 2 units along the x-axis, and -1 unit along the y-axis	[Sol] $\boldsymbol{y = 2^{x-2} - 1}$
(4) -1 unit along the x-axis, and 2 units along the y-axis	[Sol] $\boldsymbol{y = 2^{x+1} + 2}$

4. Find the functions whose graphs have been translated from $y = \left(\frac{1}{3}\right)^x$ in the following ways.

(1) 1 unit along the y-axis	[Sol] $\boldsymbol{y = \left(\frac{1}{3}\right)^x + 1}$
(2) -1 unit along the x-axis	[Sol] $\boldsymbol{y = \left(\frac{1}{3}\right)^{x+1}}$
(3) 1 unit along the x-axis, and 3 units along the y-axis	[Sol] $\boldsymbol{y = \left(\frac{1}{3}\right)^{x-1} + 3}$
(4) -3 units along the x-axis, and -1 unit along the y-axis	[Sol] $\boldsymbol{y = \left(\frac{1}{3}\right)^{x+3} - 1}$

K 187a KUMON

Graphs of Exponential Functions

1. Compare the following numbers.

Ex.

$\sqrt[3]{128}$, $\sqrt[7]{16}$

[Sol] $\sqrt[3]{128} = \sqrt[3]{2^7} = 2^{\frac{7}{3}}$, $\sqrt[7]{16} = \sqrt[7]{2^4} = 2^{\frac{4}{7}}$

Rewrite both numbers so that their bases are the same, i.e. 2.

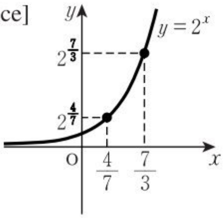
Comparing the exponents, $\frac{4}{7} < \frac{7}{3}$

When the base is greater than 1, the numbers are in the same order as the exponents.

Therefore, $2^{\frac{4}{7}} < 2^{\frac{7}{3}}$

Thus, $\sqrt[7]{16} < \sqrt[3]{128}$

[Reference]



(1) $\sqrt[7]{64}$, $\sqrt[5]{16}$

[Sol] $\sqrt[7]{64} = \sqrt[7]{2^6} = 2^{\frac{6}{7}}$

$\sqrt[5]{16} = \sqrt[5]{2^4} = 2^{\frac{4}{5}}$

Comparing the exponents,

$$\frac{4}{5} < \frac{6}{7}$$

Therefore, $2^{\frac{4}{5}} < 2^{\frac{6}{7}}$

Thus, $\boldsymbol{\sqrt[5]{16} < \sqrt[7]{64}}$

(2) $\sqrt[3]{3}$, $\sqrt[4]{9}$, $\sqrt[7]{27}$

[Sol] $\sqrt[3]{3} = 3^{\frac{1}{3}}$

$\sqrt[4]{9} = \sqrt[4]{3^2} = 3^{\frac{1}{2}}$

$\sqrt[7]{27} = \sqrt[7]{3^3} = 3^{\frac{3}{7}}$

Comparing the exponents,

$$\frac{1}{3} < \frac{3}{7} < \frac{1}{2}$$

Therefore, $3^{\frac{1}{3}} < 3^{\frac{3}{7}} < 3^{\frac{1}{2}}$

Thus, $\boldsymbol{\sqrt[3]{3} < \sqrt[7]{27} < \sqrt[4]{9}}$

Given the exponential function $y = a^x$ with $a > 1$, then as x increases, y increases.
Therefore, $p < q \Leftrightarrow a^p < a^q$. (This means $p < q$ if and only if $a^p < a^q$.)

2. Compare the following numbers.

Ex.

$\frac{1}{4}, \ \sqrt{\frac{1}{2}}$

[Sol] $\frac{1}{4}=\left(\frac{1}{2}\right)^2, \ \sqrt{\frac{1}{2}}=\left(\frac{1}{2}\right)^{\frac{1}{2}}$

Rewrite both numbers so that their bases are the same, i.e. $\frac{1}{2}$.

Comparing the exponents, $\frac{1}{2}<2$

When the base is less than 1, the numbers are in the reverse order of the exponents.

Therefore, $\left(\frac{1}{2}\right)^{\frac{1}{2}}>\left(\frac{1}{2}\right)^2$

Thus, $\frac{1}{4}<\sqrt{\frac{1}{2}}$

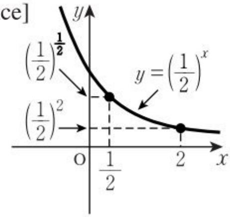
Alternative Method

Rewrite $\frac{1}{4}=2^{-2}$ and $\sqrt{\frac{1}{2}}=2^{-\frac{1}{2}}$

Comparing the exponents $-2<-\frac{1}{2}$

Therefore, $\frac{1}{4}<\sqrt{\frac{1}{2}}$

[Reference]



(1) $\sqrt[3]{\frac{1}{3}}, \ \frac{1}{9}$

[Sol] $\sqrt[3]{\frac{1}{3}}=\left(\frac{1}{3}\right)^{\frac{1}{3}}$

$\frac{1}{9}=\left(\frac{1}{3}\right)^2$

Comparing the exponents,

$\frac{1}{3}<2$

Therefore, $\left(\frac{1}{3}\right)^{\frac{1}{3}}>\left(\frac{1}{3}\right)^2$

Thus, $\mathbf{\frac{1}{9}<\sqrt[3]{\frac{1}{3}}}$

(2) $0.5^{\frac{1}{3}}, \ \sqrt{\frac{1}{2}}, \ 0.5^{-2}$

[Sol] $0.5^{\frac{1}{3}}=\left(\frac{1}{2}\right)^{\frac{1}{3}}, \ \sqrt{\frac{1}{2}}=\left(\frac{1}{2}\right)^{\frac{1}{2}}$

$0.5^{-2}=\left(\frac{1}{2}\right)^{-2}$

Comparing the exponents,

$-2<\frac{1}{3}<\frac{1}{2}$

Therefore, $\left(\frac{1}{2}\right)^{-2}>\left(\frac{1}{2}\right)^{\frac{1}{3}}>\left(\frac{1}{2}\right)^{\frac{1}{2}}$

Thus, $\mathbf{\sqrt{\frac{1}{2}}<0.5^{\frac{1}{3}}<0.5^{-2}}$

Given the exponential function $y=a^x$, with $0<a<1$, then as x increases, y decreases.
Therefore, $p<q \Leftrightarrow a^p>a^q$. (This means $p<q$ if and only if $a^p>a^q$.)

K 188a KUMON

Graphs of Exponential Functions

1. Compare the following numbers.

Ex.

