

Expansion of Polynomial Products

Formula

When m and n are positive integers:

$$a^m \times a^n = a^{m+n}$$

$$(a^m)^n = a^{mn}$$

$$(ab)^m = a^m b^m$$

Note: These formulas are often called the *properties of powers* or the *laws of indices*.

Expand the following expressions.

$$(1) \quad (a^3)^4 = a^{3 \cdot 4} = \mathbf{a^{12}}$$

$$(2) \quad (-2a^2bc^3)^3 = (-2)^3 a^{2 \cdot 3} b^{1 \cdot 3} c^{3 \cdot 3} = \mathbf{-8a^6b^3c^9}$$

$$(3) \quad 3a^2b^3 \times 5a^3b^2 = 3 \cdot 5 a^{2+3} b^{3+2} = \mathbf{15a^5b^5}$$

$$\begin{aligned} (4) \quad -4ab \left(-\frac{1}{2} ab^2 \right)^2 &= -4ab \left[\left(-\frac{1}{2} \right)^2 a^{1 \cdot 2} b^{2 \cdot 2} \right] = -4ab \left(\frac{1}{4} a^2 b^4 \right) \\ &= -4 \cdot \frac{1}{4} a^{1+2} b^{1+4} = \mathbf{-a^3b^5} \end{aligned}$$

J | b

$$(5) \quad -\frac{3}{4}ab\left(-\frac{2}{3}ab^2\right)^2 = -\frac{3}{4}ab\left(\frac{4}{9}a^2b^4\right) = -\frac{1}{3}\mathbf{a^3b^5}$$

$$(6) \quad a^7 \div a^5 = \frac{a^7}{a^5} = \mathbf{a^2}$$

$$(7) \quad (a^4)^3 \div (a^3)^4 = a^{12} \div a^{12} = \frac{a^{12}}{a^{12}} = \mathbf{1}$$

$$(8) \quad 4a^3b \div 3ab^2 = \frac{4a^3b}{3ab^2} = \frac{\mathbf{4a^2}}{\mathbf{3b}}$$

$$(9) \quad \frac{2}{3}a^4b^4 \times (-3a^2b) \div \frac{1}{4}a^5b^7 = \frac{2}{3}a^4b^4 \cdot (-3a^2b) \cdot \frac{4}{a^5b^7} = -\frac{\mathbf{8a}}{\mathbf{b^2}}$$

$$\begin{aligned}(10) \quad \frac{2}{5}x^2y^2 \div \frac{4}{15}x^4y^7 \times \frac{5}{3}x^2y^3 &= \frac{2}{5}x^2y^2 \cdot \frac{15}{4x^4y^7} \cdot \frac{5}{3}x^2y^3 \\ &= \frac{2}{5} \cdot \frac{15}{4} \cdot \frac{5}{3} \cdot \frac{x^{2+2}y^{2+3}}{x^4y^7} \\ &= \frac{5x^4y^5}{2x^4y^7} = \frac{\mathbf{5}}{\mathbf{2y^2}}\end{aligned}$$

J2a

KUMON

Expansion of Polynomial Products

Expand the following expressions.

$$\begin{aligned}
 (1) \quad & 3xy(2yz + 4xz) \\
 &= (3xy)(2yz) + (3xy)(4xz) \\
 &= \mathbf{6xy^2z + 12x^2yz}
 \end{aligned}$$

We can expand brackets as follows:

$$a(b+c) = ab+ac$$

This is called the *distributive law*.

$$\begin{aligned}
 (2) \quad & -5yz(-2xz + 3xy) \\
 &= (-5yz)(-2xz) + (-5yz)(3xy) \\
 &= \mathbf{10xyz^2 - 15xy^2z}
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad & 4x^2z(-3y^2z + xz^2) \\
 &= (4x^2z)(-3y^2z) + (4x^2z)(xz^2) \\
 &= \mathbf{-12x^2y^2z^2 + 4x^3z^3}
 \end{aligned}$$

$$\begin{aligned}
 (4) \quad & -2abc(ab + bc + ca) \\
 &= (-2abc)(ab) + (-2abc)(bc) + (-2abc)(ca) \\
 &= \mathbf{-2a^2b^2c - 2ab^2c^2 - 2a^2bc^2}
 \end{aligned}$$

J2b

$$\begin{aligned}(5) \quad & 9y^2z\left(\frac{2}{3}x^2y - \frac{5}{9}xz^2\right) \\ &= (9y^2z)\left(\frac{2}{3}x^2y\right) + (9y^2z)\left(-\frac{5}{9}xz^2\right) \\ &= \mathbf{6x^2y^3z - 5xy^2z^3}\end{aligned}$$

$$\begin{aligned}(6) \quad & 8ab^4\left(\frac{a}{2} - \frac{3}{8a^2b^2}\right) \\ &= (8ab^4)\left(\frac{a}{2}\right) + (8ab^4)\left(-\frac{3}{8a^2b^2}\right) \\ &= \mathbf{4a^2b^4 - \frac{3b^2}{a}}\end{aligned}$$

$$\begin{aligned}(7) \quad & \frac{1}{2x}(6x^2 - 8) \\ &= \left(\frac{1}{2x}\right)(6x^2) + \left(\frac{1}{2x}\right)(-8) \\ &= \mathbf{3x - \frac{4}{x}}\end{aligned}$$

$$\begin{aligned}(8) \quad & \frac{1}{3x^2y^2}(9x^3y^3 - 4x^2y^2) \\ &= \left(\frac{1}{3x^2y^2}\right)(9x^3y^3) + \left(\frac{1}{3x^2y^2}\right)(-4x^2y^2) \\ &= \mathbf{3xy - \frac{4}{3}}\end{aligned}$$

$$\begin{aligned}(9) \quad & (a^3 + 2b^2x - 3cx) \div \frac{2x}{a} \\ &= (a^3 + 2b^2x - 3cx) \times \frac{a}{2x} \\ &= \left(\frac{a}{2x}\right)(a^3) + \left(\frac{a}{2x}\right)(2b^2x) + \left(\frac{a}{2x}\right)(-3cx) \\ &= \mathbf{\frac{a^4}{2x} + ab^2 - \frac{3ac}{2}}\end{aligned}$$

J3a

Expansion of Polynomial Products

Expand the following expressions.

$$(1) \quad (a+b)(a-b) \\ = \mathbf{a^2 - b^2}$$

$$(2) \quad (-x-2y)(-x+2y) \\ = [(-x)-2y][(-x)+2y] \\ = (-x)^2 - (2y)^2 \\ = \mathbf{x^2 - 4y^2}$$

$$(3) \quad (-x-2)^2 \\ = (-x)^2 - 2 \cdot (-x) \cdot 2 + 2^2 \\ = \mathbf{x^2 + 4x + 4}$$

$$(4) \quad 2(3a^3+4)^2 \\ = 2(9a^6 + 24a^3 + 16) \\ = \mathbf{18a^6 + 48a^3 + 32}$$

$$(5) \quad -2(-a+3b)^2 \\ = -2(a^2 - 6ab + 9b^2) \\ = \mathbf{-2a^2 + 12ab - 18b^2}$$

Note: Key formulas from level I :

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a-b)^2 = a^2 - 2ab + b^2$$

$$(a+b)(a-b) = a^2 - b^2$$

J3b

$$\begin{aligned}(6) \quad & \left(x - \frac{1}{2}\right)\left(x + \frac{1}{3}\right) \\ & = x^2 - \frac{1}{6}x - \frac{1}{6}\end{aligned}$$

$$\begin{aligned}(7) \quad & 3(x-2)(x+3) \\ & = 3(x^2 + x - 6) \\ & = \mathbf{3x^2 + 3x - 18}\end{aligned}$$

$$\begin{aligned}(8) \quad & -5(3x+4)(2x-1) \\ & = -5(6x^2 + 5x - 4) \\ & = \mathbf{-30x^2 - 25x + 20}\end{aligned}$$

$$\begin{aligned}(9) \quad & (a+3b)(a+5b) - (a+4b)^2 \\ & = (a^2 + 8ab + 15b^2) - (a^2 + 8ab + 16b^2) \\ & = a^2 + 8ab + 15b^2 - a^2 - 8ab - 16b^2 \\ & = \mathbf{-b^2}\end{aligned}$$

$$\begin{aligned}(10) \quad & (3a-7)(2a+1) + (5a-3)(3a+4) \\ & = (6a^2 - 11a - 7) + (15a^2 + 11a - 12) \\ & = 6a^2 - 11a - 7 + 15a^2 + 11a - 12 \\ & = \mathbf{21a^2 - 19}\end{aligned}$$

Expansion of Polynomial Products

1. Expand the following expressions. Write the intermediate steps.

Ex.

$$\begin{aligned}
 & (a+b+c)^2 \\
 = & [(a+b)+c]^2 && \text{Treat } (a+b) \text{ and } c \text{ as separate terms.} \\
 = & (a+b)^2 + 2c(a+b) + c^2 && \text{Expand, treating } (a+b) \text{ as a single unit.} \\
 = & a^2 + 2ab + b^2 + 2ac + 2bc + c^2 && \text{Expand all } (a+b) \text{ terms.} \\
 = & a^2 + b^2 + c^2 + 2ab + 2ac + 2bc && \text{Rearrange the terms as shown.}
 \end{aligned}$$

$$\begin{aligned}
 (1) \quad & (2a-3b+c)^2 \\
 & = [(2a-3b)+c]^2 \\
 & = (2a-3b)^2 + 2c(2a-3b) + c^2 \\
 & = 4a^2 - 12ab + 9b^2 + 4ac - 6bc + c^2 \\
 & = \mathbf{4a^2 + 9b^2 + c^2 - 12ab + 4ac - 6bc}
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad & (3x-2y-z)^2 \\
 & = [(3x-2y)-z]^2 \\
 & = (3x-2y)^2 - 2z(3x-2y) + z^2 \\
 & = 9x^2 - 12xy + 4y^2 - 6xz + 4yz + z^2 \\
 & = \mathbf{9x^2 + 4y^2 + z^2 - 12xy - 6xz + 4yz}
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad & (x^2+x-3)^2 \\
 & = [(x^2+x)-3]^2 \\
 & = (x^2+x)^2 - 6(x^2+x) + 9 \\
 & = x^4 + 2x^3 + x^2 - 6x^2 - 6x + 9 \\
 & = \mathbf{x^4 + 2x^3 - 5x^2 - 6x + 9}
 \end{aligned}$$

J4b

Formula

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$$



This formula is normally written in this way (cyclical order).

2. Expand using the above formula.

$$\begin{aligned} (1) \quad (2a-3b-c)^2 &= (2a)^2 + (-3b)^2 + (-c)^2 + 2(2a)(-3b) + 2(-3b)(-c) + 2(-c)(2a) \\ &= \mathbf{4a^2 + 9b^2 + c^2 - 12ab + 6bc - 4ca} \end{aligned}$$

$$(2) \quad (2x-3y+4z)^2 = \mathbf{4x^2 + 9y^2 + 16z^2 - 12xy - 24yz + 16zx}$$

$$(3) \quad (2x+3y+5z)^2 = \mathbf{4x^2 + 9y^2 + 25z^2 + 12xy + 30yz + 20zx}$$

$$\begin{aligned} (4) \quad (x^2-2x-3)^2 &= x^4 + 4x^2 + 9 - 4x^3 + 12x - 6x^2 \\ &= \mathbf{x^4 - 4x^3 - 2x^2 + 12x + 9} \end{aligned}$$

$$\begin{aligned} (5) \quad (2x^2-3x+4)^2 &= 4x^4 + 9x^2 + 16 - 12x^3 - 24x + 16x^2 \\ &= \mathbf{4x^4 - 12x^3 + 25x^2 - 24x + 16} \end{aligned}$$

Expansion of Polynomial Products

Formula


$$(a+b)(a-b) = a^2 - b^2$$

This formula is called the *difference of two squares*.

Expand the following expressions.

Ex.

$$\begin{aligned} & (a+b+c)(a+b-c) \\ &= [(a+b)+c][(a+b)-c] \\ &= (a+b)^2 - c^2 \\ &= a^2 + 2ab + b^2 - c^2 \end{aligned}$$


 Treat $(a+b)$ and c as separate terms.
 Expand, treating $(a+b)$ as a single unit.

$$\begin{aligned} (1) \quad & (a+2b+3c)(a+2b-3c) \\ &= [(a+2b)+3c][(a+2b)-3c] \\ &= (a+2b)^2 - (3c)^2 \\ &= \mathbf{a^2 + 4ab + 4b^2 - 9c^2} \end{aligned}$$

$$\begin{aligned} (2) \quad & (a-3b+5c)(a-3b-5c) \\ &= [(a-3b)+5c][(a-3b)-5c] \\ &= (a-3b)^2 - (5c)^2 \\ &= \mathbf{a^2 - 6ab + 9b^2 - 25c^2} \end{aligned}$$

$$\begin{aligned} (3) \quad & (a-4b-2c)(a-4b+2c) \\ &= [(a-4b)-2c][(a-4b)+2c] \\ &= (a-4b)^2 - (2c)^2 \\ &= \mathbf{a^2 - 8ab + 16b^2 - 4c^2} \end{aligned}$$

J5b

$$\begin{aligned}(4) \quad & (a+4b-2c)(a-4b-2c) \\ & = [(a-2c)+4b][(a-2c)-4b] \\ & = (a-2c)^2 - (4b)^2 \\ & = \mathbf{a^2 - 4ac + 4c^2 - 16b^2}\end{aligned}$$

$$\begin{aligned}(5) \quad & (a+3b+5c)(a-3b+5c) \\ & = [(a+5c)+3b][(a+5c)-3b] \\ & = (a+5c)^2 - (3b)^2 \\ & = \mathbf{a^2 + 10ac + 25c^2 - 9b^2}\end{aligned}$$

$$\begin{aligned}(6) \quad & (a+b-c)(a-b+c) \\ & = [a+(b-c)][a-(b-c)] \\ & = a^2 - (b-c)^2 \\ & = \mathbf{a^2 - b^2 + 2bc - c^2}\end{aligned}$$

$$\begin{aligned}(7) \quad & (a+b+c+d)(a+b-c-d) \\ & = [(a+b)+(c+d)][(a+b)-(c+d)] \\ & = (a+b)^2 - (c+d)^2 \\ & = \mathbf{a^2 + 2ab + b^2 - c^2 - 2cd - d^2}\end{aligned}$$

$$\begin{aligned}(8) \quad & (a-b+c+d)(a+b+c-d) \\ & = [(a+c)-(b-d)][(a+c)+(b-d)] \\ & = (a+c)^2 - (b-d)^2 \\ & = \mathbf{a^2 + 2ac + c^2 - b^2 + 2bd - d^2}\end{aligned}$$

Expansion of Polynomial Products

Formula

$$(x+a)(x+b) = x^2 + (a+b)x + ab$$

Expand the following expressions.

Ex.

$$\begin{aligned} & (x+y+3a)(x+y+5a) \\ &= [(x+y)+3a][(x+y)+5a] \quad \text{Treat } (x+y) \text{ as a single unit.} \\ &= (x+y)^2 + 8a(x+y) + 15a^2 \\ &= x^2 + 2xy + y^2 + 8ax + 8ay + 15a^2 \end{aligned}$$

$$\begin{aligned} (1) \quad & (x+2y+3a)(x+2y+4a) \\ &= [(x+2y)+3a][(x+2y)+4a] \\ &= (x+2y)^2 + 7a(x+2y) + 12a^2 \\ &= \mathbf{x^2 + 4xy + 4y^2 + 7ax + 14ay + 12a^2} \end{aligned}$$

$$\begin{aligned} (2) \quad & (2a-4b-c)(2a+2b-c) \\ &= [(2a-c)-4b][(2a-c)+2b] \\ &= (2a-c)^2 - 2b(2a-c) - 8b^2 \\ &= \mathbf{4a^2 - 4ac + c^2 - 4ab + 2bc - 8b^2} \end{aligned}$$

J6b

$$\begin{aligned}(3) \quad & [5(a+b)-2][3(a+b)+5] \\ & = 15(a+b)^2 + 25(a+b) - 6(a+b) - 10 \\ & = 15(a+b)^2 + \boxed{19}(a+b) - 10 \\ & = \mathbf{15a^2 + 30ab + 15b^2 + 19a + 19b - 10}\end{aligned}$$

$$\begin{aligned}(4) \quad & [2(a-b)+3c][3(a-b)-5c] \\ & = 6(a-b)^2 - c(a-b) - 15c^2 \\ & = \mathbf{6a^2 - 12ab + 6b^2 - ac + bc - 15c^2}\end{aligned}$$

$$\begin{aligned}(5) \quad & (x+2)(x+3)(x+1)(x+4) \\ & = (x^2+5x+\boxed{6})(x^2+5x+\boxed{4}) \\ & = [(x^2+5x)+\boxed{6}][(x^2+5x)+\boxed{4}] \\ & = (x^2+5x)^2 + 10(x^2+5x) + 24 \\ & = x^4 + 10x^3 + 25x^2 + 10x^2 + 50x + 24 \\ & = \mathbf{x^4 + 10x^3 + 35x^2 + 50x + 24}\end{aligned}$$

$$\begin{aligned}(6) \quad & (x+1)(x+5)(x+2)(x+4) \\ & = (x^2+6x+5)(x^2+6x+8) \\ & = [(x^2+6x)+5][(x^2+6x)+8] \\ & = (x^2+6x)^2 + 13(x^2+6x) + 40 \\ & = x^4 + 12x^3 + 36x^2 + 13x^2 + 78x + 40 \\ & = \mathbf{x^4 + 12x^3 + 49x^2 + 78x + 40}\end{aligned}$$

J7a

KUMON

Expansion of Polynomial Products**Formula**

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

Expand the following expressions.

$$\begin{aligned}(1) \quad (a+1)^3 &= a^3 + 3 \cdot a^2 \cdot 1 + 3 \cdot a \cdot 1^2 + 1^3 \\ &= \mathbf{a^3 + 3a^2 + 3a + 1}\end{aligned}$$

$$(2) \quad (a+2)^3 = \mathbf{a^3 + 6a^2 + 12a + 8}$$

$$(3) \quad (a+5)^3 = \mathbf{a^3 + 15a^2 + 75a + 125}$$

$$\begin{aligned}(4) \quad (2x+1)^3 &= (2x)^3 + 3 \cdot (2x)^2 \cdot 1 + 3 \cdot 2x \cdot 1^2 + 1^3 \\ &= \mathbf{8x^3 + 12x^2 + 6x + 1}\end{aligned}$$

$$(5) \quad (x+2y)^3 = \mathbf{x^3 + 6x^2y + 12xy^2 + 8y^3}$$

J7b

Formula

$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$

$$(6) \quad (a-1)^3 = a^3 - 3 \cdot a^2 \cdot 1 + 3 \cdot a \cdot 1^2 - 1^3 \\ = a^3 - 3a^2 + 3a - 1$$

$$(7) \quad (a-3)^3 = a^3 - 9a^2 + 27a - 27$$

$$(8) \quad (2a-1)^3 = (2a)^3 - 3 \cdot (2a)^2 \cdot 1 + 3 \cdot 2a \cdot 1^2 - 1^3 \\ = 8a^3 - 12a^2 + 6a - 1$$

$$(9) \quad (2x-3)^3 = 8x^3 - 36x^2 + 54x - 27$$

$$(10) \quad (x-3y)^3 = x^3 - 9x^2y + 27xy^2 - 27y^3$$

Expansion of Polynomial Products

Formulas

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$

Expand the following expressions.

$$(1) \quad (a+4)^3 = \mathbf{a^3 + 12a^2 + 48a + 64}$$

$$(2) \quad (2a-b)^3 = (2a)^3 - 3 \cdot (2a)^2 \cdot b + 3 \cdot 2a \cdot b^2 - b^3 \\ = \mathbf{8a^3 - 12a^2b + 6ab^2 - b^3}$$

$$(3) \quad (3a+2b)^3 = (3a)^3 + 3 \cdot (3a)^2 \cdot 2b + 3 \cdot 3a \cdot (2b)^2 + (2b)^3 \\ = \mathbf{27a^3 + 54a^2b + 36ab^2 + 8b^3}$$

$$(4) \quad (x^2+1)^3 = (x^2)^3 + 3 \cdot (x^2)^2 \cdot 1 + 3 \cdot x^2 \cdot 1^2 + 1^3 \\ = \mathbf{x^6 + 3x^4 + 3x^2 + 1}$$

$$(5) \quad (2x^2-3)^3 = (2x^2)^3 - 3 \cdot (2x^2)^2 \cdot 3 + 3 \cdot (2x^2) \cdot 3^2 - 3^3 \\ = \mathbf{8x^6 - 36x^4 + 54x^2 - 27}$$

J8b

$$\begin{aligned}(6) \quad (-a-1)^3 &= [-(a+1)]^3 = -(a+1)^3 \\ &= -(a^3 + 3a^2 + 3a + 1) \\ &= \mathbf{-a^3 - 3a^2 - 3a - 1}\end{aligned}$$

$$\begin{aligned}(7) \quad (-2a-b)^3 &= [-(2a+b)]^3 = -(2a+b)^3 \\ &= -[(2a)^3 + 3 \cdot (2a)^2 \cdot b + 3 \cdot 2a \cdot b^2 + b^3] \\ &= -(8a^3 + 12a^2b + 6ab^2 + b^3) \\ &= \mathbf{-8a^3 - 12a^2b - 6ab^2 - b^3}\end{aligned}$$

$$\begin{aligned}(8) \quad (-x^2+3)^3 &= [-(x^2-3)]^3 = -(x^2-3)^3 \\ &= -[(x^2)^3 - 3 \cdot (x^2)^2 \cdot 3 + 3 \cdot x^2 \cdot 3^2 - 3^3] \\ &= -(x^6 - 9x^4 + 27x^2 - 27) \\ &= \mathbf{-x^6 + 9x^4 - 27x^2 + 27}\end{aligned}$$

$$\begin{aligned}(9) \quad \left(a + \frac{1}{3}\right)^3 &= a^3 + 3 \cdot a^2 \cdot \frac{1}{3} + 3 \cdot a \cdot \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 \\ &= \mathbf{a^3 + a^2 + \frac{1}{3}a + \frac{1}{27}}\end{aligned}$$

$$\begin{aligned}(10) \quad \left(-2a - \frac{1}{3}\right)^3 &= \left[-\left(2a + \frac{1}{3}\right)\right]^3 = -\left(2a + \frac{1}{3}\right)^3 \\ &= -\left[(2a)^3 + 3 \cdot (2a)^2 \cdot \frac{1}{3} + 3 \cdot 2a \cdot \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3\right] \\ &= -\left(8a^3 + 4a^2 + \frac{2}{3}a + \frac{1}{27}\right) \\ &= \mathbf{-8a^3 - 4a^2 - \frac{2}{3}a - \frac{1}{27}}\end{aligned}$$

Expansion of Polynomial Products

Expand as shown in the examples.