$\sqrt[3]{49}$, $\sqrt[6]{16}$

[Sol] $\sqrt[3]{49} = \sqrt[3]{7^2} = 7^{\frac{2}{3}}$, $\sqrt[6]{16} = \sqrt[6]{2^4} = 2^{\frac{4}{6}} = 2^{\frac{2}{3}}$

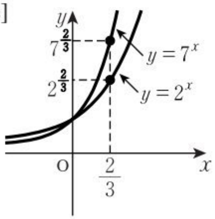
Comparing the bases, $2 < 7$

Therefore, $2^{\frac{2}{3}} < 7^{\frac{2}{3}}$

Thus, $\sqrt[6]{16} < \sqrt[3]{49}$

Since the exponents are **equal and positive**, the numbers are in the same order as the bases.

[Reference]



(1) $\sqrt[8]{25}$, $\sqrt[12]{27}$

[Sol] $\sqrt[8]{25} = \sqrt[8]{5^2} = 5^{\frac{2}{8}} = 5^{\frac{1}{4}}$

$\sqrt[12]{27} = \sqrt[12]{3^3} = 3^{\frac{3}{12}} = 3^{\frac{1}{4}}$

Comparing the bases, $3 < 5$

Therefore, $3^{\frac{1}{4}} < 5^{\frac{1}{4}}$

Thus, $\mathbf{\sqrt[12]{27} < \sqrt[8]{25}}$

(2) $\sqrt[6]{36}$, $\sqrt[12]{81}$, $\sqrt[9]{125}$

[Sol] $\sqrt[6]{36} = 6^{\frac{2}{6}} = 6^{\frac{1}{3}}$

$\sqrt[12]{81} = \sqrt[12]{3^4} = 3^{\frac{4}{12}} = 3^{\frac{1}{3}}$

$\sqrt[9]{125} = \sqrt[9]{5^3} = 5^{\frac{3}{9}} = 5^{\frac{1}{3}}$

Comparing the bases, $3 < 5 < 6$

Therefore, $3^{\frac{1}{3}} < 5^{\frac{1}{3}} < 6^{\frac{1}{3}}$

Thus, $\mathbf{\sqrt[12]{81} < \sqrt[9]{125} < \sqrt[6]{36}}$

Given the exponential functions $y = a^x$ and $y = b^x$, when $a > 0$, $b > 0$ and $p > 0$, $a < b \Leftrightarrow a^p < b^p$. (This means $a < b$ if and only if $a^p < b^p$.)

2. Compare the following numbers.

Ex.

$\sqrt{2}$, $\sqrt[3]{3}$, $\sqrt[6]{6}$

[Sol] $(\sqrt{2})^6 = 2^3 = 8$, $(\sqrt[3]{3})^6 = 3^2 = 9$, $(\sqrt[6]{6})^6 = 6$

Therefore, $(\sqrt[6]{6})^6 < (\sqrt{2})^6 < (\sqrt[3]{3})^6$

Thus, $\sqrt[6]{6} < \sqrt{2} < \sqrt[3]{3}$

Since the exponents are **equal and positive**, the numbers are in the same order as the bases.

(1) $\sqrt{3}$, $\sqrt[3]{5}$, $\sqrt[6]{7}$

[Sol] $(\sqrt{3})^6 = 3^3 = 27$, $(\sqrt[3]{5})^6 = 5^2 = 25$, $(\sqrt[6]{7})^6 = 7$

Therefore, $(\sqrt[6]{7})^6 < (\sqrt[3]{5})^6 < (\sqrt{3})^6$

Thus, $\boldsymbol{\sqrt[6]{7} < \sqrt[3]{5} < \sqrt{3}}$

Alternative Solution

Rewrite fractional exponents using the LCM.

$$\begin{cases} \sqrt{3} = 3^{\frac{1}{2}} = 3^{\frac{3}{6}} = (3^3)^{\frac{1}{6}} = 27^{\frac{1}{6}} \\ \sqrt[3]{5} = 5^{\frac{1}{3}} = 5^{\frac{2}{6}} = (5^2)^{\frac{1}{6}} = 25^{\frac{1}{6}} \\ \sqrt[6]{7} = 7^{\frac{1}{6}} \end{cases}$$

Thus, $\boldsymbol{\sqrt[6]{7} < \sqrt[3]{5} < \sqrt{3}}$

(2) $\sqrt{2}$, $\sqrt[3]{3}$, $\sqrt[4]{5}$

[Sol] $(\sqrt{2})^{12} = 2^6 = 64$, $(\sqrt[3]{3})^{12} = 3^4 = 81$, $(\sqrt[4]{5})^{12} = 5^3 = 125$

Therefore, $(\sqrt{2})^{12} < (\sqrt[3]{3})^{12} < (\sqrt[4]{5})^{12}$

Thus, $\boldsymbol{\sqrt{2} < \sqrt[3]{3} < \sqrt[4]{5}}$

Alternative Solution

$$\begin{cases} \sqrt{2} = 2^{\frac{1}{2}} = 2^{\frac{6}{12}} = 64^{\frac{1}{12}} \\ \sqrt[3]{3} = 3^{\frac{1}{3}} = 3^{\frac{4}{12}} = 81^{\frac{1}{12}} \\ \sqrt[4]{5} = 5^{\frac{1}{4}} = 5^{\frac{3}{12}} = 125^{\frac{1}{12}} \end{cases}$$

Thus, $\boldsymbol{\sqrt{2} < \sqrt[3]{3} < \sqrt[4]{5}}$

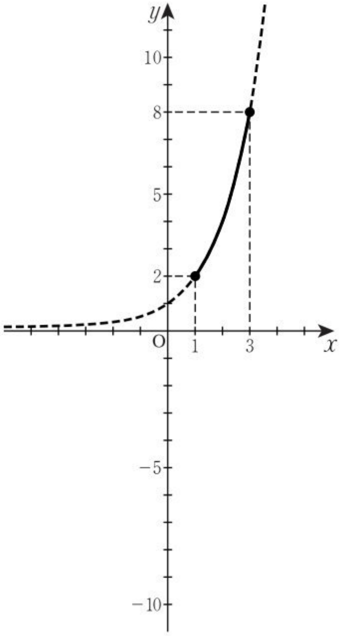
K 189a KUMON

Graphs of Exponential Functions

Graph each exponential function and find the maximum and minimum values.

Ex.

$f(x) = 2^x \quad (1 \leqq x \leqq 3)$

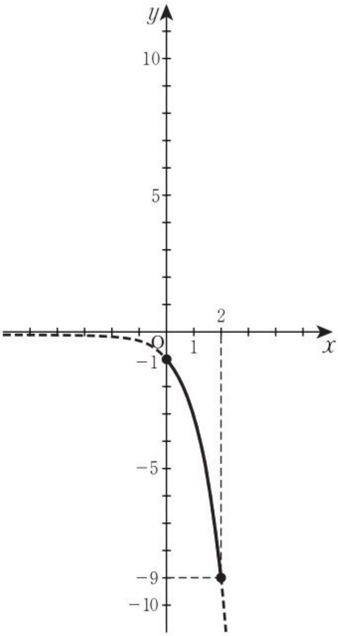


From the graph:

Maximum value: $f(3) = 2^3 = 8$

Minimum value: $f(1) = 2$

(1) $f(x) = -3^x \quad (0 \leqq x \leqq 2)$



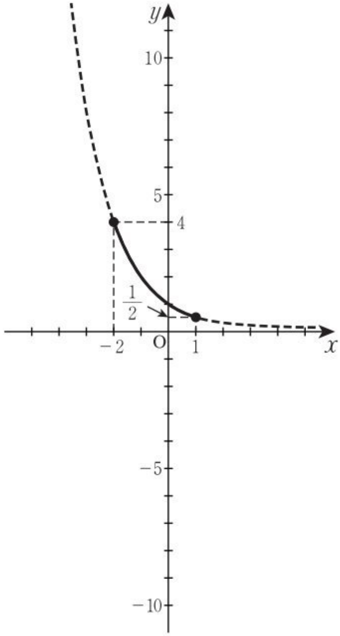
From the graph:

Maximum value: $\boldsymbol{f(0)} = -3^0 \boldsymbol{= -1}$

Minimum value: $\boldsymbol{f(2)} = -3^2 \boldsymbol{= -9}$

K 189b

(2) $f(x)=\left(\frac{1}{2}\right)^{x}$ $(-2 \leqq x \leqq 1)$



From the graph:

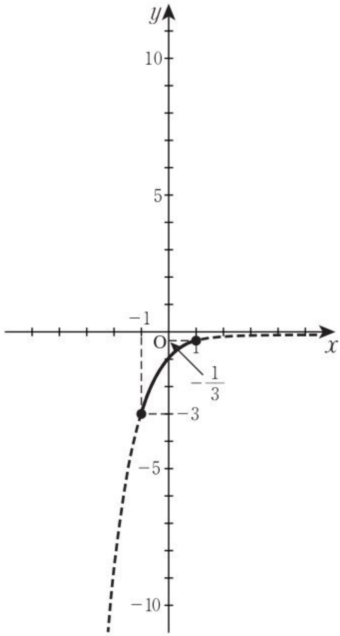
Maximum value:

$$f(-2)=\left(\frac{1}{2}\right)^{-2}=4$$

Minimum value:

$$f(1)=\frac{1}{2}$$

(3) $f(x)=-\left(\frac{1}{3}\right)^{x}$ $(-1 \leqq x \leqq 1)$



From the graph:

Maximum value:

$$f(1)=-\frac{1}{3}$$

Minimum value:

$$f(-1)=-\left(\frac{1}{3}\right)^{-1}=-3$$

K 190a KUMON

Graphs of Exponential Functions

1. For each function, write the letter (A)～(F) of the corresponding sketch.

(1) $y=2^x$ … $\boxed{\textbf{(E)}}$

(4) $y=-2^x$ … $\boxed{\textbf{(A)}}$

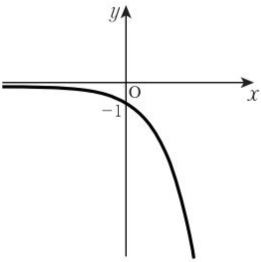
(2) $y=2\cdot 2^x$ … $\boxed{\textbf{(C)}}$

(5) $y=\left(\frac{1}{2}\right)^x$ … $\boxed{\textbf{(D)}}$

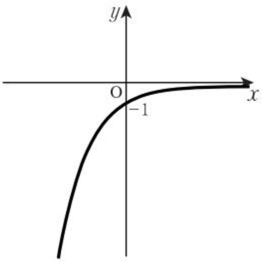
(3) $y=\frac{2^x}{2}$ … $\boxed{\textbf{(F)}}$

(6) $y=-\left(\frac{1}{2}\right)^x$ … $\boxed{\textbf{(B)}}$

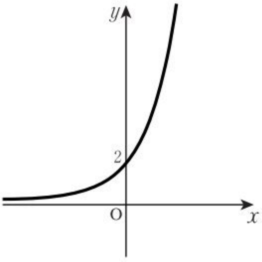
(A)



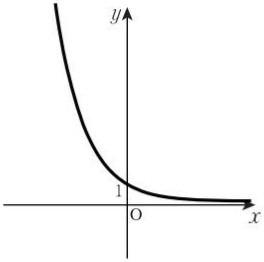
(B)



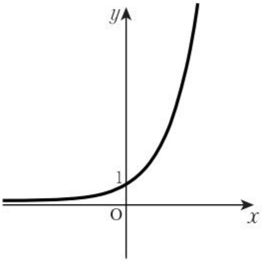
(C)



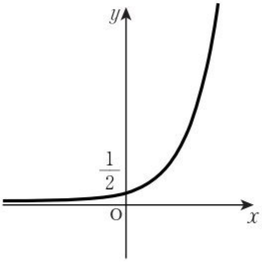
(D)



(E)



(F)



2. State how each function has been translated from $y = 3^x$.

(1) $y = 3 \cdot 3^x$

[Sol] $y = 3^{x+1}$

Therefore, the graph is translated **-1 unit along the x-axis**.

(2) $y = \dfrac{3^x}{3}$

[Sol] $y = 3^{x-1}$

Therefore, the graph is translated **1 unit along the x-axis**.