Ex.

$$\begin{aligned}
 & (x+1)^3(x-1)^3 \\
 &= [(x+1)(x-1)]^3 \\
 &= (x^2-1)^3 \quad \curvearrowright \text{Expand } (x+1)(x-1) \text{ first.} \\
 &= x^6 - 3x^4 + 3x^2 - 1
 \end{aligned}$$

Note: This is easier than expanding $(x+1)^3$ and $(x-1)^3$.

$$\begin{aligned}
 (1) \quad & (x+2)^3(x-2)^3 \\
 &= [(x+2)(x-2)]^3 \\
 &= (x^2-4)^3 \\
 &= \mathbf{x^6 - 12x^4 + 48x^2 - 64}
 \end{aligned}$$

Use the formulas
on J8a.

$$\begin{aligned}
 (2) \quad & (a+3)^3(a-3)^3 \\
 &= [(a+3)(a-3)]^3 \\
 &= (a^2-9)^3 \\
 &= \mathbf{a^6 - 27a^4 + 243a^2 - 729}
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad & (x+2y)^3(x-2y)^3 \\
 &= [(x+2y)(x-2y)]^3 \\
 &= (x^2-4y^2)^3 \\
 &= \mathbf{x^6 - 12x^4y^2 + 48x^2y^4 - 64y^6}
 \end{aligned}$$

J9b

Ex.

$$\begin{aligned} & (x+1)(x+2)(x+3) \\ &= (x^2+3x+2)(x+3) \quad \curvearrowright \text{Expand } (x+1)(x+2) \text{ first.} \\ &= x^3+3x^2+3x^2+9x+2x+6 \\ &= x^3+6x^2+11x+6 \end{aligned}$$

$$\begin{aligned} (4) \quad & (x+1)(x+3)(x+5) \\ &= (x^2+4x+3)(x+5) \\ &= x^3+5x^2+4x^2+20x+3x+15 \\ &= \mathbf{x^3+9x^2+23x+15} \end{aligned}$$

$$\begin{aligned} (5) \quad & (x-1)(x-2)(x-3) \\ &= (x^2-3x+2)(x-3) \\ &= x^3-3x^2-3x^2+9x+2x-6 \\ &= \mathbf{x^3-6x^2+11x-6} \end{aligned}$$

$$\begin{aligned} (6) \quad & (x+2)(x-3)(x+4) \\ &= (x^2-x-6)(x+4) \\ &= x^3+4x^2-x^2-4x-6x-24 \\ &= \mathbf{x^3+3x^2-10x-24} \end{aligned}$$

$$\begin{aligned} (7) \quad & (x+1)(x+2)(x+4) \\ &= (x^2+3x+2)(x+4) \\ &= x^3+4x^2+3x^2+12x+2x+8 \\ &= \mathbf{x^3+7x^2+14x+8} \end{aligned}$$

From this, we find

Formula

$$(x+a)(x+b)(x+c) = x^3 + (a+b+c)x^2 + (ab+bc+ca)x + abc$$

J10a KUMON

Expansion of Polynomial Products

Expand the following expressions.

$$(1) \left(3a + \frac{1}{3}\right)^3 = 27a^3 + 9a^2 + a + \frac{1}{27}$$

Use the formulas
on J8a.

$$(2) \left(x - \frac{2}{x}\right)^3 = x^3 - 6x + \frac{12}{x} - \frac{8}{x^3}$$

$$(3) \left(x^2 + \frac{1}{x}\right)^3 = x^6 + 3x^3 + 3 + \frac{1}{x^3}$$

$$(4) (2x+1)^3(2x-1)^3 = [(2x+1)(2x-1)]^3 \\ = (4x^2-1)^3 \\ = 64x^6 - 48x^4 + 12x^2 - 1$$

J 10b

$$(5) \quad (2a - 3b + c)^2 = \mathbf{4a^2 + 9b^2 + c^2 - 12ab - 6bc + 4ca}$$

Use the formula
on J 4b.

$$(6) \quad (2x^2 - 3x - 4)^2 = 4x^4 + 9x^2 + 16 - 12x^3 + 24x - 16x^2 \\ = \mathbf{4x^4 - 12x^3 - 7x^2 + 24x + 16}$$

$$(7) \quad (2x^2 + 3x - 4)(2x^2 - 3x + 4) = [2x^2 + (3x - 4)][2x^2 - (3x - 4)] \\ = 4x^4 - (3x - 4)^2 \\ = \mathbf{4x^4 - 9x^2 + 24x - 16}$$

$$(8) \quad (2x^2 - x + 3)(2x^2 + x - 3) = [2x^2 - (x - 3)][2x^2 + (x - 3)] \\ = 4x^4 - (x - 3)^2 \\ = \mathbf{4x^4 - x^2 + 6x - 9}$$

Factorisation I

Formula

$$x^2 + (a + b)x + ab = (x + a)(x + b)$$

Factorise the following expressions. (Review of Level I)

$$(1) \quad x^2 + 8x + 15 = (x + 5)(x + 3)$$

$$(2) \quad x^2 + 10x + 21 = (x + 7)(x + 3)$$

$$(3) \quad a^2 - 7a + 10 = (a - 5)(a - 2)$$

$$(4) \quad a^2 + 3ab - 10b^2 = (a + 5b)(a - 2b)$$

$$(5) \quad 12x^2 + 24x - 96 = 12(x^2 + 2x - 8) \\ = 12(x + 4)(x - 2)$$

First take out
the common factor 12.

$$(6) \quad ax^2 - 2ax - 8a = a(x^2 - 2x - 8) \\ = a(x - 4)(x + 2)$$

J 11 b

Ex.

$$\begin{aligned}(x+y)^2 - 3(x+y) - 10 \\ &= [(x+y) - 5][(x+y) + 2] \\ &= (x+y-5)(x+y+2)\end{aligned}$$



Treat $(x+y)$ as a single unit.

$$\begin{aligned}(7) \quad (x+y)^2 + 8(x+y) + 15 \\ &= [(x+y) + 5][(x+y) + 3] \\ &= \mathbf{(x+y+5)(x+y+3)}\end{aligned}$$

$$\begin{aligned}(8) \quad (x+y)^2 - (x+y) - 12 \\ &= [(x+y) - 4][(x+y) + 3] \\ &= \mathbf{(x+y-4)(x+y+3)}\end{aligned}$$

$$\begin{aligned}(9) \quad (2x+y)^2 + 7a(2x+y) + 10a^2 \\ &= [(2x+y) + 5a][(2x+y) + 2a] \\ &= \mathbf{(2x+y+5a)(2x+y+2a)}\end{aligned}$$

$$\begin{aligned}(10) \quad (x-3)^2 - (x-3) - 42 \\ &= [(x-3) - 7][(x-3) + 6] \\ &= \mathbf{(x-10)(x+3)}\end{aligned}$$

$$\begin{aligned}(11) \quad (x-5)^2 + (x-5) - 12 \\ &= [(x-5) + 4][(x-5) - 3] \\ &= \mathbf{(x-1)(x-8)}\end{aligned}$$

Factorisation I

Factorise the following expressions.

$$\begin{aligned}(1) \quad (x+y)^2 - 5(x+y) + 6 &= [(x+y) - 3][(x+y) - 2] \\ &= (\mathbf{x+y-3})(\mathbf{x+y-2})\end{aligned}$$

$$\begin{aligned}(2) \quad (x+y)^2 - 2(x+y) - 15 &= [(x+y) - 5][(x+y) + 3] \\ &= (\mathbf{x+y-5})(\mathbf{x+y+3})\end{aligned}$$

$$\begin{aligned}(3) \quad (x+y)^2 + 3(x+y) - 18 &= [(x+y) + 6][(x+y) - 3] \\ &= (\mathbf{x+y+6})(\mathbf{x+y-3})\end{aligned}$$

$$\begin{aligned}(4) \quad (x+y)^2 - 7(x+y) + 10 &= [(x+y) - 5][(x+y) - 2] \\ &= (\mathbf{x+y-5})(\mathbf{x+y-2})\end{aligned}$$

$$\begin{aligned}(5) \quad (x+2)^2 - 9(x+2) + 20 &= [(x+2) - 5][(x+2) - 4] \\ &= (\mathbf{x-3})(\mathbf{x-2})\end{aligned}$$

J 12b

$$\begin{aligned}(6) \quad (x-y)^2 + 4z(x-y) + 3z^2 &= [(x-y) + 3z][(x-y) + z] \\ &= \mathbf{(x-y+3z)(x-y+z)}\end{aligned}$$

$$\begin{aligned}(7) \quad (x+y)^2 - 2z(x+y) - 15z^2 &= [(x+y) - 5z][(x+y) + 3z] \\ &= \mathbf{(x+y-5z)(x+y+3z)}\end{aligned}$$

$$\begin{aligned}(8) \quad (a+b)^2 + 6c(a+b) + 5c^2 &= [(a+b) + 5c][(a+b) + c] \\ &= \mathbf{(a+b+5c)(a+b+c)}\end{aligned}$$

$$\begin{aligned}(9) \quad (a+b)^2 - 7c(a+b) + 10c^2 &= [(a+b) - 5c][(a+b) - 2c] \\ &= \mathbf{(a+b-5c)(a+b-2c)}\end{aligned}$$

$$\begin{aligned}(10) \quad (a-2b)^2 - 9c(a-2b) + 20c^2 &= [(a-2b) - 5c][(a-2b) - 4c] \\ &= \mathbf{(a-2b-5c)(a-2b-4c)}\end{aligned}$$

Factorisation I

Factorise the following expressions.

Ex.

$$2x^2 + 5x - 12 = (2x - 3)(x + 4)$$

$$\begin{array}{r} 2 \quad -3 \\ \times \\ 1 \quad 4 \end{array}$$



If you can do this mentally, it is not necessary to write this step.

(1) $2x^2 + 7x + 5 = (2x + 5)(x + 1)$

$$\begin{array}{r} 2 \quad 5 \\ \times \\ 1 \quad 1 \end{array}$$

(2) $6x^2 + 29x - 5 = (6x - 1)(x + 5)$

$$\begin{array}{r} 6 \quad -1 \\ \times \\ 1 \quad 5 \end{array}$$

(3) $10x^2 - 7x - 12 = (5x + 4)(2x - 3)$

$$\begin{array}{r} 5 \quad 4 \\ \times \\ 2 \quad -3 \end{array}$$

(4) $6x^2 - 17x + 5 = (3x - 1)(2x - 5)$

$$\begin{array}{r} 3 \quad -1 \\ \times \\ 2 \quad -5 \end{array}$$

(5) $3x^2 - 13x + 4 = (3x - 1)(x - 4)$

$$\begin{array}{r} 3 \quad -1 \\ \times \\ 1 \quad -4 \end{array}$$

Further notes on question (1)

—Looking at $2x^2$, write the factors of 2 on the left.

$$\begin{array}{r} 2 \\ \times \\ 1 \end{array}$$

—Looking at $+5$, write the factors of 5 on the right.

$$\begin{array}{r} 2 \quad 5 \\ \times \\ 1 \quad 1 \end{array}$$

—Looking at $+7x$, check these numbers multiply to give $+7$.

$$\begin{array}{r} 2 \quad 5 \longrightarrow 5 \times 1 \\ \times \\ 1 \quad 1 \longrightarrow \frac{2 \times 1}{7} \end{array}$$

—It is often necessary to try different combinations of factors and signs.

—Read off the answer directly.

$$\begin{array}{r} 2 \quad 5 \text{ gives } (2x + 5) \\ \times \\ 1 \quad 1 \text{ gives } (x + 1) \end{array}$$

J 13b

$$(6) \quad 5x^2 + 16x + 3 = (5x + 1)(x + 3)$$

$$\begin{array}{r} 5 \quad 1 \\ 1 \quad 3 \end{array}$$

You can check in two ways,

(i) multiply out each cross

$$\begin{array}{r} 5 \quad 1 \rightarrow 1 \\ 1 \quad 3 \rightarrow 15 \\ \hline 16 \end{array}$$

(ii) expand your final answer

$$(5x + 1)(x + 3) = 5x^2 + 16x + 3$$

$$(7) \quad 5x^2 - 2xy - 3y^2 = (5x + 3y)(x - y)$$

$$\begin{array}{r} 5 \quad 3y \\ 1 \quad -y \end{array}$$

$$(8) \quad 2x^2 + xy - 6y^2 = (2x - 3y)(x + 2y)$$

$$\begin{array}{r} 2 \quad -3y \\ 1 \quad 2y \end{array}$$

$$(9) \quad 4x^2 - 13x - 12 = (4x + 3)(x - 4)$$

$$\begin{array}{r} 4 \quad 3 \\ 1 \quad -4 \end{array}$$

$$(10) \quad 6x^2 - xy - 12y^2 = (3x + 4y)(2x - 3y)$$

$$\begin{array}{r} 3 \quad 4y \\ 2 \quad -3y \end{array}$$

Factorisation I

Factorise the following expressions.

Ex.

$$2(x+y)^2 + 7(x+y) + 5 = [2(x+y) + 5][(x+y) + 1] \quad \text{☞} \quad \text{Treat } (x+y) \text{ as a single unit.}$$

$$= (2x + 2y + 5)(x + y + 1)$$

$$\begin{array}{r} 2 \quad \quad 5 \\ \quad \times \quad \quad \\ 1 \quad \quad 1 \end{array}$$

☞ It is not always necessary to write this step.

$$(1) \quad 2(x+y)^2 + 5(x+y) + 3 = [2(x+y) + 3][(x+y) + 1]$$

$$= (2x + 2y + 3)(x + y + 1)$$

$$\begin{array}{r} 2 \quad \quad 3 \\ \quad \times \quad \quad \\ 1 \quad \quad 1 \end{array}$$

$$(2) \quad 2(x+y)^2 + (x+y) - 3 = [2(x+y) + 3][(x+y) - 1]$$

$$= (2x + 2y + 3)(x + y - 1)$$

$$\begin{array}{r} 2 \quad \quad 3 \\ \quad \times \quad \quad \\ 1 \quad \quad -1 \end{array}$$

$$(3) \quad 2(x+y)^2 - 9(x+y) - 5 = [2(x+y) + 1][(x+y) - 5]$$

$$= (2x + 2y + 1)(x + y - 5)$$

$$\begin{array}{r} 2 \quad \quad 1 \\ \quad \times \quad \quad \\ 1 \quad \quad -5 \end{array}$$

$$(4) \quad 3(x+y)^2 - 13(x+y) + 4 = [3(x+y) - 1][(x+y) - 4]$$

$$= (3x + 3y - 1)(x + y - 4)$$

$$\begin{array}{r} 3 \quad \quad -1 \\ \quad \times \quad \quad \\ 1 \quad \quad -4 \end{array}$$

J 14b

$$(5) \quad 2(x+y)^2 + 3(x+y) + 1 = [2(x+y) + 1][(x+y) + 1] \\ = (2x + 2y + 1)(x + y + 1)$$

$$\begin{array}{r} 2 \quad \quad 1 \\ \quad \times \quad \quad \\ 1 \quad \quad 1 \end{array}$$

$$(6) \quad 2(x+y)^2 + 7a(x+y) + 5a^2 = [2(x+y) + 5a][(x+y) + a] \\ = (2x + 2y + 5a)(x + y + a)$$

$$\begin{array}{r} 2 \quad \quad 5a \\ \quad \times \quad \quad \\ 1 \quad \quad a \end{array}$$

$$(7) \quad 3(x-y)^2 + 2a(x-y) - 5a^2 = [3(x-y) + 5a][(x-y) - a] \\ = (3x - 3y + 5a)(x - y - a)$$

$$\begin{array}{r} 3 \quad \quad 5a \\ \quad \times \quad \quad \\ 1 \quad \quad -a \end{array}$$

$$(8) \quad 7(x+y)^2 + 13a(x+y) - 2a^2 = [7(x+y) - a][(x+y) + 2a] \\ = (7x + 7y - a)(x + y + 2a)$$

$$\begin{array}{r} 7 \quad \quad -a \\ \quad \times \quad \quad \\ 1 \quad \quad 2a \end{array}$$

Factorisation I

Formulas

$$a^2 + 2ab + b^2 = (a + b)^2$$

$$a^2 - 2ab + b^2 = (a - b)^2$$

Factorise the following expressions.

$$(1) \quad 9x^2 - 12x + 4 = (3x - 2)^2$$

$$(2) \quad x^2 + 4xy + 4y^2 = (x + 2y)^2$$

$$(3) \quad 4x^2 + 12xy + 9y^2 = (2x + 3y)^2$$

$$(4) \quad -3ax^2 + 12axy - 12ay^2 = -3a(x^2 - 4xy + 4y^2) \\ = -3a(x - 2y)^2$$

$$(5) \quad -x^2 + 4x - 4 = -(x^2 - 4x + 4) = -(x - 2)^2$$

Take out the
common factor -1 .

J 15b

Ex.

$$\begin{aligned}(x+y)^2 + 6(x+y) + 9 \\ &= [(x+y) + 3]^2 \\ &= (x+y+3)^2\end{aligned}$$



Treat $(x+y)$ as a single unit.

$$\begin{aligned}(6) \quad (x+y)^2 + 4(x+y) + 4 \\ &= [(x+y) + 2]^2 \\ &= \mathbf{(x+y+2)^2}\end{aligned}$$

$$\begin{aligned}(7) \quad (x+y)^2 + 8(x+y) + 16 \\ &= [(x+y) + 4]^2 \\ &= \mathbf{(x+y+4)^2}\end{aligned}$$

$$\begin{aligned}(8) \quad (x+y)^2 - 10(x+y) + 25 \\ &= [(x+y) - 5]^2 \\ &= \mathbf{(x+y-5)^2}\end{aligned}$$

$$\begin{aligned}(9) \quad (x-y)^2 - 6a(x-y) + 9a^2 \\ &= [(x-y) - 3a]^2 \\ &= \mathbf{(x-y-3a)^2}\end{aligned}$$

$$\begin{aligned}(10) \quad (x+2y)^2 - 2z(x+2y) + z^2 \\ &= [(x+2y) - z]^2 \\ &= \mathbf{(x+2y-z)^2}\end{aligned}$$

Factorisation I

Factorise the following expressions.

Ex.

$$\begin{aligned} & x^2 + 2x(a+b) + (a+b)^2 \\ &= [x + (a+b)]^2 \\ &= (x+a+b)^2 \end{aligned}$$



Treat $(a+b)$ as a single unit.

$$\begin{aligned} (1) \quad & x^2 - 2x(a-b) + (a-b)^2 \\ &= [x - (a-b)]^2 \\ &= (\mathbf{x - a + b})^2 \end{aligned}$$

Use the formulas:

$$\begin{aligned} a^2 + 2ab + b^2 &= (a+b)^2 \\ a^2 - 2ab + b^2 &= (a-b)^2 \end{aligned}$$

$$\begin{aligned} (2) \quad & x^2 + 4x(y+z) + 4(y+z)^2 \\ &= [x + 2(y+z)]^2 \\ &= (\mathbf{x + 2y + 2z})^2 \end{aligned}$$

$$\begin{aligned} (3) \quad & a^2 + 6a(b-c) + 9(b-c)^2 \\ &= [a + 3(b-c)]^2 \\ &= (\mathbf{a + 3b - 3c})^2 \end{aligned}$$

$$\begin{aligned} (4) \quad & a^2 - 8a(b+c) + 16(b+c)^2 \\ &= [a - 4(b+c)]^2 \\ &= (\mathbf{a - 4b - 4c})^2 \end{aligned}$$

$$\begin{aligned} (5) \quad & a^2 - 6a(2b-3c) + 9(2b-3c)^2 \\ &= [a - 3(2b-3c)]^2 \\ &= (\mathbf{a - 6b + 9c})^2 \end{aligned}$$

J 16b

$$\begin{aligned}(6) \quad & x^2 + 6x(x + 3y) + 9(x + 3y)^2 \\ &= [x + 3(x + 3y)]^2 \\ &= (x + 3x + 9y)^2 \\ &= \mathbf{(4x + 9y)^2}\end{aligned}$$

$$\begin{aligned}(7) \quad & (x + 2y)^2 - 2y(x + 2y) + y^2 \\ &= [(x + 2y) - y]^2 \\ &= (x + 2y - y)^2 \\ &= \mathbf{(x + y)^2}\end{aligned}$$

$$\begin{aligned}(8) \quad & (3x - 2y)^2 + 8y(3x - 2y) + 16y^2 \\ &= [(3x - 2y) + 4y]^2 \\ &= (3x - 2y + 4y)^2 \\ &= \mathbf{(3x + 2y)^2}\end{aligned}$$

$$\begin{aligned}(9) \quad & (x + a)^2 - 2(x + a)(y + a) + (y + a)^2 \\ &= [(x + a) - (y + a)]^2 \\ &= (x + a - y - a)^2 \\ &= \mathbf{(x - y)^2}\end{aligned}$$

$$\begin{aligned}(10) \quad & (a + b)^2 + 6(a + b)(2a + b) + 9(2a + b)^2 \\ &= [(a + b) + 3(2a + b)]^2 \\ &= (a + b + 6a + 3b)^2 \\ &= \mathbf{(7a + 4b)^2}\end{aligned}$$

Factorisation I

Formula

$$a^2 - b^2 = (a + b)(a - b)$$

This formula is called the *difference of two squares*.

Factorise the following expressions.

$$(1) \quad 9a^2 - 4b^2 = (3a + 2b)(3a - 2b)$$

$$(2) \quad 5x^2 - 20 = 5(x^2 - 4) \\ = 5(x + 2)(x - 2)$$

The first step is to take out the common factor 5.

$$(3) \quad 4ax^2 - 9ay^2 = a(4x^2 - 9y^2) \\ = a(2x + 3y)(2x - 3y)$$

$$(4) \quad (x + y)^2 - 4 = [(x + y) + 2][(x + y) - 2] \\ = (x + y + 2)(x + y - 2)$$

$$(5) \quad (x + y)^2 - z^2 = [(x + y) + z][(x + y) - z] \\ = (x + y + z)(x + y - z)$$

J 17b

$$\begin{aligned}(6) \quad x^2 - (a+b)^2 &= [x+(a+b)][x-(a+b)] \\ &= \mathbf{(x+a+b)(x-a-b)}\end{aligned}$$

$$\begin{aligned}(7) \quad 9 - (x-2y)^2 &= [3+(x-2y)][3-(x-2y)] \\ &= \mathbf{(3+x-2y)(3-x+2y)}\end{aligned}$$

$$\begin{aligned}(8) \quad x^2 - (a-b)^2 &= [x+(a-b)][x-(a-b)] \\ &= \mathbf{(x+a-b)(x-a+b)}\end{aligned}$$

$$\begin{aligned}(9) \quad 4a^2 - (a-b)^2 &= [2a+(a-b)][2a-(a-b)] \\ &= (2a+a-b)(2a-a+b) \\ &= \mathbf{(3a-b)(a+b)}\end{aligned}$$


$$\begin{aligned}(10) \quad (a+b+c)^2 - b^2 &= [(a+b+c)+b][(a+b+c)-b] \\ &= (a+b+c+b)(a+b+c-b) \\ &= \mathbf{(a+2b+c)(a+c)}\end{aligned}$$

Factorisation I

Factorise the following expressions.

Ex.

$$\begin{aligned} & (a+b)^2 - (c+d)^2 \\ &= [(a+b) + (c+d)][(a+b) - (c+d)] \\ &= (a+b+c+d)(a+b-c-d) \end{aligned}$$

 Treat $(a+b)$ and $(c+d)$ as single units.

$$\begin{aligned} (1) \quad & (a+b)^2 - (c-d)^2 \\ &= [(a+b) + (c-d)][(a+b) - (c-d)] \\ &= \mathbf{(a+b+c-d)(a+b-c+d)} \end{aligned}$$

Use the formula for the difference of two squares:

$$a^2 - b^2 = (a+b)(a-b)$$

$$\begin{aligned} (2) \quad & (a-b)^2 - (c-d)^2 \\ &= [(a-b) + (c-d)][(a-b) - (c-d)] \\ &= \mathbf{(a-b+c-d)(a-b-c+d)} \end{aligned}$$

$$\begin{aligned} (3) \quad & (a-b)^2 - (c+d)^2 \\ &= [(a-b) + (c+d)][(a-b) - (c+d)] \\ &= \mathbf{(a-b+c+d)(a-b-c-d)} \end{aligned}$$

$$\begin{aligned} (4) \quad & (x-y)^2 - (a-b)^2 \\ &= [(x-y) + (a-b)][(x-y) - (a-b)] \\ &= \mathbf{(x-y+a-b)(x-y-a+b)} \end{aligned}$$

$$\begin{aligned} (5) \quad & (3x+2)^2 - (x+3)^2 \\ &= [(3x+2) + (x+3)][(3x+2) - (x+3)] \\ &= (3x+2+x+3)(3x+2-x-3) \\ &= \mathbf{(4x+5)(2x-1)} \end{aligned}$$

J 18b

Ex.

$$\begin{aligned}
 & (3a+2b)^2 - b^2 \\
 &= [(3a+2b)+b][(3a+2b)-b] \\
 &= (3a+3b)(3a+b) \\
 &= 3(a+b)(3a+b)
 \end{aligned}$$

Factorise completely.

$$\begin{aligned}
 (6) \quad & (3a+2b)^2 - (2a+3b)^2 \\
 &= [(3a+2b)+(2a+3b)][(3a+2b)-(2a+3b)] \\
 &= (3a+2b+2a+3b)(3a+2b-2a-3b) \\
 &= (5a+5b)(a-b) \\
 &= \mathbf{5(a+b)(a-b)}
 \end{aligned}$$

Make sure that your answer is factorised completely.

$$\begin{aligned}
 (7) \quad & (2a-3b)^2 - (3a-2b)^2 \\
 &= [(2a-3b)+(3a-2b)][(2a-3b)-(3a-2b)] \\
 &= (2a-3b+3a-2b)(2a-3b-3a+2b) \\
 &= (5a-5b)(-a-b) \\
 &= \mathbf{-5(a-b)(a+b)}
 \end{aligned}$$

$$\begin{aligned}
 (8) \quad & (x+2y)^2 - (2x+y)^2 \\
 &= [(x+2y)+(2x+y)][(x+2y)-(2x+y)] \\
 &= (x+2y+2x+y)(x+2y-2x-y) \\
 &= (3x+3y)(-x+y) \\
 &= \mathbf{-3(x+y)(x-y)}
 \end{aligned}$$

$$\begin{aligned}
 (9) \quad & (3a-4)^2 - (2a-1)^2 \\
 &= [(3a-4)+(2a-1)][(3a-4)-(2a-1)] \\
 &= (3a-4+2a-1)(3a-4-2a+1) \\
 &= (5a-5)(a-3) \\
 &= \mathbf{5(a-1)(a-3)}
 \end{aligned}$$

$$\begin{aligned}
 (10) \quad & (2x-5)^2 - 9 \\
 &= [(2x-5)+3][(2x-5)-3] \\
 &= (2x-5+3)(2x-5-3) \\
 &= (2x-2)(2x-8) \\
 &= \mathbf{4(x-1)(x-4)}
 \end{aligned}$$

Factorisation I

Factorise the following expressions.

Ex.

$$\begin{aligned}
 & x^4 - y^4 \\
 &= (x^2)^2 - (y^2)^2 \quad \curvearrowright \text{Treat } x^2 \text{ and } y^2 \text{ as single units.} \\
 &= (x^2 + y^2)(x^2 - y^2) \\
 &= (x^2 + y^2)(x + y)(x - y) \quad \curvearrowright \text{Factorise completely.}
 \end{aligned}$$

$$\begin{aligned}
 (1) \quad & x^4 - 16 \\
 &= (x^2)^2 - 4^2 \\
 &= (x^2 + 4)(x^2 - 4) \\
 &= \mathbf{(x^2 + 4)(x + 2)(x - 2)}
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad & 16x^4 - y^4 \\
 &= (4x^2)^2 - (y^2)^2 \\
 &= (4x^2 + y^2)(4x^2 - y^2) \\
 &= \mathbf{(4x^2 + y^2)(2x + y)(2x - y)}
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad & x^8 - y^8 \\
 &= (x^4)^2 - (y^4)^2 \\
 &= (x^4 + y^4)(x^4 - y^4) \\
 &= (x^4 + y^4)[(x^2)^2 - (y^2)^2] \\
 &= (x^4 + y^4)(x^2 + y^2)(x^2 - y^2) \\
 &= \mathbf{(x^4 + y^4)(x^2 + y^2)(x + y)(x - y)}
 \end{aligned}$$

$$\begin{aligned}
 (4) \quad & x^8 - 1 \\
 &= (x^4)^2 - 1 \\
 &= (x^4 + 1)(x^4 - 1) \\
 &= (x^4 + 1)[(x^2)^2 - 1] \\
 &= (x^4 + 1)(x^2 + 1)(x^2 - 1) \\
 &= \mathbf{(x^4 + 1)(x^2 + 1)(x + 1)(x - 1)}
 \end{aligned}$$

J 19b

$$\begin{aligned}(5) \quad & x^4 - 5x^2 - 36 \\ &= (x^2 - 9)(x^2 + 4) \\ &= (x + 3)(x - 3)(x^2 + 4)\end{aligned}$$

$$\begin{aligned}(6) \quad & x^4 - 8x^2 - 9 \\ &= (x^2 - 9)(x^2 + 1) \\ &= (x + 3)(x - 3)(x^2 + 1)\end{aligned}$$

$$\begin{aligned}(7) \quad & x^4 - 6x^2 + 8 \\ &= (x^2 - 4)(x^2 - 2) \\ &= (x + 2)(x - 2)(x^2 - 2)\end{aligned}$$

It is possible to write this as
 $(x + 2)(x - 2)(x + \sqrt{2})(x - \sqrt{2})$.

However, this is not normally done
unless solving an equation.

$$\begin{aligned}(8) \quad & 2x^4 + x^2 - 3 \\ &= (2x^2 + 3)(x^2 - 1) \\ &= (2x^2 + 3)(x + 1)(x - 1)\end{aligned}$$


$$\begin{aligned}(9) \quad & ax^4 - 3ax^2 - 4a \\ &= a(x^4 - 3x^2 - 4) \\ &= a(x^2 - 4)(x^2 + 1) \\ &= a(x + 2)(x - 2)(x^2 + 1)\end{aligned}$$

Factorisation I

Factorise the following expressions.

Ex.

$$\begin{aligned}
 & x^4 - 2x^2 + 1 \\
 &= (x^2 - 1)^2 \\
 &= [(x+1)(x-1)]^2 \\
 &= (x+1)^2(x-1)^2
 \end{aligned}$$


 Factorise inside the brackets.

$$\begin{aligned}
 (1) \quad & x^4 - 8x^2 + 16 \\
 &= (x^2 - 4)^2 \\
 &= [(x+2)(x-2)]^2 \\
 &= (\mathbf{x+2})^2(\mathbf{x-2})^2
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad & x^4 - 18x^2 + 81 \\
 &= (x^2 - 9)^2 \\
 &= [(x+3)(x-3)]^2 \\
 &= (\mathbf{x+3})^2(\mathbf{x-3})^2
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad & x^4 - 2x^2y^2 + y^4 \\
 &= (x^2 - y^2)^2 \\
 &= [(x+y)(x-y)]^2 \\
 &= (\mathbf{x+y})^2(\mathbf{x-y})^2
 \end{aligned}$$

$$\begin{aligned}
 (4) \quad & x^5 - 8x^3 + 16x \\
 &= x(x^4 - 8x^2 + 16) \\
 &= x(x^2 - 4)^2 \\
 &= x[(x+2)(x-2)]^2 \\
 &= \mathbf{x(x+2)^2(x-2)^2}
 \end{aligned}$$

J20b

$$\begin{aligned}(5) \quad x^4 - a^4 \\ &= (x^2)^2 - (a^2)^2 \\ &= (x^2 + a^2)(x^2 - a^2) \\ &= (\mathbf{x^2 + a^2})(\mathbf{x + a})(\mathbf{x - a})\end{aligned}$$

$$\begin{aligned}(6) \quad x^7 - 16x^3 \\ &= x^3(x^4 - 16) \\ &= x^3[(x^2)^2 - 4^2] \\ &= x^3(x^2 + 4)(x^2 - 4) \\ &= \mathbf{x^3(x^2 + 4)(x + 2)(x - 2)}\end{aligned}$$

$$\begin{aligned}(7) \quad x^4 - 8x^2y^2 + 16y^4 \\ &= (x^2 - 4y^2)^2 \\ &= [(x + 2y)(x - 2y)]^2 \\ &= \mathbf{(x + 2y)^2(x - 2y)^2}\end{aligned}$$

Note Summary

- For 1. $(x+y)^2 - 3(x+y) - 10 =$
and 2. $(a+b)^2 - (c+d)^2 =$
treat $(x+y)$, $(a+b)$ and $(c+d)$ as single units.

- To factorise

$$(x+y)^2 - 3(x+y) - 10,$$

the following methods can be used:

Method (i) Let $x+y = A$,
and substitute to get
 $A^2 - 3A - 10$

Method (ii) Underline or circle the $(x+y)$ terms

$$\begin{aligned}(\underline{x+y})^2 - 3(\underline{x+y}) - 10 \\ (\underline{(x+y)})^2 - 3(\underline{(x+y)}) - 10\end{aligned}$$

and treat these as single units.

Factorisation II

Factorise the following expressions.

Ex.

$$x(a-2b) + y(a-2b) = (a-2b)(x+y)$$

$$2a(x+y) + 4(x+y) = 2(x+y)(a+2) \quad \text{☞}$$

Take out the greatest common factor, $2(x+y)$.

$$(1) \quad x(2a-b) - y(2a-b) = (2a-b)(x-y)$$

$$(2) \quad 3x(a+1) + 6(a+1) = 3(a+1)(x+2)$$

The greatest common factor is $3(a+1)$.

$$(3) \quad 4a(x+3) + 6(x+3) = 2(x+3)(2a+3)$$

$$(4) \quad 3a(x-3) + 3(x-3) = 3(x-3)(a+1)$$

$$(5) \quad 3x(a-2) - 6(a-2) = 3(a-2)(x-2)$$

Ex.

$$\begin{aligned}
 & a(x-3) + b(\underline{3-x}) \\
 & = a(x-3) - b(\underline{x-3}) \\
 & = (x-3)(a-b)
 \end{aligned}$$



Write $(3-x)$ as $-(x-3)$,
then factorise.

$$\begin{aligned}
 (6) \quad & 2(x-2) + a(2-x) \\
 & = 2(x-2) - a(x-2) \\
 & = \mathbf{(x-2)(2-a)} \\
 & [= \mathbf{-(x-2)(a-2)}]
 \end{aligned}$$

$$\begin{aligned}
 (7) \quad & x(a-b) - 2(b-a) \\
 & = x(a-b) + 2(a-b) \\
 & = \mathbf{(a-b)(x+2)}
 \end{aligned}$$

$$\begin{aligned}
 (8) \quad & 5x(2a-b) + 2y(b-2a) \\
 & = 5x(2a-b) - 2y(2a-b) \\
 & = \mathbf{(2a-b)(5x-2y)}
 \end{aligned}$$

$$\begin{aligned}
 (9) \quad & 3ab(2x-y) - 5a^2(y-2x) \\
 & = 3ab(2x-y) + 5a^2(2x-y) \\
 & = \mathbf{a(2x-y)(3b+5a)} \\
 & [= \mathbf{a(2x-y)(5a+3b)}]
 \end{aligned}$$

$$\begin{aligned}
 (10) \quad & 2b(x-2y) - 4a(2y-x) \\
 & = 2b(x-2y) + 4a(x-2y) \\
 & = \mathbf{2(x-2y)(b+2a)} \\
 & [= \mathbf{2(x-2y)(2a+b)}]
 \end{aligned}$$

Factorisation II

Factorise the following expressions.

Ex.

$$\begin{aligned}
 & x^2(a-2) + y^2(2-a) \\
 = & x^2(a-2) - y^2(a-2) && \curvearrowright (2-a) = -(a-2) \\
 = & (a-2)(x^2 - y^2) && \curvearrowright \text{Factorise } (x^2 - y^2). \\
 = & (a-2)(x+y)(x-y)
 \end{aligned}$$

- (1) $9x^2(a-b) + 4(b-a)$
 $= 9x^2(a-b) - 4(a-b)$
 $= (a-b)(9x^2 - 4)$
 $= (\mathbf{a-b})(\mathbf{3x+2})(\mathbf{3x-2})$
- (2) $x^2(a-b) + 16(b-a)$
 $= x^2(a-b) - 16(a-b)$
 $= (a-b)(x^2 - 16)$
 $= (\mathbf{a-b})(\mathbf{x+4})(\mathbf{x-4})$
- (3) $4x^2(x-2y) + 9y^2(2y-x)$
 $= 4x^2(x-2y) - 9y^2(x-2y)$
 $= (x-2y)(4x^2 - 9y^2)$
 $= (\mathbf{x-2y})(\mathbf{2x+3y})(\mathbf{2x-3y})$
- (4) $a^2(x-2y) + 4b^2(2y-x)$
 $= a^2(x-2y) - 4b^2(x-2y)$
 $= (x-2y)(a^2 - 4b^2)$
 $= (\mathbf{x-2y})(\mathbf{a+2b})(\mathbf{a-2b})$
- (5) $-a^2(3y-x) - 9(x-3y)$
 $= a^2(x-3y) - 9(x-3y)$
 $= (x-3y)(a^2 - 9)$
 $= (\mathbf{x-3y})(\mathbf{a+3})(\mathbf{a-3})$

J22b

Ex.

$$\begin{aligned}
 & x^2(a-b) + x(a-b) + 2(b-a) \\
 &= x^2(a-b) + x(a-b) - 2(a-b) \quad \curvearrowright (b-a) = -(a-b) \\
 &= (a-b)(x^2 + x - 2) \\
 &= (a-b)(x+2)(x-1)
 \end{aligned}$$

$$\begin{aligned}
 (6) \quad & x^2(a-b) + x(b-a) + 2(b-a) \\
 &= x^2(a-b) - x(a-b) - 2(a-b) \\
 &= (a-b)(x^2 - x - 2) \\
 &= \mathbf{(a-b)(x-2)(x+1)}
 \end{aligned}$$

$$\begin{aligned}
 (7) \quad & x^2(a-b) - 6x(a-b) - 9(b-a) \\
 &= x^2(a-b) - 6x(a-b) + 9(a-b) \\
 &= (a-b)(x^2 - 6x + 9) \\
 &= \mathbf{(a-b)(x-3)^2}
 \end{aligned}$$

Use the formula:

$$a^2 - 2ab + b^2 = (a-b)^2$$

$$\begin{aligned}
 (8) \quad & 2x^2(a-b) + x(b-a) + 6(b-a) \\
 &= 2x^2(a-b) - x(a-b) - 6(a-b) \\
 &= (a-b)(2x^2 - x - 6) \\
 &= \mathbf{(a-b)(2x+3)(x-2)}
 \end{aligned}$$

$$\begin{aligned}
 (9) \quad & (a-b)(x^2-5) + (b-a)(3x+5) \\
 &= (a-b)(x^2-5) - (a-b)(3x+5) \\
 &= (a-b)(x^2-5-3x-5) \\
 &= (a-b)(x^2-3x-10) \\
 &= \mathbf{(a-b)(x-5)(x+2)}
 \end{aligned}$$

$$\begin{aligned}
 (10) \quad & (a-b)(2x^2+9) - (b-a)(7x-6) \\
 &= (a-b)(2x^2+9) + (a-b)(7x-6) \\
 &= (a-b)(2x^2+9+7x-6) \\
 &= (a-b)(2x^2+7x+3) \\
 &= \mathbf{(a-b)(2x+1)(x+3)}
 \end{aligned}$$

Factorisation II

Factorise the following expressions.