3. Compare the following numbers.

$$\left(\frac{1}{2}\right)^{-2}, \quad \sqrt[3]{4}, \quad 0.25, \quad \sqrt{\frac{1}{2}}$$

[Sol] $\left(\dfrac{1}{2}\right)^{-2} = 2^2$, $\sqrt[3]{4} = 2^{\frac{2}{3}}$, $0.25 = 2^{-2}$, $\sqrt{\dfrac{1}{2}} = 2^{-\frac{1}{2}}$

Comparing the exponents,

$$-2 < -\frac{1}{2} < \frac{2}{3} < 2$$

Therefore,

$$2^{-2} < 2^{-\frac{1}{2}} < 2^{\frac{2}{3}} < 2^2$$

Thus,

$$\mathbf{0.25 < \sqrt{\frac{1}{2}} < \sqrt[3]{4} < \left(\frac{1}{2}\right)^{-2}}$$

Alternative Solution

$$\sqrt[3]{4} = 4^{\frac{1}{3}} = 2^{\frac{2}{3}} = \left(\frac{1}{2}\right)^{-\frac{2}{3}},$$

$$0.25 = \frac{1}{4} = \left(\frac{1}{2}\right)^2, \ \sqrt{\frac{1}{2}} = \left(\frac{1}{2}\right)^{\frac{1}{2}}$$

Comparing the exponents,

$$-2 < -\frac{2}{3} < \frac{1}{2} < 2$$

Therefore,

$$\left(\frac{1}{2}\right)^{-2} > \left(\frac{1}{2}\right)^{-\frac{2}{3}} > \left(\frac{1}{2}\right)^{\frac{1}{2}} > \left(\frac{1}{2}\right)^2$$

Thus,

$$\mathbf{\left(\frac{1}{2}\right)^{-2} > \sqrt[3]{4} > \sqrt{\frac{1}{2}} > 0.25}$$

Let's try this!

How thick, in mm, will a newspaper page be if we fold it 10 times?

(The thickness of a newspaper page is 0.071mm, and $2^{10} = 1024$.)

Answer: 72.704 mm

K 191a KUMON

Exponential Equations and Inequalities

Solve the following equations, and mark each solution on the graph.

Ex.

$2^x = 16$

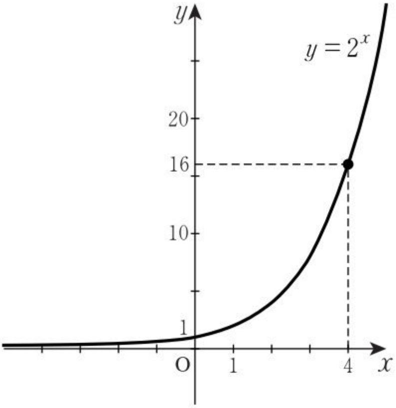
[Sol] Let $y = 2^x$

When $y = 16$,

$16 = 2^x$

$2^4 = 2^x$

Therefore, $x = 4$



(1) $3^x = 27$

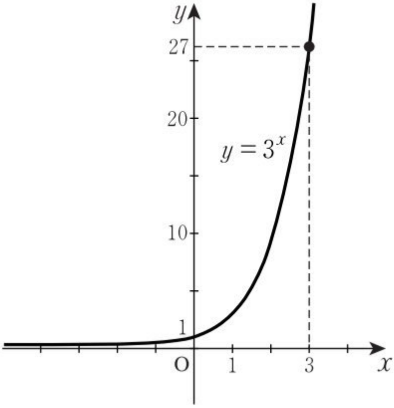
[Sol] Let $y = 3^x$

When $y = 27$,

$27 = 3^x$

$3^3 = 3^x$

Therefore, $\boldsymbol{x = 3}$



K191b

(2) $\left(\frac{1}{2}\right)^x = \frac{1}{16}$

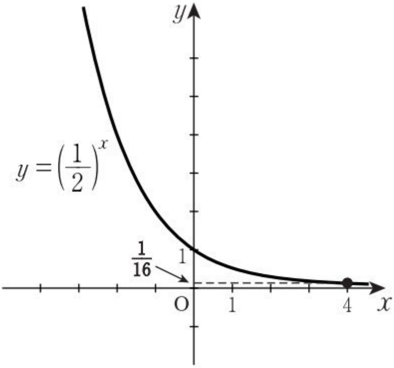
[Sol] Let $y = \left(\frac{1}{2}\right)^x$

When $y = \frac{1}{16}$,

$$\frac{1}{16} = \left(\frac{1}{2}\right)^x$$

$$\left(\frac{1}{2}\right)^4 = \left(\frac{1}{2}\right)^x$$

Therefore, $\boldsymbol{x = 4}$



Alternative Solution

$$2^{-4} = 2^{-x}$$
$$-x = -4$$
$$x = 4$$

(3) $2^x = \frac{1}{8}$

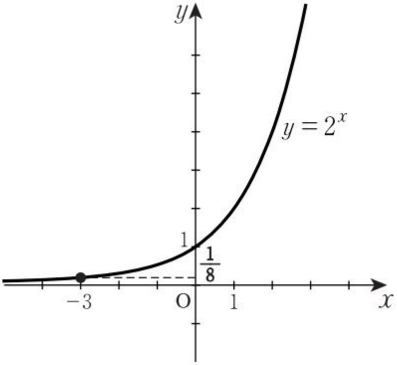
[Sol] Let $y = 2^x$

When $y = \frac{1}{8}$,

$$\frac{1}{8} = 2^x$$

$$2^{-3} = 2^x$$

Therefore, $\boldsymbol{x = -3}$



K 192a KUMON

Exponential Equations and Inequalities

1. Solve the following equations.

Ex.

$9^x = 27$

[Sol] $3^{2x} = 3^3$ ☜ Rewrite with base 3.

$2x = 3$

$x = \frac{3}{2}$

Ex.

$\left(\frac{1}{4}\right)^x = 8$

[Sol] $2^{-2x} = 2^3$ ☜ Rewrite with base 2.

$-2x = 3$

$x = -\frac{3}{2}$

(1) $4^x = 8$

[Sol] $2^{2x} = 2^3$

$2x = 3$

$\boldsymbol{x = \frac{3}{2}}$

(2) $4^x = \sqrt{8}$

[Sol] $2^{2x} = 2^{\frac{3}{2}}$

$2x = \frac{3}{2}$

$\boldsymbol{x = \frac{3}{4}}$

(3) $\left(\frac{1}{27}\right)^x = 9$

[Sol] $3^{-3x} = 3^2$

$-3x = 2$

$\boldsymbol{x = -\frac{2}{3}}$

(4) $\left(\frac{1}{9}\right)^x = \sqrt{3}$

[Sol] $3^{-2x} = 3^{\frac{1}{2}}$

$-2x = \frac{1}{2}$

$\boldsymbol{x = -\frac{1}{4}}$

In the term a^x, a is called the ***base***.

2. Solve the following equations.

Ex.