Ex.

$$\begin{aligned}
 & 2(a-b) + (b-a)^2 \\
 = & 2(a-b) + (a-b)^2 \\
 = & (a-b)[2 + (a-b)] \\
 = & (a-b)(2 + a - b)
 \end{aligned}$$

$(b-a)^2 = (a-b)^2$
 The sign of the term does not change.

Note: $(b-a)^2 = [-(a-b)]^2$
 $= (-1)^2(a-b)^2$
 $= (a-b)^2$

The answer can also be written $(a-b)(a-b+2)$.

$$\begin{aligned}
 (1) \quad & 3(a-b) + (b-a)^2 \\
 & = 3(a-b) + (a-b)^2 \\
 & = (a-b)[3 + (a-b)] \\
 & = \mathbf{(a-b)(3+a-b)}
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad & 3(a-b) - (b-a)^2 \\
 & = 3(a-b) - (a-b)^2 \\
 & = (a-b)[3 - (a-b)] \\
 & = \mathbf{(a-b)(3-a+b)}
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad & (y-x)^2 + 3(x-y) \\
 & = (x-y)^2 + 3(x-y) \\
 & = (x-y)[(x-y) + 3] \\
 & = \mathbf{(x-y)(x-y+3)}
 \end{aligned}$$

[Why the sign does not change.]

If we compare

$$(b-a)^2 = b^2 - 2ba + a^2$$

$$(a-b)^2 = a^2 - 2ab + b^2$$

both give the same result.

J 23b

$$\begin{aligned}(4) \quad & 2(2y-x)^2 + 4a(x-2y) \\ &= 2(x-2y)^2 + 4a(x-2y) \\ &= 2(x-2y)[(x-2y) + 2a] \\ &= \mathbf{2(x-2y)(x-2y+2a)}\end{aligned}$$

$$\begin{aligned}(5) \quad & a(2y-x)^2 + 2a^2(x-2y) \\ &= a(x-2y)^2 + 2a^2(x-2y) \\ &= a(x-2y)[(x-2y) + 2a] \\ &= \mathbf{a(x-2y)(x-2y+2a)}\end{aligned}$$

$$\begin{aligned}(6) \quad & x(3b-2a)^2 - 2x^2(2a-3b) \\ &= x(2a-3b)^2 - 2x^2(2a-3b) \\ &= x(2a-3b)[(2a-3b) - 2x] \\ &= \mathbf{x(2a-3b)(2a-3b-2x)}\end{aligned}$$

$$\begin{aligned}(7) \quad & 2(x-y) - (y-x)^2 \\ &= 2(x-y) - (x-y)^2 \\ &= (x-y)[2 - (x-y)] \\ &= \mathbf{(x-y)(2-x+y)}\end{aligned}$$

$$\begin{aligned}(8) \quad & 2a^2(x-3y) + a(3y-x)^2 \\ &= 2a^2(x-3y) + a(x-3y)^2 \\ &= a(x-3y)[2a + (x-3y)] \\ &= \mathbf{a(x-3y)(2a+x-3y)}\end{aligned}$$

Factorisation II

Factorise the following expressions.

Ex.

$$\begin{aligned}
 & xy(x-y) + 3y(y-x)^2 \\
 = & xy(x-y) + 3y(x-y)^2 \quad \leftarrow (y-x)^2 = (x-y)^2 \\
 = & y(x-y)[x + 3(x-y)] \\
 = & y(x-y)(4x-3y)
 \end{aligned}$$

- (1) $2x(x-y) - 3(y-x)^2$
 $= 2x(x-y) - 3(x-y)^2$
 $= (x-y)[2x - 3(x-y)]$
 $= (x-y)(2x - 3x + 3y)$
 $= \mathbf{(x-y)(-x+3y)}$
 $[= \mathbf{-(x-y)(x-3y)}]$
- (2) $x(2y-x)^2 + 2x^2(x-2y)$
 $= x(x-2y)^2 + 2x^2(x-2y)$
 $= x(x-2y)[(x-2y) + 2x]$
 $= x(x-2y)(x-2y+2x)$
 $= \mathbf{x(x-2y)(3x-2y)}$
- (3) $4x^2(x-3y) - 2x(3y-x)^2$
 $= 4x^2(x-3y) - 2x(x-3y)^2$
 $= 2x(x-3y)[2x - (x-3y)]$
 $= 2x(x-3y)(2x-x+3y)$
 $= \mathbf{2x(x-3y)(x+3y)}$
- (4) $xy(x-2y) + 3y(2y-x)^2$
 $= xy(x-2y) + 3y(x-2y)^2$
 $= y(x-2y)[x + 3(x-2y)]$
 $= y(x-2y)(x+3x-6y)$
 $= y(x-2y)(4x-6y)$
 $= \mathbf{2y(x-2y)(2x-3y)}$

J 24b

$$\begin{aligned}(5) \quad & 2a^2(a-3b) + a(3b-a)^2 \\ &= 2a^2(a-3b) + a(a-3b)^2 \\ &= a(a-3b)[2a + (a-3b)] \\ &= a(a-3b)(2a+a-3b) \\ &= a(a-3b)(3a-3b) \\ &= \mathbf{3a(a-3b)(a-b)}\end{aligned}$$

$$\begin{aligned}(6) \quad & (a-b)^2(3x-5y) - (b-a)^2(x-y) \\ &= (a-b)^2(3x-5y) - (a-b)^2(x-y) \\ &= (a-b)^2[(3x-5y) - (x-y)] \\ &= (a-b)^2(3x-5y-x+y) \\ &= (a-b)^2(2x-4y) \\ &= \mathbf{2(a-b)^2(x-2y)}\end{aligned}$$

$$\begin{aligned}(7) \quad & 3x^2y(x-y) - 6xy^2(y-x)^2 \\ &= 3x^2y(x-y) - 6xy^2(x-y)^2 \\ &= 3xy(x-y)[x - 2y(x-y)] \\ &= \mathbf{3xy(x-y)(x-2xy+2y^2)}\end{aligned}$$

$$\begin{aligned}(8) \quad & 5xy^2(2y-3x) - 15x^2(3x-2y)^2 \\ &= 5xy^2(2y-3x) - 15x^2(2y-3x)^2 \\ &= 5x(2y-3x)[y^2 - 3x(2y-3x)] \\ &= 5x(2y-3x)(y^2 - 6xy + 9x^2) \\ &= \mathbf{5x(2y-3x)(y-3x)^2} \\ &[\mathbf{= -5x(3x-2y)(3x-y)^2}] \end{aligned}$$

Factorisation II

Factorise the following expressions.

Ex.

$$\begin{aligned}
 & 2(x-y)^2 + \underbrace{(y-x)^3} \\
 = & 2(x-y)^2 - \underbrace{(x-y)^3} \quad \left(\begin{array}{l} (y-x)^3 = -(x-y)^3 \\ \text{The sign of the term changes.} \end{array} \right. \\
 = & (x-y)^2 [2 - (x-y)] \\
 = & (x-y)^2 (2-x+y)
 \end{aligned}$$

Note: $(y-x)^3 = [-(x-y)]^3 = (-1)^3(x-y)^3 = -(x-y)^3$

$$\begin{aligned}
 (1) \quad & (x-y)^2 + 3(y-x)^3 \\
 & = (x-y)^2 - 3(x-y)^3 \\
 & = (x-y)^2 [1 - 3(x-y)] \\
 & = \mathbf{(x-y)^2(1-3x+3y)}
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad & x(a-b)^2 + 2y(b-a)^3 \\
 & = x(a-b)^2 - 2y(a-b)^3 \\
 & = (a-b)^2 [x - 2y(a-b)] \\
 & = \mathbf{(a-b)^2(x-2ay+2by)}
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad & 4a^2(x-3)^2 - 2(3-x)^3 \\
 & = 4a^2(x-3)^2 + 2(x-3)^3 \\
 & = 2(x-3)^2 [2a^2 + (x-3)] \\
 & = \mathbf{2(x-3)^2(2a^2+x-3)}
 \end{aligned}$$

$$\begin{aligned}
 (4) \quad & x^2y(x-3y)^2 + xy^2(3y-x)^3 \\
 & = x^2y(x-3y)^2 - xy^2(x-3y)^3 \\
 & = xy(x-3y)^2 [x - y(x-3y)] \\
 & = \mathbf{xy(x-3y)^2(x-xy+3y^2)}
 \end{aligned}$$

J25b

$$\begin{aligned}(5) \quad & 6xy^2(3y-x) - 3x^2(x-3y)^2 \\ & = 6xy^2(3y-x) - 3x^2(3y-x)^2 \\ & = 3x(3y-x)[2y^2 - x(3y-x)] \\ & = 3x(3y-x)(2y^2 - 3xy + x^2) \\ & = \mathbf{3x(3y-x)(2y-x)(y-x)} \\ & [= -\mathbf{3x(x-3y)(x-2y)(x-y)}]\end{aligned}$$

$$\begin{aligned}(6) \quad & 5x^2y(x-y)^3 - 10xy^3(y-x)^2 \\ & = 5x^2y(x-y)^3 - 10xy^3(x-y)^2 \\ & = 5xy(x-y)^2[x(x-y) - 2y^2] \\ & = 5xy(x-y)^2(x^2 - xy - 2y^2) \\ & = \mathbf{5xy(x-y)^2(x-2y)(x+y)}\end{aligned}$$

$$\begin{aligned}(7) \quad & 2x^2(x-2y)^2 + 6xy^2(2y-x) \\ & = 2x^2(2y-x)^2 + 6xy^2(2y-x) \\ & = 2x(2y-x)[x(2y-x) + 3y^2] \\ & = 2x(2y-x)(3y^2 + 2xy - x^2) \\ & = \mathbf{2x(2y-x)(3y-x)(y+x)} \\ & [= \mathbf{2x(x-2y)(x-3y)(x+y)}]\end{aligned}$$

Note Summary

A. 1. When the power is an even number, the sign of the term remains the same:

$$(b-a)^2 = (a-b)^2, \quad (b-a)^4 = (a-b)^4$$

2. When the power is an odd number, the sign of the term changes:

$$(b-a) = -(a-b), \quad (b-a)^3 = -(a-b)^3$$

B. There is often a choice in how to rewrite:

$$1. a(x-3y)^2 - b(3y-x)^3 = a(x-3y)^2 + b(\mathbf{x-3y})^3$$

$$2. a(x-3y)^2 - b(3y-x)^3 = a(\mathbf{3y-x})^2 - b(3y-x)^3$$


Either way is correct.

Factorisation II

Factorise the following expressions.

Ex.

$$\begin{aligned}
 & ax + ay + bx + by \\
 = & a(x + y) + b(x + y) \\
 = & (x + y)(a + b)
 \end{aligned}$$


ax and *ay* have *a* in common.
bx and *by* have *b* in common.

$$\begin{aligned}
 (1) \quad & ax - ay + bx - by \\
 & = a(x - y) + b(x - y) \\
 & = (x - y)(a + b)
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad & ax + ay - bx - by \\
 & = a(x + y) - b(x + y) \\
 & = (x + y)(a - b)
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad & ax - ay - bx + by \\
 & = a(x - y) - b(x - y) \\
 & = (x - y)(a - b)
 \end{aligned}$$

$$\begin{aligned}
 (4) \quad & ab + ac + bd + cd \\
 & = a(b + c) + d(b + c) \\
 & = (b + c)(a + d)
 \end{aligned}$$

$$\begin{aligned}
 (5) \quad & ab - ac + bd - cd \\
 & = a(b - c) + d(b - c) \\
 & = (b - c)(a + d)
 \end{aligned}$$

Note: In the example, we could get the same result by grouping the expression into *x* and *y* terms.

$$ax + ay + bx + by = x(a + b) + y(a + b) = (a + b)(x + y)$$

J26b

$$\begin{aligned}(6) \quad & ab + cd - ac - bd \\ &= ab - ac - bd + cd \\ &= a(b - c) - d(b - c) \\ &= \mathbf{(b - c)(a - d)}\end{aligned}$$

$$\begin{aligned}(7) \quad & ab + cd + bc + ad \\ &= ab + ad + bc + cd \\ &= a(b + d) + c(b + d) \\ &= \mathbf{(b + d)(a + c)}\end{aligned}$$

$$\begin{aligned}(8) \quad & ab + cd - bd - ac \\ &= ab - bd - ac + cd \\ &= b(a - d) - c(a - d) \\ &= \mathbf{(a - d)(b - c)}\end{aligned}$$

$$\begin{aligned}(9) \quad & ab - cd + bd - ac \\ &= ab + bd - ac - cd \\ &= b(a + d) - c(a + d) \\ &= \mathbf{(a + d)(b - c)}\end{aligned}$$

$$\begin{aligned}(10) \quad & ab - cd - bd + ac \\ &= ab - bd + ac - cd \\ &= b(a - d) + c(a - d) \\ &= \mathbf{(a - d)(b + c)}\end{aligned}$$

Factorisation II

Factorise the following expressions.

$$\begin{aligned}(1) \quad & a^2 + xy + ax + ay \\ &= a^2 + ax + ay + xy \\ &= a(a+x) + y(a+x) \\ &= \mathbf{(a+x)(a+y)}\end{aligned}$$

$$\begin{aligned}(2) \quad & a^2 - xy + ax - ay \\ &= a^2 + ax - ay - xy \\ &= a(a+x) - y(a+x) \\ &= \mathbf{(a+x)(a-y)}\end{aligned}$$

$$\begin{aligned}(3) \quad & a^2 - xy - ax + ay \\ &= a^2 - ax + ay - xy \\ &= a(a-x) + y(a-x) \\ &= \mathbf{(a-x)(a+y)}\end{aligned}$$

$$\begin{aligned}(4) \quad & 2ax + 3by + 3bx + 2ay \\ &= 2ax + 2ay + 3bx + 3by \\ &= 2a(x+y) + 3b(x+y) \\ &= \mathbf{(x+y)(2a+3b)}\end{aligned}$$

$$\begin{aligned}(5) \quad & 2ax - 3by + 3bx - 2ay \\ &= 2ax - 2ay + 3bx - 3by \\ &= 2a(x-y) + 3b(x-y) \\ &= \mathbf{(x-y)(2a+3b)}\end{aligned}$$

J27b

$$\begin{aligned}(6) \quad & a^2 + 2ax + ab + 2bx \\ & = a(a + 2x) + b(a + 2x) \\ & = \mathbf{(a + 2x)(a + b)}\end{aligned}$$

$$\begin{aligned}(7) \quad & a^2 + 2ax - ab - 2bx \\ & = a(a + 2x) - b(a + 2x) \\ & = \mathbf{(a + 2x)(a - b)}\end{aligned}$$

$$\begin{aligned}(8) \quad & 2ax - ab - a^2 + 2bx \\ & = 2ax + 2bx - a^2 - ab \\ & = 2x(a + b) - a(a + b) \\ & = \mathbf{(a + b)(2x - a)}\end{aligned}$$

$$\begin{aligned}(9) \quad & 3ax + 2bx + 3ay + 2by \\ & = 3ax + 3ay + 2bx + 2by \\ & = 3a(x + y) + 2b(x + y) \\ & = \mathbf{(x + y)(3a + 2b)}\end{aligned}$$


$$\begin{aligned}(10) \quad & 2ax - 3by - 6bx + ay \\ & = 2ax + ay - 6bx - 3by \\ & = a(2x + y) - 3b(2x + y) \\ & = \mathbf{(2x + y)(a - 3b)}\end{aligned}$$

Factorisation II

Factorise the following expressions.

Ex.

$$\begin{aligned}
 & x^3 + 4x^2 + 2x + 8 \\
 &= x^2(x+4) + 2(x+4) \\
 &= (x+4)(x^2+2)
 \end{aligned}$$


 x^3 and $4x^2$ have x^2 in common.
 $2x$ and 8 have 2 in common.

$$\begin{aligned}
 (1) \quad & x^3 - 4x^2 + 2x - 8 \\
 &= x^2(x-4) + 2(x-4) \\
 &= (x-4)(x^2+2)
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad & x^3 + 4x^2 - 2x - 8 \\
 &= x^2(x+4) - 2(x+4) \\
 &= (x+4)(x^2-2)
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad & x^3 - 2x - 4x^2 + 8 \\
 &= x(x^2-2) - 4(x^2-2) \\
 &= (x^2-2)(x-4)
 \end{aligned}$$

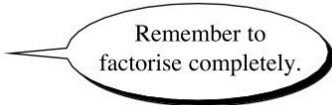
$$\begin{aligned}
 (4) \quad & x^3 + x^2 + x + 1 \\
 &= x^2(x+1) + (x+1) \\
 &= (x+1)(x^2+1)
 \end{aligned}$$

$$\begin{aligned}
 (5) \quad & x^3 - x^2 + x - 1 \\
 &= x^2(x-1) + (x-1) \\
 &= (x-1)(x^2+1)
 \end{aligned}$$

J28b

$$\begin{aligned}(6) \quad & x^3 - 2x^2 + x - 2 \\ &= x^2(x-2) + (x-2) \\ &= \mathbf{(x-2)(x^2+1)}\end{aligned}$$

$$\begin{aligned}(7) \quad & x^3 + x^2 - 4x - 4 \\ &= x^2(x+1) - 4(x+1) \\ &= (x+1)(x^2-4) \\ &= \mathbf{(x+1)(x+2)(x-2)}\end{aligned}$$



Remember to
factorise completely.

$$\begin{aligned}(8) \quad & x^3 - x - x^2 + 1 \\ &= x(x^2-1) - (x^2-1) \\ &= (x^2-1)(x-1) \\ &= \mathbf{(x+1)(x-1)^2}\end{aligned}$$

$$\begin{aligned}(9) \quad & x^2y^2 - x^2 + y^2 - 1 \\ &= x^2(y^2-1) + (y^2-1) \\ &= (y^2-1)(x^2+1) \\ &= \mathbf{(y+1)(y-1)(x^2+1)}\end{aligned}$$

$$\begin{aligned}(10) \quad & x^2y^2 - x^2 - y^2 + 1 \\ &= x^2(y^2-1) - (y^2-1) \\ &= (y^2-1)(x^2-1) \\ &= \mathbf{(y+1)(y-1)(x+1)(x-1)}\end{aligned}$$

Factorisation II

Factorise the following expressions.

$$\begin{aligned}(1) \quad & x^2 - xy - x + y \\ &= x(x - y) - (x - y) \\ &= \mathbf{(x - y)(x - 1)}\end{aligned}$$

$$\begin{aligned}(2) \quad & x^3 + x^2y - xy^2 - y^3 \\ &= x^2(x + y) - y^2(x + y) \\ &= (x + y)(x^2 - y^2) \\ &= (x + y)(x + y)(x - y) \\ &= \mathbf{(x + y)^2(x - y)}\end{aligned}$$

$$\begin{aligned}(3) \quad & xy + 1 + x + y \\ &= xy + x + y + 1 \\ &= x(y + 1) + (y + 1) \\ &= \mathbf{(y + 1)(x + 1)}\end{aligned}$$


$$\begin{aligned}(4) \quad & x^2y + y^2z + x^2z + y^3 \\ &= x^2y + x^2z + y^3 + y^2z \\ &= x^2(y + z) + y^2(y + z) \\ &= \mathbf{(y + z)(x^2 + y^2)}\end{aligned}$$

$$\begin{aligned}(5) \quad & 1 - x - x^2 + x^3 \\ &= x^3 - x^2 - x + 1 \\ &= x^2(x - 1) - (x - 1) \\ &= (x - 1)(x^2 - 1) \\ &= (x - 1)(x + 1)(x - 1) \\ &= \mathbf{(x - 1)^2(x + 1)}\end{aligned}$$

J 29b

Ex.

$$\begin{aligned}
 & a^2 + 2ab + b^2 + ax + bx \\
 &= (a+b)^2 + x(a+b) \\
 &= (a+b)[(a+b) + x] \\
 &= (a+b)(a+b+x)
 \end{aligned}$$


 $a^2 + 2ab + b^2 = (a+b)^2$

$$\begin{aligned}
 (6) \quad & a^2 - 2ab + b^2 - ax + bx \\
 &= (a-b)^2 - x(a-b) \\
 &= (a-b)[(a-b) - x] \\
 &= \mathbf{(a-b)(a-b-x)}
 \end{aligned}$$

$$\begin{aligned}
 (7) \quad & ab - ac - b^2 + 2bc - c^2 \\
 &= a(b-c) - (b^2 - 2bc + c^2) \\
 &= a(b-c) - (b-c)^2 \\
 &= (b-c)[a - (b-c)] \\
 &= \mathbf{(b-c)(a-b+c)}
 \end{aligned}$$

$$\begin{aligned}
 (8) \quad & a^2 - b^2 - ac + bc \\
 &= (a+b)(a-b) - c(a-b) \\
 &= (a-b)[(a+b) - c] \\
 &= \mathbf{(a-b)(a+b-c)}
 \end{aligned}$$

$$\begin{aligned}
 (9) \quad & ab + ac - b^2 - 2bc - c^2 \\
 &= a(b+c) - (b^2 + 2bc + c^2) \\
 &= a(b+c) - (b+c)^2 \\
 &= (b+c)[a - (b+c)] \\
 &= \mathbf{(b+c)(a-b-c)}
 \end{aligned}$$

Factorisation II

Factorise the following expressions.

$$\begin{aligned}
 (1) \quad & x(2y-x)^2 + 2x^2(x-2y) \\
 &= x(x-2y)^2 + 2x^2(x-2y) \\
 &= x(x-2y)[(x-2y) + 2x] \\
 &= x(x-2y)(x-2y+2x) \\
 &= \mathbf{x(x-2y)(3x-2y)}
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad & xy(x-2y) + 3y(2y-x)^2 \\
 &= xy(x-2y) + 3y(x-2y)^2 \\
 &= y(x-2y)[x + 3(x-2y)] \\
 &= y(x-2y)(x+3x-6y) \\
 &= y(x-2y)(4x-6y) \\
 &= \mathbf{2y(x-2y)(2x-3y)}
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad & (a-2b)(3x-5y) + (2b-a)(x-y) \\
 &= (a-2b)(3x-5y) - (a-2b)(x-y) \\
 &= (a-2b)[(3x-5y) - (x-y)] \\
 &= (a-2b)(3x-5y-x+y) \\
 &= (a-2b)(2x-4y) \\
 &= \mathbf{2(a-2b)(x-2y)}
 \end{aligned}$$

$$\begin{aligned}
 (4) \quad & 4xy^2(3y-x) - 2x^2(x-3y)^2 \\
 &= 4xy^2(3y-x) - 2x^2(3y-x)^2 \\
 &= 2x(3y-x)[2y^2 - x(3y-x)] \\
 &= 2x(3y-x)(2y^2 - 3xy + x^2) \\
 &= \mathbf{2x(3y-x)(2y-x)(y-x)} \\
 &[\mathbf{= -2x(x-3y)(x-2y)(x-y)}]
 \end{aligned}$$

$$\begin{aligned}
 (5) \quad & 5xy^2(2y-3x) - 15x^2(3x-2y)^2 \\
 &= 5xy^2(2y-3x) - 15x^2(2y-3x)^2 \\
 &= 5x(2y-3x)[y^2 - 3x(2y-3x)] \\
 &= 5x(2y-3x)(y^2 - 6xy + 9x^2) \\
 &= \mathbf{5x(2y-3x)(y-3x)^2} \\
 &[\mathbf{= -5x(3x-2y)(3x-y)^2}]
 \end{aligned}$$

J 30b

$$\begin{aligned}(6) \quad & 2xy - 4z^2 - 2xz + 4yz \\ & = 2(xy - xz + 2yz - 2z^2) \\ & = 2[x(y - z) + 2z(y - z)] \\ & = \mathbf{2(y - z)(x + 2z)}\end{aligned}$$

$$\begin{aligned}(7) \quad & 6 - 9x^2 + 12y - 18x^2y \\ & = 3(2 - 3x^2 + 4y - 6x^2y) \\ & = 3[(2 - 3x^2) + 2y(2 - 3x^2)] \\ & = \mathbf{3(2 - 3x^2)(1 + 2y)}\end{aligned}$$

$$\begin{aligned}(8) \quad & ax^2y^2 - ax^2 - ay^2 + a \\ & = a(x^2y^2 - x^2 - y^2 + 1) \\ & = a[x^2(y^2 - 1) - (y^2 - 1)] \\ & = a(y^2 - 1)(x^2 - 1) \\ & = \mathbf{a(y + 1)(y - 1)(x + 1)(x - 1)}\end{aligned}$$

Note Summary

- A.** Group like terms to get common factors.

For example, for $ax + ay + bx + by$,

a can be taken from $ax + ay$ to give $a(x + y)$

and b can be taken from $bx + by$ to give $b(x + y)$.

Then we can see that $(x + y)$ is the common factor.

- B.** We can group using a and b :

$$ax + ay + bx + by = a(x + y) + b(x + y) = (x + y)(a + b)$$

or using x and y :

$$ax + ay + bx + by = x(a + b) + y(a + b) = (a + b)(x + y)$$

Either way is correct.

Factorisation II

Factorise the following expressions.

Ex.

$$x(a-2b) + y(a-2b) = (a-2b)(x+y)$$

$$2a(x+y) + 4(x+y) = 2(x+y)(a+2)$$

Take out the greatest common factor, $2(x+y)$.

$$(1) \quad x(2a-b) - y(2a-b) = (2a-b)(x-y)$$

$$(2) \quad 3x(a+1) + 6(a+1) = 3(a+1)(x+2)$$

The greatest common factor is $3(a+1)$.

$$(3) \quad 4a(x+3) + 6(x+3) = 2(x+3)(2a+3)$$

$$(4) \quad 3a(x-3) + 3(x-3) = 3(x-3)(a+1)$$

$$(5) \quad 3x(a-2) - 6(a-2) = 3(a-2)(x-2)$$

Ex.

$$\begin{aligned}
 & a(x-3) + b(\underline{3-x}) \\
 & = a(x-3) - b(\underline{x-3}) \\
 & = (x-3)(a-b)
 \end{aligned}$$

Write $(3-x)$ as $-(x-3)$,
then factorise.

$$\begin{aligned}
 (6) \quad & 2(x-2) + a(2-x) \\
 & = 2(x-2) - a(x-2) \\
 & = \mathbf{(x-2)(2-a)} \\
 & [= \mathbf{-(x-2)(a-2)}]
 \end{aligned}$$

$$\begin{aligned}
 (7) \quad & x(a-b) - 2(b-a) \\
 & = x(a-b) + 2(a-b) \\
 & = \mathbf{(a-b)(x+2)}
 \end{aligned}$$

$$\begin{aligned}
 (8) \quad & 5x(2a-b) + 2y(b-2a) \\
 & = 5x(2a-b) - 2y(2a-b) \\
 & = \mathbf{(2a-b)(5x-2y)}
 \end{aligned}$$

$$\begin{aligned}
 (9) \quad & 3ab(2x-y) - 5a^2(y-2x) \\
 & = 3ab(2x-y) + 5a^2(2x-y) \\
 & = \mathbf{a(2x-y)(3b+5a)} \\
 & [= \mathbf{a(2x-y)(5a+3b)}]
 \end{aligned}$$

$$\begin{aligned}
 (10) \quad & 2b(x-2y) - 4a(2y-x) \\
 & = 2b(x-2y) + 4a(x-2y) \\
 & = \mathbf{2(x-2y)(b+2a)} \\
 & [= \mathbf{2(x-2y)(2a+b)}]
 \end{aligned}$$

Factorisation II

Factorise the following expressions.

Ex.

$$\begin{aligned}
 & x^2(a-2) + y^2(2-a) \\
 = & x^2(a-2) - y^2(a-2) && \curvearrowright (2-a) = -(a-2) \\
 = & (a-2)(x^2 - y^2) && \curvearrowright \text{Factorise } (x^2 - y^2). \\
 = & (a-2)(x+y)(x-y)
 \end{aligned}$$

- (1) $9x^2(a-b) + 4(b-a)$
 $= 9x^2(a-b) - 4(a-b)$
 $= (a-b)(9x^2 - 4)$
 $= (\mathbf{a-b})(\mathbf{3x+2})(\mathbf{3x-2})$
- (2) $x^2(a-b) + 16(b-a)$
 $= x^2(a-b) - 16(a-b)$
 $= (a-b)(x^2 - 16)$
 $= (\mathbf{a-b})(\mathbf{x+4})(\mathbf{x-4})$
- (3) $4x^2(x-2y) + 9y^2(2y-x)$
 $= 4x^2(x-2y) - 9y^2(x-2y)$
 $= (x-2y)(4x^2 - 9y^2)$
 $= (\mathbf{x-2y})(\mathbf{2x+3y})(\mathbf{2x-3y})$
- (4) $a^2(x-2y) + 4b^2(2y-x)$
 $= a^2(x-2y) - 4b^2(x-2y)$
 $= (x-2y)(a^2 - 4b^2)$
 $= (\mathbf{x-2y})(\mathbf{a+2b})(\mathbf{a-2b})$
- (5) $-a^2(3y-x) - 9(x-3y)$
 $= a^2(x-3y) - 9(x-3y)$
 $= (x-3y)(a^2 - 9)$
 $= (\mathbf{x-3y})(\mathbf{a+3})(\mathbf{a-3})$

J22b

Ex.

$$\begin{aligned}
 & x^2(a-b) + x(a-b) + 2(b-a) \\
 &= x^2(a-b) + x(a-b) - 2(a-b) \quad \curvearrowright (b-a) = -(a-b) \\
 &= (a-b)(x^2 + x - 2) \\
 &= (a-b)(x+2)(x-1)
 \end{aligned}$$

$$\begin{aligned}
 (6) \quad & x^2(a-b) + x(b-a) + 2(b-a) \\
 &= x^2(a-b) - x(a-b) - 2(a-b) \\
 &= (a-b)(x^2 - x - 2) \\
 &= \mathbf{(a-b)(x-2)(x+1)}
 \end{aligned}$$

$$\begin{aligned}
 (7) \quad & x^2(a-b) - 6x(a-b) - 9(b-a) \\
 &= x^2(a-b) - 6x(a-b) + 9(a-b) \\
 &= (a-b)(x^2 - 6x + 9) \\
 &= \mathbf{(a-b)(x-3)^2}
 \end{aligned}$$

Use the formula:

$$a^2 - 2ab + b^2 = (a-b)^2$$

$$\begin{aligned}
 (8) \quad & 2x^2(a-b) + x(b-a) + 6(b-a) \\
 &= 2x^2(a-b) - x(a-b) - 6(a-b) \\
 &= (a-b)(2x^2 - x - 6) \\
 &= \mathbf{(a-b)(2x+3)(x-2)}
 \end{aligned}$$

$$\begin{aligned}
 (9) \quad & (a-b)(x^2-5) + (b-a)(3x+5) \\
 &= (a-b)(x^2-5) - (a-b)(3x+5) \\
 &= (a-b)(x^2-5-3x-5) \\
 &= (a-b)(x^2-3x-10) \\
 &= \mathbf{(a-b)(x-5)(x+2)}
 \end{aligned}$$

$$\begin{aligned}
 (10) \quad & (a-b)(2x^2+9) - (b-a)(7x-6) \\
 &= (a-b)(2x^2+9) + (a-b)(7x-6) \\
 &= (a-b)(2x^2+9+7x-6) \\
 &= (a-b)(2x^2+7x+3) \\
 &= \mathbf{(a-b)(2x+1)(x+3)}
 \end{aligned}$$

Factorisation II

Factorise the following expressions.

Ex.

$$\begin{aligned}
 & 2(a-b) + (b-a)^2 \\
 = & 2(a-b) + (a-b)^2 \\
 = & (a-b)[2 + (a-b)] \\
 = & (a-b)(2 + a - b)
 \end{aligned}$$

$(b-a)^2 = (a-b)^2$
 The sign of the term does not change.

Note: $(b-a)^2 = [-(a-b)]^2$
 $= (-1)^2(a-b)^2$
 $= (a-b)^2$

The answer can also be written $(a-b)(a-b+2)$.

$$\begin{aligned}
 (1) \quad & 3(a-b) + (b-a)^2 \\
 & = 3(a-b) + (a-b)^2 \\
 & = (a-b)[3 + (a-b)] \\
 & = \mathbf{(a-b)(3+a-b)}
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad & 3(a-b) - (b-a)^2 \\
 & = 3(a-b) - (a-b)^2 \\
 & = (a-b)[3 - (a-b)] \\
 & = \mathbf{(a-b)(3-a+b)}
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad & (y-x)^2 + 3(x-y) \\
 & = (x-y)^2 + 3(x-y) \\
 & = (x-y)[(x-y) + 3] \\
 & = \mathbf{(x-y)(x-y+3)}
 \end{aligned}$$

[Why the sign does not change.]

If we compare

$$(b-a)^2 = b^2 - 2ba + a^2$$

$$(a-b)^2 = a^2 - 2ab + b^2$$

both give the same result.

J 23b

$$\begin{aligned}(4) \quad & 2(2y-x)^2 + 4a(x-2y) \\ &= 2(x-2y)^2 + 4a(x-2y) \\ &= 2(x-2y)[(x-2y) + 2a] \\ &= \mathbf{2(x-2y)(x-2y+2a)}\end{aligned}$$

$$\begin{aligned}(5) \quad & a(2y-x)^2 + 2a^2(x-2y) \\ &= a(x-2y)^2 + 2a^2(x-2y) \\ &= a(x-2y)[(x-2y) + 2a] \\ &= \mathbf{a(x-2y)(x-2y+2a)}\end{aligned}$$

$$\begin{aligned}(6) \quad & x(3b-2a)^2 - 2x^2(2a-3b) \\ &= x(2a-3b)^2 - 2x^2(2a-3b) \\ &= x(2a-3b)[(2a-3b) - 2x] \\ &= \mathbf{x(2a-3b)(2a-3b-2x)}\end{aligned}$$

$$\begin{aligned}(7) \quad & 2(x-y) - (y-x)^2 \\ &= 2(x-y) - (x-y)^2 \\ &= (x-y)[2 - (x-y)] \\ &= \mathbf{(x-y)(2-x+y)}\end{aligned}$$

$$\begin{aligned}(8) \quad & 2a^2(x-3y) + a(3y-x)^2 \\ &= 2a^2(x-3y) + a(x-3y)^2 \\ &= a(x-3y)[2a + (x-3y)] \\ &= \mathbf{a(x-3y)(2a+x-3y)}\end{aligned}$$

Factorisation II

Factorise the following expressions.

Ex.

$$\begin{aligned}
 & xy(x-y) + 3y(y-x)^2 \\
 = & xy(x-y) + 3y(x-y)^2 \quad \leftarrow (y-x)^2 = (x-y)^2 \\
 = & y(x-y)[x + 3(x-y)] \\
 = & y(x-y)(4x-3y)
 \end{aligned}$$

- (1) $2x(x-y) - 3(y-x)^2$
 $= 2x(x-y) - 3(x-y)^2$
 $= (x-y)[2x - 3(x-y)]$
 $= (x-y)(2x - 3x + 3y)$
 $= \mathbf{(x-y)(-x+3y)}$
 $[= \mathbf{-(x-y)(x-3y)}]$
- (2) $x(2y-x)^2 + 2x^2(x-2y)$
 $= x(x-2y)^2 + 2x^2(x-2y)$
 $= x(x-2y)[(x-2y) + 2x]$
 $= x(x-2y)(x-2y+2x)$
 $= \mathbf{x(x-2y)(3x-2y)}$
- (3) $4x^2(x-3y) - 2x(3y-x)^2$
 $= 4x^2(x-3y) - 2x(x-3y)^2$
 $= 2x(x-3y)[2x - (x-3y)]$
 $= 2x(x-3y)(2x-x+3y)$
 $= \mathbf{2x(x-3y)(x+3y)}$
- (4) $xy(x-2y) + 3y(2y-x)^2$
 $= xy(x-2y) + 3y(x-2y)^2$
 $= y(x-2y)[x + 3(x-2y)]$
 $= y(x-2y)(x+3x-6y)$
 $= y(x-2y)(4x-6y)$
 $= \mathbf{2y(x-2y)(2x-3y)}$

J 24b

$$\begin{aligned}(5) \quad & 2a^2(a-3b) + a(3b-a)^2 \\ &= 2a^2(a-3b) + a(a-3b)^2 \\ &= a(a-3b)[2a + (a-3b)] \\ &= a(a-3b)(2a+a-3b) \\ &= a(a-3b)(3a-3b) \\ &= \mathbf{3a(a-3b)(a-b)}\end{aligned}$$

$$\begin{aligned}(6) \quad & (a-b)^2(3x-5y) - (b-a)^2(x-y) \\ &= (a-b)^2(3x-5y) - (a-b)^2(x-y) \\ &= (a-b)^2[(3x-5y) - (x-y)] \\ &= (a-b)^2(3x-5y-x+y) \\ &= (a-b)^2(2x-4y) \\ &= \mathbf{2(a-b)^2(x-2y)}\end{aligned}$$

$$\begin{aligned}(7) \quad & 3x^2y(x-y) - 6xy^2(y-x)^2 \\ &= 3x^2y(x-y) - 6xy^2(x-y)^2 \\ &= 3xy(x-y)[x - 2y(x-y)] \\ &= \mathbf{3xy(x-y)(x-2xy+2y^2)}\end{aligned}$$

$$\begin{aligned}(8) \quad & 5xy^2(2y-3x) - 15x^2(3x-2y)^2 \\ &= 5xy^2(2y-3x) - 15x^2(2y-3x)^2 \\ &= 5x(2y-3x)[y^2 - 3x(2y-3x)] \\ &= 5x(2y-3x)(y^2 - 6xy + 9x^2) \\ &= \mathbf{5x(2y-3x)(y-3x)^2} \\ &[\mathbf{= -5x(3x-2y)(3x-y)^2}] \end{aligned}$$

Factorisation II

Factorise the following expressions.

Ex.

$$\begin{aligned}
 & 2(x-y)^2 + \underbrace{(y-x)^3} \\
 = & 2(x-y)^2 - \underbrace{(x-y)^3} \quad \left(\begin{array}{l} (y-x)^3 = -(x-y)^3 \\ \text{The sign of the term changes.} \end{array} \right. \\
 = & (x-y)^2 [2 - (x-y)] \\
 = & (x-y)^2 (2-x+y)
 \end{aligned}$$

Note: $(y-x)^3 = [-(x-y)]^3 = (-1)^3(x-y)^3 = -(x-y)^3$

$$\begin{aligned}
 (1) \quad & (x-y)^2 + 3(y-x)^3 \\
 & = (x-y)^2 - 3(x-y)^3 \\
 & = (x-y)^2 [1 - 3(x-y)] \\
 & = \mathbf{(x-y)^2(1-3x+3y)}
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad & x(a-b)^2 + 2y(b-a)^3 \\
 & = x(a-b)^2 - 2y(a-b)^3 \\
 & = (a-b)^2 [x - 2y(a-b)] \\
 & = \mathbf{(a-b)^2(x-2ay+2by)}
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad & 4a^2(x-3)^2 - 2(3-x)^3 \\
 & = 4a^2(x-3)^2 + 2(x-3)^3 \\
 & = 2(x-3)^2 [2a^2 + (x-3)] \\
 & = \mathbf{2(x-3)^2(2a^2+x-3)}
 \end{aligned}$$

$$\begin{aligned}
 (4) \quad & x^2y(x-3y)^2 + xy^2(3y-x)^3 \\
 & = x^2y(x-3y)^2 - xy^2(x-3y)^3 \\
 & = xy(x-3y)^2 [x - y(x-3y)] \\
 & = \mathbf{xy(x-3y)^2(x-xy+3y^2)}
 \end{aligned}$$

J25b

$$\begin{aligned}(5) \quad & 6xy^2(3y-x) - 3x^2(x-3y)^2 \\ & = 6xy^2(3y-x) - 3x^2(3y-x)^2 \\ & = 3x(3y-x)[2y^2 - x(3y-x)] \\ & = 3x(3y-x)(2y^2 - 3xy + x^2) \\ & = \mathbf{3x(3y-x)(2y-x)(y-x)} \\ & [= -\mathbf{3x(x-3y)(x-2y)(x-y)}]\end{aligned}$$

$$\begin{aligned}(6) \quad & 5x^2y(x-y)^3 - 10xy^3(y-x)^2 \\ & = 5x^2y(x-y)^3 - 10xy^3(x-y)^2 \\ & = 5xy(x-y)^2[x(x-y) - 2y^2] \\ & = 5xy(x-y)^2(x^2 - xy - 2y^2) \\ & = \mathbf{5xy(x-y)^2(x-2y)(x+y)}\end{aligned}$$

$$\begin{aligned}(7) \quad & 2x^2(x-2y)^2 + 6xy^2(2y-x) \\ & = 2x^2(2y-x)^2 + 6xy^2(2y-x) \\ & = 2x(2y-x)[x(2y-x) + 3y^2] \\ & = 2x(2y-x)(3y^2 + 2xy - x^2) \\ & = \mathbf{2x(2y-x)(3y-x)(y+x)} \\ & [= \mathbf{2x(x-2y)(x-3y)(x+y)}]\end{aligned}$$

Note Summary

A. 1. When the power is an even number, the sign of the term remains the same:

$$(b-a)^2 = (a-b)^2, \quad (b-a)^4 = (a-b)^4$$

2. When the power is an odd number, the sign of the term changes:

$$(b-a) = -(a-b), \quad (b-a)^3 = -(a-b)^3$$

B. There is often a choice in how to rewrite:

$$1. a(x-3y)^2 - b(3y-x)^3 = a(x-3y)^2 + b(\mathbf{x-3y})^3$$

$$2. a(x-3y)^2 - b(3y-x)^3 = a(\mathbf{3y-x})^2 - b(3y-x)^3$$


Either way is correct.