$8^x = 2^{x+1}$

[Sol] $2^{3x} = 2^{x+1}$

$3x = x+1$

$2x = 1$

$x = \frac{1}{2}$

(1) $9^x = 3^{x-1}$

[Sol] $3^{2x} = 3^{x-1}$

$2x = x-1$

$\boldsymbol{x = -1}$

(2) $2^{2x} = 16^{x-1}$

[Sol] $2^{2x} = 2^{4(x-1)}$

$2x = 4(x-1)$

$2x = 4$

$\boldsymbol{x = 2}$

(3) $\left(\frac{1}{3}\right)^{x-2} = 9^x$

[Sol] $3^{-(x-2)} = 3^{2x}$

$-(x-2) = 2x$

$3x = 2$

$\boldsymbol{x = \frac{2}{3}}$

(4) $\left(\frac{1}{4}\right)^x = 8^{1-x}$

[Sol] $2^{-2x} = 2^{3(1-x)}$

$-2x = 3(1-x)$

$\boldsymbol{x = 3}$

(5) $(0.1)^x = 10^{x+1}$

[Sol] $10^{-x} = 10^{x+1}$

$-x = x+1$

$2x = -1$

$\boldsymbol{x = -\frac{1}{2}}$

K 193a KUMON

Exponential Equations and Inequalities

Solve the following equations.

Ex.

$9^x = 3\sqrt{3}$

[Sol] $3^{2x} = 3^{\frac{3}{2}}$ ☜ Rewrite with base 3.

$$2x = \frac{3}{2}$$

$$x = \frac{3}{4}$$

(2) $\left(\frac{1}{2}\right)^{x-1} = 2\sqrt[3]{2}$

[Sol] $2^{-(x-1)} = 2^{\frac{4}{3}}$

$$-(x-1) = \frac{4}{3}$$

$$\boldsymbol{x = -\frac{1}{3}}$$

(1) $8^x = \frac{\sqrt{2}}{4}$

[Sol] $2^{3x} = 2^{-\frac{3}{2}}$

$$3x = -\frac{3}{2}$$

$$\boldsymbol{x = -\frac{1}{2}}$$

(3) $\left(\frac{1}{4}\right)^x = \frac{\sqrt[4]{2}}{2}$

[Sol] $2^{-2x} = 2^{-\frac{3}{4}}$

$$-2x = -\frac{3}{4}$$

$$\boldsymbol{x = \frac{3}{8}}$$

K 193b

(4) $8^{x+1}=4\sqrt[3]{2}$

[Sol] $2^{3(x+1)}=2^{\frac{7}{3}}$

$$3(x+1)=\frac{7}{3}$$

$$3x=-\frac{2}{3}$$

$$\boldsymbol{x=-\frac{2}{9}}$$

(5) $2^{x-1}=0.25$

[Sol] $2^{x-1}=\frac{1}{4}$

$$2^{x-1}=2^{-2}$$

$$x-1=-2$$

$$\boldsymbol{x=-1}$$

(6) $10^{3x-1}=0.01$

[Sol] $10^{3x-1}=10^{-2}$

$$3x-1=-2$$

$$3x=-1$$

$$\boldsymbol{x=-\frac{1}{3}}$$

(7) $\left(\frac{2}{3}\right)^{x}=\left(\frac{3}{2}\right)^{x-1}$ *Hint*

[Sol] $\left(\frac{2}{3}\right)^{x}=\left(\frac{2}{3}\right)^{-x+1}$

$$x=-x+1$$

$$2x=1$$

$$\boldsymbol{x=\frac{1}{2}}$$

Hint

Rewrite so that both sides have the same base, either $\frac{2}{3}$ or $\frac{3}{2}$.

K 194a KUMON

Exponential Equations and Inequalities

Solve the following equations.

Ex.

$$2^{2x+1}+3\cdot 2^x-2=0$$

[Sol]

$$2(2^x)^2+3\cdot 2^x-2=0$$

Let $2^x=X$ (where $X>0$)

$$2X^2+3X-2=0$$

$$(2X-1)(X+2)=0$$

$$X=\frac{1}{2},\ -2$$

Since $X>0$, $X=-2$ is an extraneous solution.

Therefore,

$$X=\frac{1}{2}$$

Since $X=2^x$.

$$2^x=\frac{1}{2}$$

$$x=-1$$

Note: $X=-2$ corresponds to $2^x=-2$, which has no solution.

(1) $4^x-2^{x+1}-8=0$

[Sol] $2^{2x}-2\cdot 2^x-8=0$

$$(2^x)^2-2\cdot 2^x-8=0$$

Let $2^x=X$ (where $X>0$)

$$X^2-2X-8=0$$

$$(X-4)(X+2)=0$$

$$X=4,\ -2$$

Since $X>0$, $X=-2$ is an extraneous solution.

Therefore,

$$X=4$$

$$2^x=4$$

$$\boldsymbol{x=2}$$

K 194b

(2) $4^x - 3 \cdot 2^{x+1} + 8 = 0$

[Sol] $2^{2x} - 6 \cdot 2^x + 8 = 0$

$(2^x)^2 - 6 \cdot 2^x + 8 = 0$

Let $2^x = X$ (where $X > 0$)

$X^2 - 6X + 8 = 0$

$(X-4)(X-2) = 0$

$X = 4,\ 2$

Since $X > 0$, both solutions apply.

Therefore,

When $X = 4$

$2^x = 4$

$x = 2$

When $X = 2$

$2^x = 2$

$x = 1$

Therefore, $\boldsymbol{x = 2,\ 1}$

(3) $3^{2x+1} + 2 \cdot 3^x = 1$

[Sol] $3 \cdot 3^{2x} + 2 \cdot 3^x - 1 = 0$

$3(3^x)^2 + 2 \cdot 3^x - 1 = 0$

Let $3^x = X$ (where $X > 0$)

$3X^2 + 2X - 1 = 0$

$(3X-1)(X+1) = 0$

$X = \frac{1}{3},\ -1$

Since $X > 0$, $X = -1$ is an extraneous solution.

Therefore,

$X = \frac{1}{3}$

$3^x = \frac{1}{3}$

$\boldsymbol{x = -1}$

K 195a KUMON

Exponential Equations and Inequalities

1. Solve the following equations.

(1) $9^x - 2 \cdot 3^{x+1} = 27$

[Sol] $3^{2x} - 6 \cdot 3^x - 27 = 0$

$(3^x)^2 - 6 \cdot 3^x - 27 = 0$

Let $3^x = X$ (where $X > 0$)

$X^2 - 6X - 27 = 0$

$(X-9)(X+3) = 0$

$X = 9, \ -3$

Since $X > 0$, $X = -3$ is an extraneous solution.