Factorisation II

Factorise the following expressions.

Ex.

$$\begin{aligned}
 & ax + ay + bx + by \\
 = & a(x + y) + b(x + y) \\
 = & (x + y)(a + b)
 \end{aligned}$$


ax and *ay* have *a* in common.
bx and *by* have *b* in common.

$$\begin{aligned}
 (1) \quad & ax - ay + bx - by \\
 & = a(x - y) + b(x - y) \\
 & = (x - y)(a + b)
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad & ax + ay - bx - by \\
 & = a(x + y) - b(x + y) \\
 & = (x + y)(a - b)
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad & ax - ay - bx + by \\
 & = a(x - y) - b(x - y) \\
 & = (x - y)(a - b)
 \end{aligned}$$

$$\begin{aligned}
 (4) \quad & ab + ac + bd + cd \\
 & = a(b + c) + d(b + c) \\
 & = (b + c)(a + d)
 \end{aligned}$$

$$\begin{aligned}
 (5) \quad & ab - ac + bd - cd \\
 & = a(b - c) + d(b - c) \\
 & = (b - c)(a + d)
 \end{aligned}$$

Note: In the example, we could get the same result by grouping the expression into *x* and *y* terms.

$$ax + ay + bx + by = x(a + b) + y(a + b) = (a + b)(x + y)$$

J26b

$$\begin{aligned}(6) \quad & ab + cd - ac - bd \\ &= ab - ac - bd + cd \\ &= a(b - c) - d(b - c) \\ &= \mathbf{(b - c)(a - d)}\end{aligned}$$

$$\begin{aligned}(7) \quad & ab + cd + bc + ad \\ &= ab + ad + bc + cd \\ &= a(b + d) + c(b + d) \\ &= \mathbf{(b + d)(a + c)}\end{aligned}$$

$$\begin{aligned}(8) \quad & ab + cd - bd - ac \\ &= ab - bd - ac + cd \\ &= b(a - d) - c(a - d) \\ &= \mathbf{(a - d)(b - c)}\end{aligned}$$

$$\begin{aligned}(9) \quad & ab - cd + bd - ac \\ &= ab + bd - ac - cd \\ &= b(a + d) - c(a + d) \\ &= \mathbf{(a + d)(b - c)}\end{aligned}$$

$$\begin{aligned}(10) \quad & ab - cd - bd + ac \\ &= ab - bd + ac - cd \\ &= b(a - d) + c(a - d) \\ &= \mathbf{(a - d)(b + c)}\end{aligned}$$

Factorisation II

Factorise the following expressions.

$$\begin{aligned}(1) \quad & a^2 + xy + ax + ay \\ &= a^2 + ax + ay + xy \\ &= a(a+x) + y(a+x) \\ &= \mathbf{(a+x)(a+y)}\end{aligned}$$

$$\begin{aligned}(2) \quad & a^2 - xy + ax - ay \\ &= a^2 + ax - ay - xy \\ &= a(a+x) - y(a+x) \\ &= \mathbf{(a+x)(a-y)}\end{aligned}$$

$$\begin{aligned}(3) \quad & a^2 - xy - ax + ay \\ &= a^2 - ax + ay - xy \\ &= a(a-x) + y(a-x) \\ &= \mathbf{(a-x)(a+y)}\end{aligned}$$

$$\begin{aligned}(4) \quad & 2ax + 3by + 3bx + 2ay \\ &= 2ax + 2ay + 3bx + 3by \\ &= 2a(x+y) + 3b(x+y) \\ &= \mathbf{(x+y)(2a+3b)}\end{aligned}$$

$$\begin{aligned}(5) \quad & 2ax - 3by + 3bx - 2ay \\ &= 2ax - 2ay + 3bx - 3by \\ &= 2a(x-y) + 3b(x-y) \\ &= \mathbf{(x-y)(2a+3b)}\end{aligned}$$

J27b

$$\begin{aligned}(6) \quad & a^2 + 2ax + ab + 2bx \\ &= a(a + 2x) + b(a + 2x) \\ &= \mathbf{(a + 2x)(a + b)}\end{aligned}$$

$$\begin{aligned}(7) \quad & a^2 + 2ax - ab - 2bx \\ &= a(a + 2x) - b(a + 2x) \\ &= \mathbf{(a + 2x)(a - b)}\end{aligned}$$

$$\begin{aligned}(8) \quad & 2ax - ab - a^2 + 2bx \\ &= 2ax + 2bx - a^2 - ab \\ &= 2x(a + b) - a(a + b) \\ &= \mathbf{(a + b)(2x - a)}\end{aligned}$$

$$\begin{aligned}(9) \quad & 3ax + 2bx + 3ay + 2by \\ &= 3ax + 3ay + 2bx + 2by \\ &= 3a(x + y) + 2b(x + y) \\ &= \mathbf{(x + y)(3a + 2b)}\end{aligned}$$


$$\begin{aligned}(10) \quad & 2ax - 3by - 6bx + ay \\ &= 2ax + ay - 6bx - 3by \\ &= a(2x + y) - 3b(2x + y) \\ &= \mathbf{(2x + y)(a - 3b)}\end{aligned}$$

Factorisation II

Factorise the following expressions.

Ex.

$$\begin{aligned}
 & x^3 + 4x^2 + 2x + 8 \\
 &= x^2(x+4) + 2(x+4) \\
 &= (x+4)(x^2+2)
 \end{aligned}$$


 x^3 and $4x^2$ have x^2 in common.
 $2x$ and 8 have 2 in common.

$$\begin{aligned}
 (1) \quad & x^3 - 4x^2 + 2x - 8 \\
 &= x^2(x-4) + 2(x-4) \\
 &= (x-4)(x^2+2)
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad & x^3 + 4x^2 - 2x - 8 \\
 &= x^2(x+4) - 2(x+4) \\
 &= (x+4)(x^2-2)
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad & x^3 - 2x - 4x^2 + 8 \\
 &= x(x^2-2) - 4(x^2-2) \\
 &= (x^2-2)(x-4)
 \end{aligned}$$

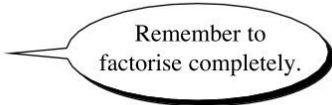
$$\begin{aligned}
 (4) \quad & x^3 + x^2 + x + 1 \\
 &= x^2(x+1) + (x+1) \\
 &= (x+1)(x^2+1)
 \end{aligned}$$

$$\begin{aligned}
 (5) \quad & x^3 - x^2 + x - 1 \\
 &= x^2(x-1) + (x-1) \\
 &= (x-1)(x^2+1)
 \end{aligned}$$

J28b

$$\begin{aligned}(6) \quad & x^3 - 2x^2 + x - 2 \\ &= x^2(x-2) + (x-2) \\ &= \mathbf{(x-2)(x^2+1)}\end{aligned}$$

$$\begin{aligned}(7) \quad & x^3 + x^2 - 4x - 4 \\ &= x^2(x+1) - 4(x+1) \\ &= (x+1)(x^2-4) \\ &= \mathbf{(x+1)(x+2)(x-2)}\end{aligned}$$



Remember to
factorise completely.

$$\begin{aligned}(8) \quad & x^3 - x - x^2 + 1 \\ &= x(x^2-1) - (x^2-1) \\ &= (x^2-1)(x-1) \\ &= \mathbf{(x+1)(x-1)^2}\end{aligned}$$

$$\begin{aligned}(9) \quad & x^2y^2 - x^2 + y^2 - 1 \\ &= x^2(y^2-1) + (y^2-1) \\ &= (y^2-1)(x^2+1) \\ &= \mathbf{(y+1)(y-1)(x^2+1)}\end{aligned}$$

$$\begin{aligned}(10) \quad & x^2y^2 - x^2 - y^2 + 1 \\ &= x^2(y^2-1) - (y^2-1) \\ &= (y^2-1)(x^2-1) \\ &= \mathbf{(y+1)(y-1)(x+1)(x-1)}\end{aligned}$$

Factorisation II

Factorise the following expressions.

$$\begin{aligned}(1) \quad & x^2 - xy - x + y \\ &= x(x - y) - (x - y) \\ &= \mathbf{(x - y)(x - 1)}\end{aligned}$$

$$\begin{aligned}(2) \quad & x^3 + x^2y - xy^2 - y^3 \\ &= x^2(x + y) - y^2(x + y) \\ &= (x + y)(x^2 - y^2) \\ &= (x + y)(x + y)(x - y) \\ &= \mathbf{(x + y)^2(x - y)}\end{aligned}$$

$$\begin{aligned}(3) \quad & xy + 1 + x + y \\ &= xy + x + y + 1 \\ &= x(y + 1) + (y + 1) \\ &= \mathbf{(y + 1)(x + 1)}\end{aligned}$$


$$\begin{aligned}(4) \quad & x^2y + y^2z + x^2z + y^3 \\ &= x^2y + x^2z + y^3 + y^2z \\ &= x^2(y + z) + y^2(y + z) \\ &= \mathbf{(y + z)(x^2 + y^2)}\end{aligned}$$

$$\begin{aligned}(5) \quad & 1 - x - x^2 + x^3 \\ &= x^3 - x^2 - x + 1 \\ &= x^2(x - 1) - (x - 1) \\ &= (x - 1)(x^2 - 1) \\ &= (x - 1)(x + 1)(x - 1) \\ &= \mathbf{(x - 1)^2(x + 1)}\end{aligned}$$

J 29b

Ex.

$$\begin{aligned}
 & a^2 + 2ab + b^2 + ax + bx \\
 &= (a+b)^2 + x(a+b) \\
 &= (a+b)[(a+b) + x] \\
 &= (a+b)(a+b+x)
 \end{aligned}$$


 $a^2 + 2ab + b^2 = (a+b)^2$

$$\begin{aligned}
 (6) \quad & a^2 - 2ab + b^2 - ax + bx \\
 &= (a-b)^2 - x(a-b) \\
 &= (a-b)[(a-b) - x] \\
 &= \mathbf{(a-b)(a-b-x)}
 \end{aligned}$$

$$\begin{aligned}
 (7) \quad & ab - ac - b^2 + 2bc - c^2 \\
 &= a(b-c) - (b^2 - 2bc + c^2) \\
 &= a(b-c) - (b-c)^2 \\
 &= (b-c)[a - (b-c)] \\
 &= \mathbf{(b-c)(a-b+c)}
 \end{aligned}$$

$$\begin{aligned}
 (8) \quad & a^2 - b^2 - ac + bc \\
 &= (a+b)(a-b) - c(a-b) \\
 &= (a-b)[(a+b) - c] \\
 &= \mathbf{(a-b)(a+b-c)}
 \end{aligned}$$

$$\begin{aligned}
 (9) \quad & ab + ac - b^2 - 2bc - c^2 \\
 &= a(b+c) - (b^2 + 2bc + c^2) \\
 &= a(b+c) - (b+c)^2 \\
 &= (b+c)[a - (b+c)] \\
 &= \mathbf{(b+c)(a-b-c)}
 \end{aligned}$$

Factorisation II

Factorise the following expressions.

$$\begin{aligned}
 (1) \quad & x(2y-x)^2 + 2x^2(x-2y) \\
 &= x(x-2y)^2 + 2x^2(x-2y) \\
 &= x(x-2y)[(x-2y) + 2x] \\
 &= x(x-2y)(x-2y+2x) \\
 &= \mathbf{x(x-2y)(3x-2y)}
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad & xy(x-2y) + 3y(2y-x)^2 \\
 &= xy(x-2y) + 3y(x-2y)^2 \\
 &= y(x-2y)[x + 3(x-2y)] \\
 &= y(x-2y)(x+3x-6y) \\
 &= y(x-2y)(4x-6y) \\
 &= \mathbf{2y(x-2y)(2x-3y)}
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad & (a-2b)(3x-5y) + (2b-a)(x-y) \\
 &= (a-2b)(3x-5y) - (a-2b)(x-y) \\
 &= (a-2b)[(3x-5y) - (x-y)] \\
 &= (a-2b)(3x-5y-x+y) \\
 &= (a-2b)(2x-4y) \\
 &= \mathbf{2(a-2b)(x-2y)}
 \end{aligned}$$

$$\begin{aligned}
 (4) \quad & 4xy^2(3y-x) - 2x^2(x-3y)^2 \\
 &= 4xy^2(3y-x) - 2x^2(3y-x)^2 \\
 &= 2x(3y-x)[2y^2 - x(3y-x)] \\
 &= 2x(3y-x)(2y^2 - 3xy + x^2) \\
 &= \mathbf{2x(3y-x)(2y-x)(y-x)} \\
 &[= -\mathbf{2x(x-3y)(x-2y)(x-y)}]
 \end{aligned}$$

$$\begin{aligned}
 (5) \quad & 5xy^2(2y-3x) - 15x^2(3x-2y)^2 \\
 &= 5xy^2(2y-3x) - 15x^2(2y-3x)^2 \\
 &= 5x(2y-3x)[y^2 - 3x(2y-3x)] \\
 &= 5x(2y-3x)(y^2 - 6xy + 9x^2) \\
 &= \mathbf{5x(2y-3x)(y-3x)^2} \\
 &[= -\mathbf{5x(3x-2y)(3x-y)^2}]
 \end{aligned}$$

J 30b

$$\begin{aligned}(6) \quad & 2xy - 4z^2 - 2xz + 4yz \\ & = 2(xy - xz + 2yz - 2z^2) \\ & = 2[x(y - z) + 2z(y - z)] \\ & = \mathbf{2(y - z)(x + 2z)}\end{aligned}$$

$$\begin{aligned}(7) \quad & 6 - 9x^2 + 12y - 18x^2y \\ & = 3(2 - 3x^2 + 4y - 6x^2y) \\ & = 3[(2 - 3x^2) + 2y(2 - 3x^2)] \\ & = \mathbf{3(2 - 3x^2)(1 + 2y)}\end{aligned}$$

$$\begin{aligned}(8) \quad & ax^2y^2 - ax^2 - ay^2 + a \\ & = a(x^2y^2 - x^2 - y^2 + 1) \\ & = a[x^2(y^2 - 1) - (y^2 - 1)] \\ & = a(y^2 - 1)(x^2 - 1) \\ & = \mathbf{a(y + 1)(y - 1)(x + 1)(x - 1)}\end{aligned}$$

Note Summary

- A.** Group like terms to get common factors.

For example, for $ax + ay + bx + by$,

a can be taken from $ax + ay$ to give $a(x + y)$

and b can be taken from $bx + by$ to give $b(x + y)$.

Then we can see that $(x + y)$ is the common factor.

- B.** We can group using a and b :

$$ax + ay + bx + by = a(x + y) + b(x + y) = (x + y)(a + b)$$

or using x and y :

$$ax + ay + bx + by = x(a + b) + y(a + b) = (a + b)(x + y)$$

Either way is correct.

Factorisation III

Factorise the following expressions.

Ex.

$$x^2 + (2a + b)x + 2ab$$

$$= (x + 2a)(x + b)$$



Look for two terms that multiply to $2ab$ and add to $(2a + b)$.
The terms are $2a$ and b .

$$(1) \quad x^2 + (a + 3b)x + 3ab$$

$$= (x + a)(x + 3b)$$

$$(2) \quad x^2 - (2a + b)x + 2ab$$

$$= (x - 2a)(x - b)$$

$$(3) \quad x^2 + (2a - b)x - 2ab$$

$$= (x + 2a)(x - b)$$

$$(4) \quad x^2 - (2a - b)x - 2ab$$

$$= (x - 2a)(x + b)$$

$$(5) \quad x^2 - (a - 3b)x - 3ab$$

$$= (x - a)(x + 3b)$$

J3|b

Ex.

$$\begin{aligned} & x^2 + 2x - a(a+2) \\ &= (x-a)[x+(a+2)] \\ &= (x-a)(x+a+2) \end{aligned}$$



Consider which factor, a or $(a+2)$, will take the negative sign, $[-]$.

$$\begin{cases} a - (a+2) = -2 \\ -a + (a+2) = 2 \end{cases}$$

$$\begin{aligned} (6) \quad & x^2 + 3x - a(a+3) \\ &= (x-a)[x+(a+3)] \\ &= \mathbf{(x-a)(x+a+3)} \end{aligned}$$

$$\begin{aligned} (7) \quad & x^2 + bx - a(a+b) \\ &= (x-a)[x+(a+b)] \\ &= \mathbf{(x-a)(x+a+b)} \end{aligned}$$

$$\begin{aligned} (8) \quad & x^2 - bx - a(a+b) \\ &= (x+a)[x-(a+b)] \\ &= \mathbf{(x+a)(x-a-b)} \end{aligned}$$

$$\begin{aligned} (9) \quad & x^2 + 2x - a(a-2) \\ &= (x+a)[x-(a-2)] \\ &= \mathbf{(x+a)(x-a+2)} \end{aligned}$$

J 32 a

Factorisation III

Factorise the following expressions.

Ex.

$$\begin{aligned} & x^2 + (2y+5)x + (y+6)(y-1) \\ &= [x + (y+6)][x + (y-1)] \\ &= (x+y+6)(x+y-1) \end{aligned}$$



When multiplied: $(y+6)(y-1)$

When added: $(2y+5)$

$$\begin{aligned} (1) \quad & x^2 + (3y+4)x + (2y+3)(y+1) \\ &= [x + (2y+3)][x + (y+1)] \\ &= (\mathbf{x+2y+3})(\mathbf{x+y+1}) \end{aligned}$$

$$\begin{aligned} (2) \quad & x^2 + (3y+5)x + (2y+3)(y+2) \\ &= [x + (2y+3)][x + (y+2)] \\ &= (\mathbf{x+2y+3})(\mathbf{x+y+2}) \end{aligned}$$

$$\begin{aligned} (3) \quad & x^2 - (2y+5)x + (y+6)(y-1) \\ &= [x - (y+6)][x - (y-1)] \\ &= (\mathbf{x-y-6})(\mathbf{x-y+1}) \end{aligned}$$

$$\begin{aligned} (4) \quad & x^2 - (3y+4)x + (2y+3)(y+1) \\ &= [x - (2y+3)][x - (y+1)] \\ &= (\mathbf{x-2y-3})(\mathbf{x-y-1}) \end{aligned}$$

J 32b

Ex.

$$\begin{aligned} & x^2 + (y+4)x - (2y+1)(3y+5) \\ &= [x - (2y+1)][x + (3y+5)] \\ &= (x - 2y - 1)(x + 3y + 5) \end{aligned}$$

Consider which factor, $(2y+1)$ or $(3y+5)$, will take the negative sign, $[-]$.

$$\begin{aligned} & [(2y+1) - (3y+5) = -y-4] \\ & [-(2y+1) + (3y+5) = y+4] \end{aligned}$$

$$\begin{aligned} (5) \quad & x^2 + (y-7)x - (y+6)(2y-1) \\ &= [x - (y+6)][x + (2y-1)] \\ &= \mathbf{(x - y - 6)(x + 2y - 1)} \end{aligned}$$

$$\begin{aligned} (6) \quad & x^2 - (y+2)x - (2y+3)(y+1) \\ &= [x - (2y+3)][x + (y+1)] \\ &= \mathbf{(x - 2y - 3)(x + y + 1)} \end{aligned}$$

$$\begin{aligned} (7) \quad & x^2 + (y+1)x - (2y+3)(y+2) \\ &= [x + (2y+3)][x - (y+2)] \\ &= \mathbf{(x + 2y + 3)(x - y - 2)} \end{aligned}$$

$$\begin{aligned} (8) \quad & x^2 - (y-4)x - (2y-1)(3y-5) \\ &= [x + (2y-1)][x - (3y-5)] \\ &= \mathbf{(x + 2y - 1)(x - 3y + 5)} \end{aligned}$$

J 33a

Factorisation III

Factorise the following expressions.

Ex.

$$\begin{aligned}
 & x^2 + (3y + 4)x + (2y^2 + 5y + 3) \\
 &= x^2 + (3y + 4)x + (2y + 3)(y + 1) \\
 &= [x + (2y + 3)][x + (y + 1)] \\
 &= (x + 2y + 3)(x + y + 1)
 \end{aligned}$$

Factorise $(2y^2 + 5y + 3)$ first.

$$\begin{aligned}
 (1) \quad & x^2 + (3y + 5)x + (2y^2 + 7y + 6) \\
 &= x^2 + (3y + 5)x + (2y + 3)(y + 2) \\
 &= [x + (2y + 3)][x + (y + 2)] \\
 &= (x + 2y + 3)(x + y + 2)
 \end{aligned}$$

Factorise the “non- x ”
terms first.

$$\begin{aligned}
 (2) \quad & x^2 - (3y + 4)x + (2y^2 + 3y - 5) \\
 &= x^2 - (3y + 4)x + (2y + 5)(y - 1) \\
 &= [x - (2y + 5)][x - (y - 1)] \\
 &= (x - 2y - 5)(x - y + 1)
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad & x^2 - (5y - 6)x + (6y^2 - 13y + 5) \\
 &= x^2 - (5y - 6)x + (3y - 5)(2y - 1) \\
 &= [x - (3y - 5)][x - (2y - 1)] \\
 &= (x - 3y + 5)(x - 2y + 1)
 \end{aligned}$$

J 33b

$$\begin{aligned}(4) \quad & x^2 + yx - (6y^2 - 5y + 1) \\ &= x^2 + yx - (3y - 1)(2y - 1) \\ &= [x + (3y - 1)][x - (2y - 1)] \\ &= \mathbf{(x + 3y - 1)(x - 2y + 1)}\end{aligned}$$

$$\begin{aligned}(5) \quad & x^2 - (y + 4)x - (2y^2 + y - 3) \\ &= x^2 - (y + 4)x - (2y + 3)(y - 1) \\ &= [x - (2y + 3)][x + (y - 1)] \\ &= \mathbf{(x - 2y - 3)(x + y - 1)}\end{aligned}$$

$$\begin{aligned}(6) \quad & x^2 - (y - 1)x - (2y^2 + 11y + 12) \\ &= x^2 - (y - 1)x - (2y + 3)(y + 4) \\ &= [x - (2y + 3)][x + (y + 4)] \\ &= \mathbf{(x - 2y - 3)(x + y + 4)}\end{aligned}$$

$$\begin{aligned}(7) \quad & x^2 - (2y + 1)x - (3y^2 - 11y + 6) \\ &= x^2 - (2y + 1)x - (3y - 2)(y - 3) \\ &= [x - (3y - 2)][x + (y - 3)] \\ &= \mathbf{(x - 3y + 2)(x + y - 3)}\end{aligned}$$

$$\begin{aligned}(8) \quad & x^2 - yx - (6y^2 - 5y + 1) \\ &= x^2 - yx - (3y - 1)(2y - 1) \\ &= [x - (3y - 1)][x + (2y - 1)] \\ &= \mathbf{(x - 3y + 1)(x + 2y - 1)}\end{aligned}$$

J 34 a

Factorisation III

Factorise the following expressions.

Ex.

$$\begin{aligned}
 & x^2 + 5xy + 6x + 6y^2 + 13y + 5 \\
 &= x^2 + (5y + 6)x + (6y^2 + 13y + 5) \\
 &= x^2 + (5y + 6)x + (3y + 5)(2y + 1) \\
 &= [x + (3y + 5)][x + (2y + 1)] \\
 &= (x + 3y + 5)(x + 2y + 1)
 \end{aligned}$$

Arrange in descending powers of x .

Factorise the “non- x ” terms first.

$$\begin{aligned}
 (1) \quad & x^2 + 3xy + 2y^2 + 5x + 7y + 6 \\
 &= x^2 + (3y + 5)x + (2y^2 + 7y + 6) \\
 &= x^2 + (3y + 5)x + (2y + 3)(y + 2) \\
 &= [x + (2y + 3)][x + (y + 2)] \\
 &= \mathbf{(x + 2y + 3)(x + y + 2)}
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad & x^2 - 3xy + 2y^2 - 5x + 7y + 6 \\
 &= x^2 - (3y + 5)x + (2y^2 + 7y + 6) \\
 &= x^2 - (3y + 5)x + (2y + 3)(y + 2) \\
 &= [x - (2y + 3)][x - (y + 2)] \\
 &= \mathbf{(x - 2y - 3)(x - y - 2)}
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad & x^2 + 3xy + 2y^2 - 5x - 7y + 6 \\
 &= x^2 + (3y - 5)x + (2y^2 - 7y + 6) \\
 &= x^2 + (3y - 5)x + (2y - 3)(y - 2) \\
 &= [x + (2y - 3)][x + (y - 2)] \\
 &= \mathbf{(x + 2y - 3)(x + y - 2)}
 \end{aligned}$$

J 34b

$$\begin{aligned}(4) \quad & x^2 - 5xy + 6y^2 + 6x - 13y + 5 \\ &= x^2 - (5y - 6)x + (6y^2 - 13y + 5) \\ &= x^2 - (5y - 6)x + (3y - 5)(2y - 1) \\ &= [x - (3y - 5)][x - (2y - 1)] \\ &= \mathbf{(x - 3y + 5)(x - 2y + 1)}\end{aligned}$$

$$\begin{aligned}(5) \quad & x^2 + 2y^2 + 3 + 3xy + 4x + 5y \\ &= x^2 + (3y + 4)x + (2y^2 + 5y + 3) \\ &= x^2 + (3y + 4)x + (2y + 3)(y + 1) \\ &= [x + (2y + 3)][x + (y + 1)] \\ &= \mathbf{(x + 2y + 3)(x + y + 1)}\end{aligned}$$

$$\begin{aligned}(6) \quad & x^2 - xy - 2y^2 - 4x + 11y - 5 \\ &= x^2 - (y + 4)x - (2y^2 - 11y + 5) \\ &= x^2 - (y + 4)x - (2y - 1)(y - 5) \\ &= [x - (2y - 1)][x + (y - 5)] \\ &= \mathbf{(x - 2y + 1)(x + y - 5)}\end{aligned}$$

$$\begin{aligned}(7) \quad & x^2 + xy - 2y^2 - x - 11y - 12 \\ &= x^2 + (y - 1)x - (2y^2 + 11y + 12) \\ &= x^2 + (y - 1)x - (2y + 3)(y + 4) \\ &= [x + (2y + 3)][x - (y + 4)] \\ &= \mathbf{(x + 2y + 3)(x - y - 4)}\end{aligned}$$

J 35a

KUMON

Factorisation III

Factorise the following expressions.

$$\begin{aligned}
 (1) \quad & x^2 + 3xy + 2y^2 + 4x + 7y + 3 \\
 &= x^2 + (3y + 4)x + (2y^2 + 7y + 3) \\
 &= x^2 + (3y + 4)x + (2y + 1)(y + 3) \\
 &= [x + (2y + 1)][x + (y + 3)] \\
 &= \mathbf{(x + 2y + 1)(x + y + 3)}
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad & x^2 - 5xy + 6y^2 - 3x + 7y + 2 \\
 &= x^2 - (5y + 3)x + (6y^2 + 7y + 2) \\
 &= x^2 - (5y + 3)x + (3y + 2)(2y + 1) \\
 &= [x - (3y + 2)][x - (2y + 1)] \\
 &= \mathbf{(x - 3y - 2)(x - 2y - 1)}
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad & x^2 - 3y^2 + 2xy + 4x - 8y - 5 \\
 &= x^2 + 2(y + 2)x - (3y^2 + 8y + 5) \\
 &= x^2 + 2(y + 2)x - (3y + 5)(y + 1) \\
 &= [x + (3y + 5)][x - (y + 1)] \\
 &= \mathbf{(x + 3y + 5)(x - y - 1)}
 \end{aligned}$$

$$\begin{aligned}
 (4) \quad & x^2 + 6x + 5 - 2y^2 - xy - 9y \\
 &= x^2 - (y - 6)x - (2y^2 + 9y - 5) \\
 &= x^2 - (y - 6)x - (2y - 1)(y + 5) \\
 &= [x - (2y - 1)][x + (y + 5)] \\
 &= \mathbf{(x - 2y + 1)(x + y + 5)}
 \end{aligned}$$

J 35b

- (5) $x^2 - 4y + 3y^2 + 4xy - 4$
 $= x^2 + 4yx + (3y^2 - 4y - 4)$
 $= x^2 + 4yx + (3y + 2)(y - 2)$
 $= [x + (3y + 2)][x + (y - 2)]$
 $= \mathbf{(x + 3y + 2)(x + y - 2)}$
- (6) $x^2 - xy - 6y^2 - x + 13y - 6$
 $= x^2 - (y + 1)x - (6y^2 - 13y + 6)$
 $= x^2 - (y + 1)x - (3y - 2)(2y - 3)$
 $= [x - (3y - 2)][x + (2y - 3)]$
 $= \mathbf{(x - 3y + 2)(x + 2y - 3)}$
- (7) $x^2 + 3y^2 - 4xy + 4x - 16y - 12$
 $= x^2 - 4(y - 1)x + (3y^2 - 16y - 12)$
 $= x^2 - 4(y - 1)x + (3y + 2)(y - 6)$
 $= [x - (3y + 2)][x - (y - 6)]$
 $= \mathbf{(x - 3y - 2)(x - y + 6)}$
- (8) $x^2 - y - 6y^2 + xy - 7x + 12$
 $= x^2 + (y - 7)x - (6y^2 + y - 12)$
 $= x^2 + (y - 7)x - (3y - 4)(2y + 3)$
 $= [x + (3y - 4)][x - (2y + 3)]$
 $= \mathbf{(x + 3y - 4)(x - 2y - 3)}$

Factorisation III

Factorise the following expressions.

Ex.

$$\begin{aligned}
 & x^2 + 4xy + 4x + 4y^2 + 8y + 4 \\
 &= x^2 + 4(y+1)x + 4(y^2 + 2y + 1) \\
 &= x^2 + 4(y+1)x + 4(y+1)^2 \\
 &= [x + 2(y+1)]^2 \\
 &= (x + 2y + 2)^2
 \end{aligned}$$

Use the formula
 $a^2 + 2ab + b^2 = (a + b)^2$.

$$\begin{aligned}
 (1) \quad & x^2 + 2xy + 2x + y^2 + 2y + 1 \\
 &= x^2 + 2(y+1)x + (y^2 + 2y + 1) \\
 &= x^2 + 2(y+1)x + (y+1)^2 \\
 &= [x + (y+1)]^2 \\
 &= (\mathbf{x + y + 1})^2
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad & x^2 - 2xy - 2x + y^2 + 2y + 1 \\
 &= x^2 - 2(y+1)x + (y^2 + 2y + 1) \\
 &= x^2 - 2(y+1)x + (y+1)^2 \\
 &= [x - (y+1)]^2 \\
 &= (\mathbf{x - y - 1})^2
 \end{aligned}$$


$$\begin{aligned}
 (3) \quad & x^2 - 4xy - 4x + 4y^2 + 8y + 4 \\
 &= x^2 - 4(y+1)x + 4(y^2 + 2y + 1) \\
 &= x^2 - 4(y+1)x + 4(y+1)^2 \\
 &= [x - 2(y+1)]^2 \\
 &= (\mathbf{x - 2y - 2})^2
 \end{aligned}$$

$$\begin{aligned}
 (4) \quad & x^2 - 6xy - 6x + 9y^2 + 18y + 9 \\
 &= x^2 - 6(y+1)x + 9(y^2 + 2y + 1) \\
 &= x^2 - 6(y+1)x + 9(y+1)^2 \\
 &= [x - 3(y+1)]^2 \\
 &= (\mathbf{x - 3y - 3})^2
 \end{aligned}$$

J 36b

Ex.

$$\begin{aligned} & x^2 + y^2 + z^2 + 2xy + 2yz + 2zx \\ &= x^2 + 2(y+z)x + (y^2 + 2yz + z^2) \\ &= x^2 + 2(y+z)x + (y+z)^2 \\ &= [x + (y+z)]^2 \\ &= (x+y+z)^2 \end{aligned}$$

 Arrange in powers of x .

$$\begin{aligned} (5) \quad & x^2 + y^2 + 9z^2 + 2xy + 6xz + 6yz \\ &= x^2 + 2(y+3z)x + (y^2 + 6yz + 9z^2) \\ &= x^2 + 2(y+3z)x + (y+3z)^2 \\ &= [x + (y+3z)]^2 \\ &= (\mathbf{x+y+3z})^2 \end{aligned}$$

$$\begin{aligned} (6) \quad & x^2 + 4y^2 + z^2 + 4xy + 2xz + 4yz \\ &= x^2 + 2(2y+z)x + (4y^2 + 4yz + z^2) \\ &= x^2 + 2(2y+z)x + (2y+z)^2 \\ &= [x + (2y+z)]^2 \\ &= (\mathbf{x+2y+z})^2 \end{aligned}$$


$$\begin{aligned} (7) \quad & x^2 + 4y^2 + 9z^2 + 4xy + 6xz + 12yz \\ &= x^2 + 2(2y+3z)x + (4y^2 + 12yz + 9z^2) \\ &= x^2 + 2(2y+3z)x + (2y+3z)^2 \\ &= [x + (2y+3z)]^2 \\ &= (\mathbf{x+2y+3z})^2 \end{aligned}$$

Factorisation III

Factorise the following expressions.

Ex.

$$\begin{aligned}
 & x^2 + 3xy + 5x + 2y^2 + 8y + 6 \\
 &= x^2 + (3y + 5)x + 2(y^2 + 4y + 3) \\
 &= x^2 + (3y + 5)x + 2(y + 3)(y + 1) \\
 &= [x + 2(y + 1)][x + (y + 3)] \\
 &= (x + 2y + 2)(x + y + 3)
 \end{aligned}$$


 When multiplied: $2(y + 3)(y + 1)$
 When added: $(3y + 5)$
 The terms are $2(y + 1)$ and $(y + 3)$.

$$\begin{aligned}
 (1) \quad & x^2 + 3xy - 3x + 2y^2 - 8y - 10 \\
 &= x^2 + 3(y - 1)x + 2(y^2 - 4y - 5) \\
 &= x^2 + 3(y - 1)x + 2(y - 5)(y + 1) \\
 &= [x + 2(y + 1)][x + (y - 5)] \\
 &= \mathbf{(x + 2y + 2)(x + y - 5)}
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad & x^2 + 3xy - 3x + 2y^2 - 2y - 4 \\
 &= x^2 + 3(y - 1)x + 2(y^2 - y - 2) \\
 &= x^2 + 3(y - 1)x + 2(y - 2)(y + 1) \\
 &= [x + 2(y - 2)][x + (y + 1)] \\
 &= \mathbf{(x + 2y - 4)(x + y + 1)}
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad & x^2 - xy - x - 2y^2 + 14y - 20 \\
 &= x^2 - (y + 1)x - 2(y^2 - 7y + 10) \\
 &= x^2 - (y + 1)x - 2(y - 5)(y - 2) \\
 &= [x - 2(y - 2)][x + (y - 5)] \\
 &= \mathbf{(x - 2y + 4)(x + y - 5)}
 \end{aligned}$$

J 37b

Ex.

$$\begin{aligned}
 & x^2 + 4xy + 4x + 3y^2 + 6y + 3 \\
 &= x^2 + 4(y+1)x + 3(y^2 + 2y + 1) \\
 &= x^2 + 4(y+1)x + 3(y+1)^2 \\
 &= [x + 3(y+1)][x + (y+1)] \quad \left\{ \begin{array}{l} \text{When multiplied: } 3(y+1)^2 \\ \text{When added: } 4(y+1) \\ \text{The terms are } 3(y+1) \text{ and } (y+1). \end{array} \right. \\
 &= (x + 3y + 3)(x + y + 1)
 \end{aligned}$$

$$\begin{aligned}
 (4) \quad & x^2 + 5xy + 5x + 6y^2 + 12y + 6 \\
 &= x^2 + 5(y+1)x + 6(y^2 + 2y + 1) \\
 &= x^2 + 5(y+1)x + 6(y+1)^2 \\
 &= [x + 3(y+1)][x + 2(y+1)] \\
 &= \mathbf{(x + 3y + 3)(x + 2y + 2)}
 \end{aligned}$$

$$\begin{aligned}
 (5) \quad & x^2 + 5xy + 5x - 6y^2 - 12y - 6 \\
 &= x^2 + 5(y+1)x - 6(y^2 + 2y + 1) \\
 &= x^2 + 5(y+1)x - 6(y+1)^2 \\
 &= [x - (y+1)][x + 6(y+1)] \\
 &= \mathbf{(x - y - 1)(x + 6y + 6)}
 \end{aligned}$$

$$\begin{aligned}
 (6) \quad & x^2 + 4xy + 4x - 12y^2 - 24y - 12 \\
 &= x^2 + 4(y+1)x - 12(y^2 + 2y + 1) \\
 &= x^2 + 4(y+1)x - 12(y+1)^2 \\
 &= [x - 2(y+1)][x + 6(y+1)] \\
 &= \mathbf{(x - 2y - 2)(x + 6y + 6)}
 \end{aligned}$$

Factorisation III

Factorise the following expressions.

Ex.

$$2x^2 + (a + 6b)x + 3ab$$

Factorise into $(2x + \Delta)(x + \square)$.

$$= (2x + a)(x + 3b)$$

Two possibilities are:

$$\left[\begin{array}{l} 2 \times a \rightarrow a \\ 1 \times 3b \rightarrow 3b \\ \hline a + 6b \end{array} \right]$$

$$\left[\begin{array}{l} 2 \times 3b \rightarrow 3b \\ 1 \times a \rightarrow 2a \\ \hline 2a + 3b \end{array} \right]$$

The first is correct.

$$(1) \quad 2x^2 - (a + 6b)x + 3ab$$

$$= (2x - a)(x - 3b)$$

$$\left[\begin{array}{l} 2 \times -a \rightarrow -a \\ 1 \times -3b \rightarrow -3b \\ \hline -(a + 6b) \end{array} \right]$$

$$(2) \quad 2x^2 + (a - 6b)x - 3ab$$

$$= (2x + a)(x - 3b)$$

$$\left[\begin{array}{l} 2 \times a \rightarrow a \\ 1 \times -3b \rightarrow -3b \\ \hline a - 6b \end{array} \right]$$

$$(3) \quad 2x^2 + (2a + 3b)x + 3ab$$

$$= (2x + 3b)(x + a)$$

$$\left[\begin{array}{l} 2 \times 3b \rightarrow 3b \\ 1 \times a \rightarrow 2a \\ \hline 2a + 3b \end{array} \right]$$

Further notes on question (1).

—Looking at $2x^2$, write the factors of 2 on the left.

$$\begin{array}{l} 2 \\ \times \\ 1 \end{array}$$

—Looking at $3ab$, write the factors of $3ab$ on the right.

$$\begin{array}{r} 2 \\ \times -a \\ 1 \times -3b \end{array}$$

—Looking at $-(a + 6b)x$, check these terms multiply to give $-(a + 6b)$.

$$\begin{array}{r} 2 \times -a \rightarrow -a \\ 1 \times -3b \rightarrow -3b \\ \hline -(a + 6b) \end{array}$$

—It is often necessary to try different combinations of factors and signs.