Therefore,

$X = 9$

$3^x = 9$

$\boldsymbol{x = 2}$

(2) $9^{x^2} - 3^{x+1} = 0$

[Sol] $3^{2x^2} = 3^{x+1}$

$2x^2 = x + 1$

$2x^2 - x - 1 = 0$

$(2x+1)(x-1) = 0$

$\boldsymbol{x = -\frac{1}{2}, \ 1}$

2. Solve the following equations.

Ex.

$$\begin{cases} 2^x + 2^y = 6 \\ 2^{x+y} = 8 \end{cases}$$

[Sol] $\begin{cases} 2^x + 2^y = 6 \\ 2^x \cdot 2^y = 8 \end{cases}$

Let $2^x = X$ (where $X > 0$)

$2^y = Y$ (where $Y > 0$)

Substituting,

$$\begin{cases} X + Y = 6 & \cdots ① \\ XY = 8 & \cdots ② \end{cases}$$

From ① and ②,

$$\begin{cases} X = 2 \\ Y = 4 \end{cases} \quad \begin{cases} X = 4 \\ Y = 2 \end{cases}$$

When $\begin{cases} X = 2 \\ Y = 4 \end{cases}$

$\begin{cases} 2^x = 2 \\ 2^y = 4 \end{cases}$ therefore, $\begin{cases} x = 1 \\ y = 2 \end{cases}$

When $\begin{cases} X = 4 \\ Y = 2 \end{cases}$

$\begin{cases} 2^x = 4 \\ 2^y = 2 \end{cases}$ therefore, $\begin{cases} x = 2 \\ y = 1 \end{cases}$

Ans. $\begin{cases} x = 1 \\ y = 2 \end{cases} \quad \begin{cases} x = 2 \\ y = 1 \end{cases}$

(1) $\begin{cases} 3^x - 3^y = 6 \\ 3^{x+y} = 27 \end{cases}$

[Sol] $\begin{cases} 3^x - 3^y = 6 \\ 3^x \cdot 3^y = 27 \end{cases}$

Let $3^x = X$ (where $X > 0$)

$3^y = Y$ (where $Y > 0$)

Substituting,

$$\begin{cases} X - Y = 6 & \cdots ① \\ XY = 27 & \cdots ② \end{cases}$$

From ① and ②,

$$\begin{cases} X = -3 \\ Y = -9 \end{cases} \quad \begin{cases} X = 9 \\ Y = 3 \end{cases}$$

Since $X > 0$, $Y > 0$

$\begin{cases} X = -3 \\ Y = -9 \end{cases}$ is an extraneous solution.

When $\begin{cases} X = 9 \\ Y = 3 \end{cases}$

$\begin{cases} 3^x = 9 \\ 3^y = 3 \end{cases}$ therefore, $\begin{cases} \boldsymbol{x = 2} \\ \boldsymbol{y = 1} \end{cases}$

K 196a KUMON

Exponential Equations and Inequalities

1. Solve the following inequalities.

Ex.

$9^x < 27$

[Sol] $3^{2x} < 3^3$

$2x < 3$

$x < \frac{3}{2}$

Since the bases are **equal and greater than 1**, the exponents follow the same order as the original terms.

(1) $3^{2x-1} < 27$

[Sol] $3^{2x-1} < 3^3$

$2x - 1 < 3$

$\boldsymbol{x < 2}$

(2) $9^x > 3\sqrt{3}$

[Sol] $3^{2x} > 3^{\frac{3}{2}}$

$2x > \frac{3}{2}$

$\boldsymbol{x > \frac{3}{4}}$

(3) $4^{x+2} \geqslant 8^x$

[Sol] $2^{2(x+2)} \geqslant 2^{3x}$

$2(x+2) \geqslant 3x$

$\boldsymbol{x \leqslant 4}$

(4) $3^{x+3} \leqslant \left(\frac{1}{3}\right)^x$

[Sol] $3^{x+3} \leqslant 3^{-x}$

$x + 3 \leqslant -x$

$2x \leqslant -3$

$\boldsymbol{x \leqslant -\frac{3}{2}}$

2. Solve the following inequalities.

Ex.

$$\left(\frac{1}{4}\right)^{x} < \frac{1}{8}$$

[Sol] $\left(\frac{1}{2}\right)^{2x} < \left(\frac{1}{2}\right)^{3}$

$$2x > 3$$

$$x > \frac{3}{2}$$

Since the bases are **equal and less than 1**, the exponents are in the reverse order of the original terms.

(1) $\left(\frac{1}{9}\right)^{2x} < \frac{1}{27}$

[Sol] $\left(\frac{1}{3}\right)^{4x} < \left(\frac{1}{3}\right)^{3}$

$$4x > 3$$

$$\boldsymbol{x > \frac{3}{4}}$$

(2) $\left(\frac{2}{3}\right)^{x} \geqslant \left(\frac{3}{2}\right)^{x-1}$

[Sol] $\left(\frac{2}{3}\right)^{x} \geqslant \left(\frac{2}{3}\right)^{-x+1}$

$$x \leqslant -x+1$$

$$2x \leqslant 1$$

$$\boldsymbol{x \leqslant \frac{1}{2}}$$

(3) $\left(\frac{1}{2}\right)^{x+1} > 0.125$

[Sol] $\left(\frac{1}{2}\right)^{x+1} > \left(\frac{1}{2}\right)^{3}$

$$x+1 < 3$$

$$\boldsymbol{x < 2}$$

(4) $\left(\frac{1}{3}\right)^{1-x} \leqslant \sqrt[3]{9}$

[Sol] $\left(\frac{1}{3}\right)^{1-x} \leqslant \left(\frac{1}{3}\right)^{-\frac{2}{3}}$

$$1-x \geqslant -\frac{2}{3}$$

$$\boldsymbol{x \leqslant \frac{5}{3}}$$

Alternative Solution

$$3^{-1+x} \leqslant 3^{\frac{2}{3}}$$

$$-1+x \leqslant \frac{2}{3}$$

$$\boldsymbol{x \leqslant \frac{5}{3}}$$

K 197a KUMON

Exponential Equations and Inequalities

Solve the following inequalities.

Ex.

$2^{2x+1}+3\cdot 2^x-2<0$

[Sol]

$2(2^x)^2+3\cdot 2^x-2<0$

Let $2^x=X$ (where $X>0$)

$2X^2+3X-2<0$

$(X+2)(2X-1)<0$

$-2<X<\frac{1}{2}$

Since $X>0$,

$0<X<\frac{1}{2}$

$0<2^x<\frac{1}{2}$

$x<-1$

(1) $4^x-2^{x+1}-8>0$

[Sol] $2^{2x}-2\cdot 2^x-8>0$

$(2^x)^2-2\cdot 2^x-8>0$

Let $2^x=X$ (where $X>0$)

$X^2-2X-8>0$

$(X+2)(X-4)>0$

$X<-2,\ X>4$

Since $X>0$,

$X>4$

$2^x>4$

$\boldsymbol{x>2}$

K 197b

(2) $4^x - 3 \cdot 2^{x+1} + 8 < 0$

[Sol] $2^{2x} - 6 \cdot 2^x + 8 < 0$

$(2^x)^2 - 6 \cdot 2^x + 8 < 0$

Let $2^x = X$ (where $X > 0$)

$X^2 - 6X + 8 < 0$

$(X-2)(X-4) < 0$

$2 < X < 4$

$2 < 2^x < 4$

$\boldsymbol{1 < x < 2}$

(3) $3^{2x+1} + 2 \cdot 3^x \geqq 1$

[Sol] $3 \cdot 3^{2x} + 2 \cdot 3^x - 1 \geqq 0$

$3(3^x)^2 + 2 \cdot 3^x - 1 \geqq 0$

Let $3^x = X$ (where $X > 0$)

$3X^2 + 2X - 1 \geqq 0$

$(X+1)(3X-1) \geqq 0$

$X \leqq -1, \ X \geqq \frac{1}{3}$

Since $X > 0$,

$X \geqq \frac{1}{3}$

$3^x \geqq \frac{1}{3}$

$\boldsymbol{x \geqq -1}$

K 198a KUMON

Exponential Equations and Inequalities

1. For each function, find the maximum and minimum values within the given domain.

Ex.

$y = 3 \cdot 2^x \quad (-1 \leqslant x \leqslant 1)$

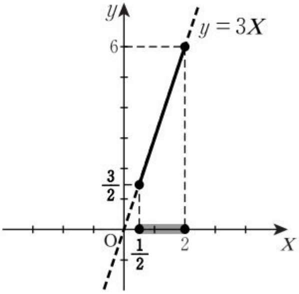
[Sol] Let $2^x = X$, and thus, $y = 3X$

Determining the range of values of X:

When $x = -1$, $X = \frac{1}{2}$

When $x = 1$, $X = 2$

Therefore, $\frac{1}{2} \leqslant X \leqslant 2$



(Draw the graph using X on the horizontal axis.)

From the graph of $y = 3X$:

The maximum value, when $X = 2$, i.e. when $x = 1$, is 6. ☜ Write the condition for x.

The minimum value, when $X = \frac{1}{2}$, i.e. when $x = -1$, is $\frac{3}{2}$.

(1) $y = 2^x + 1 \quad (-1 \leqslant x \leqslant 1)$

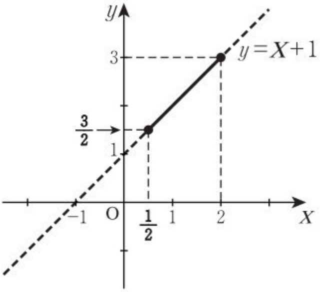
[Sol] Let $2^x = X$, and thus, $y = X + 1$

Determining the range of values of X:

When $x = -1$, $X = \frac{1}{2}$

When $x = 1$, $X = 2$

Therefore, $\frac{1}{2} \leqslant X \leqslant 2$



(Draw the graph using X on the horizontal axis.)

From the graph of $y = X + 1$:

The maximum value, when $X = 2$, i.e. **when $x = 1$, is 3.**

The minimum value, when $X = \frac{1}{2}$, i.e. **when $x = -1$, is $\frac{3}{2}$.**

K 198b

2. For each function, find the maximum and minimum values within the given domain.

Ex.

$y=2^{2x}\quad(-1\leqslant x\leqslant 1)$

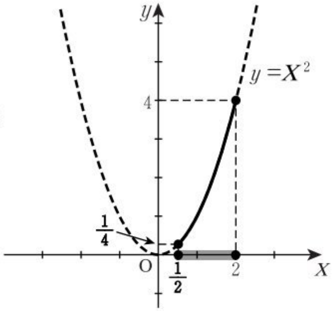
[Sol] Let $2^x=X$, and thus, $y=X^2$

Determining the range of values of X:

When $x=-1$, $X=\dfrac{1}{2}$

When $x=1$, $X=2$

Therefore, $\dfrac{1}{2}\leqslant X\leqslant 2$



(Draw the graph using X on the horizontal axis.)

From the graph of $y=X^2$:

The maximum value, when $X=2$, i.e. when $x=1$, is 4.

The minimum value, when $X=\dfrac{1}{2}$, i.e. when $x=-1$, is $\dfrac{1}{4}$.

(1) $y=3\cdot 2^{2x}-4\quad(-1\leqslant x\leqslant 1)$

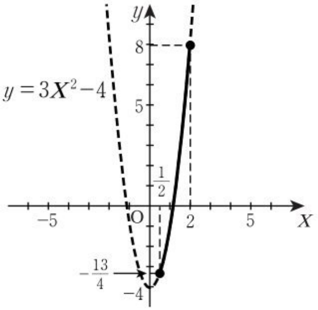
[Sol] Let $2^x=X$, and thus, $y=3X^2-4$

Determining the range of values of X:

When $x=-1$, $X=\dfrac{1}{2}$

When $x=1$, $X=2$

Therefore, $\dfrac{1}{2}\leqslant X\leqslant 2$



(Draw the graph using X on the horizontal axis.)

From the graph of $y=3X^2-4$:

The maximum value, when $X=2$, i.e. **when $x=1$, is 8.**

The minimum value, when $X=\dfrac{1}{2}$, i.e. **when $x=-1$, is $-\dfrac{13}{4}$.**

K 199a KUMON

Exponential Equations and Inequalities

For each function, find the maximum and minimum values within the given domain.

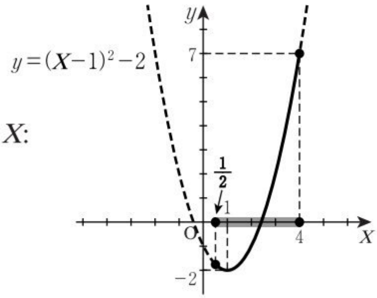
Ex.

$y=2^{2x}-2^{x+1}-1 \quad (-1\leqslant x\leqslant 2)$

[Sol] $y=(2^x)^2-2\cdot 2^x-1$

Let $2^x=X$ (where $X>0$)

$$y=X^2-2X-1=(X-1)^2-2$$



(Draw the graph using X on the horizontal axis.)

Determining the range of values of X:

When $x=-1$, $X=\frac{1}{2}$

When $x=2$, $X=4$

Therefore, $\frac{1}{2}\leqslant X\leqslant 4$

From the graph:

The maximum value, when $X=4$, ☜ $(4-1)^2-2=7$

i.e. when $x=2$, is 7.

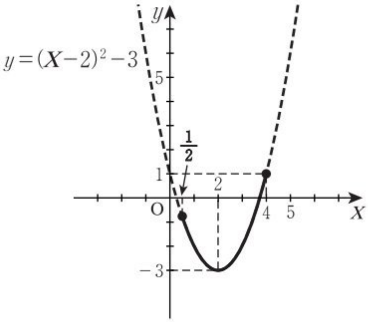
The minimum value, when $X=1$, i.e. when $x=0$, is -2.

(1) $y=2^{2x}-4\cdot 2^x+1 \quad (-1\leqslant x\leqslant 2)$

[Sol] $y=(2^x)^2-4\cdot 2^x+1$

Let $2^x=X$ (where $X>0$)

$$y=X^2-4X+1=(X-2)^2-3$$



(Draw the graph using X on the horizontal axis.)

Determining the range of values of X:

When $x=-1$, $X=\frac{1}{2}$

When $x=2$, $X=4$

Therefore, $\frac{1}{2}\leqslant X\leqslant 4$

From the graph:

The maximum value, when $X=4$, i.e. **when $x=2$, is 1.**

The minimum value, when $X=2$, i.e. **when $x=1$, is -3.**

K 199b

(2) $y=4^x+2^{x+1}-1 \quad (-1 \leqq x \leqq 1)$

[Sol] $y=(2^x)^2+2\cdot 2^x-1$

Let $2^x=X$ (where $X>0$)

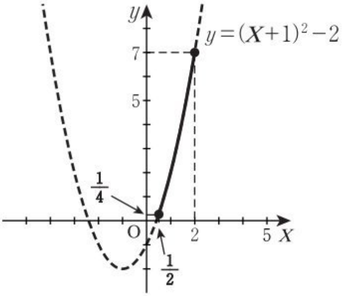
$$y=X^2+2X-1=(X+1)^2-2$$

Determining the range of values of X:

When $x=-1$, $X=\frac{1}{2}$

When $x=1$, $X=2$

Therefore, $\frac{1}{2} \leqq X \leqq 2$



(Draw the graph using
X on the horizontal axis.)

From the graph:

The maximum value, when $X=2$, i.e. **when $x=1$, is 7.**

The minimum value, when $X=\frac{1}{2}$, i.e. **when $x=-1$, is $\frac{1}{4}$.**

(3) $y=-3^{2x}+2\cdot 3^x+4 \quad (-1 \leqq x \leqq 1)$

[Sol] $y=-(3^x)^2+2\cdot 3^x+4$

Let $3^x=X$ (where $X>0$)

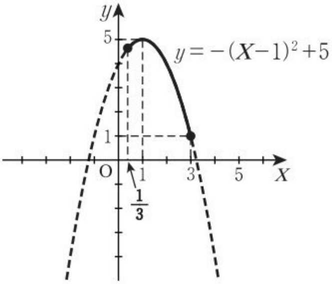
$$y=-X^2+2X+4=-(X-1)^2+5$$

Determining the range of values of X:

When $x=-1$, $X=\frac{1}{3}$

When $x=1$, $X=3$

Therefore, $\frac{1}{3} \leqq X \leqq 3$



(Draw the graph using
X on the horizontal axis.)

From the graph:

The maximum value, when $X=1$, i.e. **when $x=0$, is 5.**

The minimum value, when $X=3$, i.e. **when $x=1$, is 1.**

K 200a KUMON

Exponential Equations and Inequalities

1. Solve the following exponential equations and inequalities.

(1) $4^{x+1} = 8\sqrt{2}$

[Sol] $2^{2(x+1)} = 2^{\frac{7}{2}}$

$2(x+1) = \frac{7}{2}$

$\boldsymbol{x = \frac{3}{4}}$

(3) $\left(\frac{1}{4}\right)^{x} > \left(\frac{1}{2}\right)^{x-1}$

[Sol] $\left(\frac{1}{2}\right)^{2x} > \left(\frac{1}{2}\right)^{x-1}$

$2x < x-1$

$\boldsymbol{x < -1}$

[Alternative Solution

$2^{-2x} > 2^{-x+1}$

$-2x > -x+1$

$-x > 1$

$\boldsymbol{x < -1}$]

(2) $2^{2x} - 2^{x+1} - 8 = 0$

[Sol] $(2^x)^2 - 2 \cdot 2^x - 8 = 0$

Let $2^x = X$ (where $X > 0$)

$X^2 - 2X - 8 = 0$

$(X-4)(X+2) = 0$

$X = 4, \ -2$

Since $X > 0$, $X = -2$ is an extraneous solution.

Therefore,

$X = 4$

$2^x = 4$

$\boldsymbol{x = 2}$

(4) $9^x - 3^{x+1} - 54 > 0$

[Sol] $3^{2x} - 3 \cdot 3^x - 54 > 0$

$(3^x)^2 - 3 \cdot 3^x - 54 > 0$

Let $3^x = X$ (where $X > 0$)

$X^2 - 3X - 54 > 0$

$(X+6)(X-9) > 0$

$X < -6, \ X > 9$

Since $X > 0$,

$X > 9$

$3^x > 9$

$\boldsymbol{x > 2}$

K 200b

2. For the following function, find the maximum and minimum values within the given domain.

$y = -4^x + 2^{x+1} + 4 \quad (-2 \leqslant x \leqslant 1)$

[Sol] $y = -(2^x)^2 + 2 \cdot 2^x + 4$

Let $2^x = X$ (where $X > 0$)

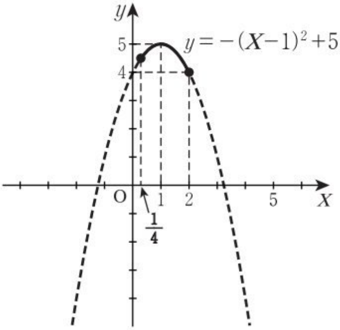
$$y = -X^2 + 2X + 4 = -(X-1)^2 + 5$$

Determining the range of values of X:

When $x = -2$, $X = \frac{1}{4}$

When $x = 1$, $X = 2$

Therefore, $\frac{1}{4} \leqslant X \leqslant 2$



(Draw the graph using X on the horizontal axis.)

From the graph:

The maximum value, when $X = 1$, i.e. **when $x = 0$, is 5.**

The minimum value, when $X = 2$, i.e. **when $x = 1$, is 4.**