—Read off the answer directly.

$$\begin{array}{r} 2 \times -a \text{ gives } (2x - a) \\ 1 \times -3b \text{ gives } (x - 3b) \end{array}$$

J 38b

$$(4) \quad 2x^2 + (2a - 3b)x - 3ab \\ = (2x - 3b)(x + a)$$

$$\left[\begin{array}{r} 2 \quad \times \quad -3b \rightarrow -3b \\ 1 \quad \times \quad a \quad \rightarrow \frac{2a}{2a-3b} \end{array} \right]$$

$$(5) \quad 2x^2 - (2a - 3b)x - 3ab \\ = (2x + 3b)(x - a)$$

$$\left[\begin{array}{r} 2 \quad \times \quad 3b \rightarrow 3b \\ 1 \quad \times \quad -a \rightarrow -2a \\ \hline \quad \quad \quad -(2a-3b) \end{array} \right]$$

$$(6) \quad 2x^2 + (3a + 8)x + a(a + 4) \\ = (2x + a)(x + a + 4)$$

$$\left[\begin{array}{r} 2 \quad \times \quad a \rightarrow a \\ 1 \quad \times \quad a+4 \rightarrow \frac{2a+8}{3a+8} \end{array} \right]$$

$$(7) \quad 2x^2 + (a + 6)x - a(a + 3) \\ = (2x - a)(x + a + 3)$$

$$\left[\begin{array}{r} 2 \quad \times \quad -a \rightarrow -a \\ 1 \quad \times \quad a+3 \rightarrow \frac{2a+6}{a+6} \end{array} \right]$$

$$(8) \quad 2x^2 + (3a + 4b)x + a(a + 4b) \\ = (2x + a + 4b)(x + a)$$

$$\left[\begin{array}{r} 2 \quad \times \quad a+4b \rightarrow a+4b \\ 1 \quad \times \quad a \quad \rightarrow \frac{2a}{3a+4b} \end{array} \right]$$

Factorisation III

Factorise the following expressions.

Ex.

$$\begin{aligned}
 & 2x^2 + 5xy + 3y^2 + 6x + 7y + 4 \\
 &= 2x^2 + (5y + 6)x + (3y^2 + 7y + 4) \\
 &= 2x^2 + (5y + 6)x + (3y + 4)(y + 1) \\
 &= [2x + (3y + 4)][x + (y + 1)] \\
 &= (2x + 3y + 4)(x + y + 1)
 \end{aligned}$$

Note:
 $2 \times (y + 1) + 1 \times (3y + 4) = 5y + 6$

$$\begin{aligned}
 (1) \quad & 2x^2 + 7xy + 3y^2 + 9x + 7y + 4 \\
 &= 2x^2 + (7y + 9)x + (3y^2 + 7y + 4) \\
 &= 2x^2 + (7y + 9)x + (3y + 4)(y + 1) \\
 &= [2x + (y + 1)][x + (3y + 4)] \\
 &= (2x + y + 1)(x + 3y + 4)
 \end{aligned}$$

$$\left[\begin{array}{l} 2 \times y + 1 \rightarrow y + 1 \\ 1 \times 3y + 4 \rightarrow 6y + 8 \\ \hline 7y + 9 \end{array} \right]$$

$$\begin{aligned}
 (2) \quad & 2x^2 + 7xy + 3y^2 - 9x - 7y + 4 \\
 &= 2x^2 + (7y - 9)x + (3y^2 - 7y + 4) \\
 &= 2x^2 + (7y - 9)x + (3y - 4)(y - 1) \\
 &= [2x + (y - 1)][x + (3y - 4)] \\
 &= (2x + y - 1)(x + 3y - 4)
 \end{aligned}$$

$$\left[\begin{array}{l} 2 \times y - 1 \rightarrow y - 1 \\ 1 \times 3y - 4 \rightarrow 6y - 8 \\ \hline 7y - 9 \end{array} \right]$$

$$\begin{aligned}
 (3) \quad & 2x^2 + 8xy + 6y^2 + 11x + 13y + 5 \\
 &= 2x^2 + (8y + 11)x + (6y^2 + 13y + 5) \\
 &= 2x^2 + (8y + 11)x + (3y + 5)(2y + 1) \\
 &= [2x + (2y + 1)][x + (3y + 5)] \\
 &= (2x + 2y + 1)(x + 3y + 5)
 \end{aligned}$$

$$\left[\begin{array}{l} 2 \times 2y + 1 \rightarrow 2y + 1 \\ 1 \times 3y + 5 \rightarrow 6y + 10 \\ \hline 8y + 11 \end{array} \right]$$

J 39b

$$\begin{aligned}
 (4) \quad & 3x^2 + 7xy + 2y^2 + 11x + 7y + 6 \\
 & = 3x^2 + (7y + 11)x + (2y^2 + 7y + 6) \\
 & = 3x^2 + (7y + 11)x + (2y + 3)(y + 2) \\
 & = [3x + (y + 2)][x + (2y + 3)] \\
 & = \mathbf{(3x + y + 2)(x + 2y + 3)}
 \end{aligned}$$

$$\left[\begin{array}{r} 3 \quad \times \quad y+2 \rightarrow y+2 \\ 1 \quad \times \quad 2y+3 \rightarrow \frac{6y+9}{7y+11} \end{array} \right]$$

$$\begin{aligned}
 (5) \quad & 2x^2 - 7xy + 3y^2 - 9x + 7y + 4 \\
 & = 2x^2 - (7y + 9)x + (3y^2 + 7y + 4) \\
 & = 2x^2 - (7y + 9)x + (3y + 4)(y + 1) \\
 & = [2x - (y + 1)][x - (3y + 4)] \\
 & = \mathbf{(2x - y - 1)(x - 3y - 4)}
 \end{aligned}$$

$$\left[\begin{array}{r} 2 \quad \times \quad -(y+1) \rightarrow -y-1 \\ 1 \quad \times \quad -(3y+4) \rightarrow \frac{-6y-8}{-(7y+9)} \end{array} \right]$$

$$\begin{aligned}
 (6) \quad & 2x^2 - 8xy + 6y^2 - 11x + 13y + 5 \\
 & = 2x^2 - (8y + 11)x + (6y^2 + 13y + 5) \\
 & = 2x^2 - (8y + 11)x + (3y + 5)(2y + 1) \\
 & = [2x - (2y + 1)][x - (3y + 5)] \\
 & = \mathbf{(2x - 2y - 1)(x - 3y - 5)}
 \end{aligned}$$

$$\left[\begin{array}{r} 2 \quad \times \quad -(2y+1) \rightarrow -2y-1 \\ 1 \quad \times \quad -(3y+5) \rightarrow \frac{-6y-10}{-(8y+11)} \end{array} \right]$$

$$\begin{aligned}
 (7) \quad & 3x^2 - 7xy + 2y^2 - 11x + 7y + 6 \\
 & = 3x^2 - (7y + 11)x + (2y^2 + 7y + 6) \\
 & = 3x^2 - (7y + 11)x + (2y + 3)(y + 2) \\
 & = [3x - (y + 2)][x - (2y + 3)] \\
 & = \mathbf{(3x - y - 2)(x - 2y - 3)}
 \end{aligned}$$

$$\left[\begin{array}{r} 3 \quad \times \quad -(y+2) \rightarrow -y-2 \\ 1 \quad \times \quad -(2y+3) \rightarrow \frac{-6y-9}{-(7y+11)} \end{array} \right]$$

J40a

KUMON

Factorisation III

Factorise the following expressions.

$$\begin{aligned}
 (1) \quad & x^2 + 3(y+z)x + (y+2z)(2y+z) \\
 &= [x + (y+2z)][x + (2y+z)] \\
 &= \mathbf{(x+y+2z)(x+2y+z)}
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad & x^2 + (y-z)x - (y+2z)(2y+z) \\
 &= [x - (y+2z)][x + (2y+z)] \\
 &= \mathbf{(x-y-2z)(x+2y+z)}
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad & x^2 - (y-z)x - (y+2z)(2y+z) \\
 &= [x + (y+2z)][x - (2y+z)] \\
 &= \mathbf{(x+y+2z)(x-2y-z)}
 \end{aligned}$$

$$\begin{aligned}
 (4) \quad & a^2 - 2b^2 - 3c^2 - ab - 2ac - 5bc \\
 &= a^2 - (b+2c)a - (2b^2 + 5bc + 3c^2) \\
 &= a^2 - (b+2c)a - (2b+3c)(b+c) \\
 &= [a - (2b+3c)][a + (b+c)] \\
 &= \mathbf{(a-2b-3c)(a+b+c)}
 \end{aligned}$$

J 40b

$$\begin{aligned}
 (5) \quad & a^2 - 8b^2 + 2c^2 + 2ab + 3ac \\
 &= a^2 + (2b + 3c)a - 2(4b^2 - c^2) \\
 &= a^2 + (2b + 3c)a - 2(2b + c)(2b - c) \\
 &= [a + 2(2b + c)][a - (2b - c)] \\
 &= \mathbf{(a + 4b + 2c)(a - 2b + c)}
 \end{aligned}$$

$$\begin{aligned}
 (6) \quad & x^2 - 4y^2 + 3x - 2y + 2 \\
 &= x^2 + 3x - 2(2y^2 + y - 1) \\
 &= x^2 + 3x - 2(2y - 1)(y + 1) \\
 &= [x - (2y - 1)][x + 2(y + 1)] \\
 &= \mathbf{(x - 2y + 1)(x + 2y + 2)}
 \end{aligned}$$

$$\begin{aligned}
 (7) \quad & 2x^2 - (5a - 4b)x - (a + 2b)(3a - b) \\
 &= [2x + (a + 2b)][x - (3a - b)] \\
 &= \mathbf{(2x + a + 2b)(x - 3a + b)}
 \end{aligned}$$

$$\left[\begin{array}{l} 2 \times a + 2b \rightarrow a + 2b \\ 1 \times -(3a - b) \rightarrow \frac{-6a + 2b}{-(5a - 4b)} \end{array} \right]$$

$$\begin{aligned}
 (8) \quad & 2x^2 - 7xy + 3y^2 + 9x - 7y + 4 \\
 &= 2x^2 - (7y - 9)x + (3y^2 - 7y + 4) \\
 &= 2x^2 - (7y - 9)x + (3y - 4)(y - 1) \\
 &= [2x - (y - 1)][x - (3y - 4)] \\
 &= \mathbf{(2x - y + 1)(x - 3y + 4)}
 \end{aligned}$$

$$\left[\begin{array}{l} 2 \times -(y - 1) \rightarrow -y + 1 \\ 1 \times -(3y - 4) \rightarrow \frac{-6y + 8}{-(7y - 9)} \end{array} \right]$$

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Factorisation IV

Factorise the following expressions.

$$\begin{aligned}
 (1) \quad & 2x^2 - 3xy - 2y^2 - 2x - 11y - 12 \\
 & = 2x^2 - (3y+2)x - (2y^2 + 11y + 12) \\
 & = 2x^2 - (3y+2)x - (2y+3)(y+4) \\
 & = [2x + (y+4)][x - (2y+3)] \\
 & = \mathbf{(2x + y + 4)(x - 2y - 3)}
 \end{aligned}$$

$$\left[\begin{array}{l} 2 \quad \times \quad y+4 \quad \rightarrow \quad y+4 \\ 1 \quad \times \quad -(2y+3) \rightarrow \quad \frac{-4y-6}{-(3y+2)} \end{array} \right]$$

$$\begin{aligned}
 (2) \quad & 2x^2 + 3xy - 2y^2 + 2x - 11y - 12 \\
 & = 2x^2 + (3y+2)x - (2y^2 + 11y + 12) \\
 & = 2x^2 + (3y+2)x - (2y+3)(y+4) \\
 & = [2x - (y+4)][x + (2y+3)] \\
 & = \mathbf{(2x - y - 4)(x + 2y + 3)}
 \end{aligned}$$

$$\left[\begin{array}{l} 2 \quad \times \quad -(y+4) \rightarrow \quad -y-4 \\ 1 \quad \times \quad 2y+3 \quad \rightarrow \quad \frac{4y+6}{3y+2} \end{array} \right]$$

$$\begin{aligned}
 (3) \quad & 2x^2 + 7xy + 3y^2 + 13x + 14y + 15 \\
 & = 2x^2 + (7y+13)x + (3y^2 + 14y + 15) \\
 & = 2x^2 + (7y+13)x + (3y+5)(y+3) \\
 & = [2x + (y+3)][x + (3y+5)] \\
 & = \mathbf{(2x + y + 3)(x + 3y + 5)}
 \end{aligned}$$

$$\left[\begin{array}{l} 2 \quad \times \quad y+3 \quad \rightarrow \quad y+3 \\ 1 \quad \times \quad 3y+5 \rightarrow \quad \frac{6y+10}{7y+13} \end{array} \right]$$

J4|b

$$\begin{aligned}
 (4) \quad & 2x^2 - 7xy + 6y^2 + 7x - 11y + 3 \\
 & = 2x^2 - 7(y-1)x + (6y^2 - 11y + 3) \\
 & = 2x^2 - 7(y-1)x + (3y-1)(2y-3) \\
 & = [2x - (3y-1)][x - (2y-3)] \\
 & = \mathbf{(2x - 3y + 1)(x - 2y + 3)}
 \end{aligned}$$

$$\left[\begin{array}{r} 2 \times - (3y-1) \rightarrow -3y+1 \\ 1 \times - (2y-3) \rightarrow \frac{-4y+6}{-7(y-1)} \end{array} \right]$$

$$\begin{aligned}
 (5) \quad & 2x^2 + xy - y^2 + 3x - 3y - 2 \\
 & = 2x^2 + (y+3)x - (y^2 + 3y + 2) \\
 & = 2x^2 + (y+3)x - (y+2)(y+1) \\
 & = [2x - (y+1)][x + (y+2)] \\
 & = \mathbf{(2x - y - 1)(x + y + 2)}
 \end{aligned}$$

$$\left[\begin{array}{r} 2 \times - (y+1) \rightarrow -y-1 \\ 1 \times y+2 \rightarrow \frac{2y+4}{y+3} \end{array} \right]$$

$$\begin{aligned}
 (6) \quad & 2x^2 + xy - y^2 + 3x + 1 \\
 & = 2x^2 + (y+3)x - (y^2 - 1) \\
 & = 2x^2 + (y+3)x - (y+1)(y-1) \\
 & = [2x - (y-1)][x + (y+1)] \\
 & = \mathbf{(2x - y + 1)(x + y + 1)}
 \end{aligned}$$

$$\left[\begin{array}{r} 2 \times - (y-1) \rightarrow -y+1 \\ 1 \times y+1 \rightarrow \frac{2y+2}{y+3} \end{array} \right]$$

$$\begin{aligned}
 (7) \quad & 2x^2 - 5xy - 3y^2 - 14y - 8 \\
 & = 2x^2 - 5yx - (3y^2 + 14y + 8) \\
 & = 2x^2 - 5yx - (3y+2)(y+4) \\
 & = [2x + (y+4)][x - (3y+2)] \\
 & = \mathbf{(2x + y + 4)(x - 3y - 2)}
 \end{aligned}$$

$$\left[\begin{array}{r} 2 \times y+4 \rightarrow y+4 \\ 1 \times - (3y+2) \rightarrow \frac{-6y-4}{-5y} \end{array} \right]$$

J42a

KUMON

Factorisation IV

Arrange in descending powers of a , then factorise.

Ex.

$$\begin{aligned}
 & 2a^2 + 2b^2 + c^2 + 4ab + 3ac + 3bc \\
 &= 2a^2 + (4b + 3c)a + (2b^2 + 3bc + c^2) \\
 &= 2a^2 + (4b + 3c)a + (2b + c)(b + c) \\
 &= [2a + (2b + c)][a + (b + c)] \\
 &= (2a + 2b + c)(a + b + c)
 \end{aligned}$$

$$\left[\begin{array}{l} 2 \times \begin{array}{l} 2b+c \rightarrow 2b+c \\ b+c \rightarrow 2b+2c \\ \hline 4b+3c \end{array} \\ 1 \end{array} \right]$$

$$\begin{aligned}
 (1) \quad & 2a^2 + b^2 + 2c^2 + 3ab + 5ac + 3bc \\
 &= 2a^2 + (3b + 5c)a + (b^2 + 3bc + 2c^2) \\
 &= 2a^2 + (3b + 5c)a + (b + 2c)(b + c) \\
 &= [2a + (b + c)][a + (b + 2c)] \\
 &= (2a + b + c)(a + b + 2c)
 \end{aligned}$$

$$\left[\begin{array}{l} 2 \times \begin{array}{l} b+c \rightarrow b+c \\ b+2c \rightarrow 2b+4c \\ \hline 3b+5c \end{array} \\ 1 \end{array} \right]$$

$$\begin{aligned}
 (2) \quad & 2a^2 + 2b^2 + 2c^2 - 4ab - 5ac + 5bc \\
 &= 2a^2 - (4b + 5c)a + (2b^2 + 5bc + 2c^2) \\
 &= 2a^2 - (4b + 5c)a + (2b + c)(b + 2c) \\
 &= [2a - (2b + c)][a - (b + 2c)] \\
 &= (2a - 2b - c)(a - b - 2c)
 \end{aligned}$$

$$\left[\begin{array}{l} 2 \times \begin{array}{l} -(2b+c) \rightarrow -2b-c \\ -(b+2c) \rightarrow -2b-4c \\ \hline -(4b+5c) \end{array} \\ 1 \end{array} \right]$$

J42b

$$\begin{aligned}
 (3) \quad & 2a^2 - 3b^2 - 4c^2 + 5ab - 2ac + 8bc \\
 &= 2a^2 + (5b - 2c)a - (3b^2 - 8bc + 4c^2) \\
 &= 2a^2 + (5b - 2c)a - (3b - 2c)(b - 2c) \\
 &= [2a - (b - 2c)][a + (3b - 2c)] \\
 &= \mathbf{(2a - b + 2c)(a + 3b - 2c)}
 \end{aligned}$$

$$\left[\begin{array}{r} 2 \\ 1 \end{array} \begin{array}{l} \times \\ \times \end{array} \begin{array}{l} -(b-2c) \rightarrow -b+2c \\ 3b-2c \rightarrow \frac{6b-4c}{5b-2c} \end{array} \right]$$

$$\begin{aligned}
 (4) \quad & 3a^2 - 6b^2 - 2c^2 - 7ab + 5ac + 7bc \\
 &= 3a^2 - (7b - 5c)a - (6b^2 - 7bc + 2c^2) \\
 &= 3a^2 - (7b - 5c)a - (3b - 2c)(2b - c) \\
 &= [3a + (2b - c)][a - (3b - 2c)] \\
 &= \mathbf{(3a + 2b - c)(a - 3b + 2c)}
 \end{aligned}$$

$$\left[\begin{array}{r} 3 \\ 1 \end{array} \begin{array}{l} \times \\ \times \end{array} \begin{array}{l} 2b-c \rightarrow 2b-c \\ -(3b-2c) \rightarrow \frac{-9b+6c}{-(7b-5c)} \end{array} \right]$$

$$\begin{aligned}
 (5) \quad & 2a^2 - 2b^2 - 3c^2 + 3ab + 5ac - 5bc \\
 &= 2a^2 + (3b + 5c)a - (2b^2 + 5bc + 3c^2) \\
 &= 2a^2 + (3b + 5c)a - (2b + 3c)(b + c) \\
 &= [2a - (b + c)][a + (2b + 3c)] \\
 &= \mathbf{(2a - b - c)(a + 2b + 3c)}
 \end{aligned}$$

$$\left[\begin{array}{r} 2 \\ 1 \end{array} \begin{array}{l} \times \\ \times \end{array} \begin{array}{l} -(b+c) \rightarrow -b-c \\ 2b+3c \rightarrow \frac{4b+6c}{3b+5c} \end{array} \right]$$

$$\begin{aligned}
 (6) \quad & 2a^2 - 2b^2 - c^2 - 3ab + ac + 3bc \\
 &= 2a^2 - (3b - c)a - (2b^2 - 3bc + c^2) \\
 &= 2a^2 - (3b - c)a - (2b - c)(b - c) \\
 &= [2a + (b - c)][a - (2b - c)] \\
 &= \mathbf{(2a + b - c)(a - 2b + c)}
 \end{aligned}$$

$$\left[\begin{array}{r} 2 \\ 1 \end{array} \begin{array}{l} \times \\ \times \end{array} \begin{array}{l} b-c \rightarrow b-c \\ -(2b-c) \rightarrow \frac{-4b+2c}{-(3b-c)} \end{array} \right]$$

J43a

Factorisation IV

1. Arrange in descending powers of b , then factorise.

Ex.

$$\begin{aligned}
 & 2a^2 + 2b^2 + c^2 + 4ab + 3ac + 3bc \\
 &= 2b^2 + (4a + 3c)b + (2a^2 + 3ac + c^2) \\
 &= 2b^2 + (4a + 3c)b + (2a + c)(a + c) \\
 &= [2b + (2a + c)][b + (a + c)] \\
 &= (2b + 2a + c)(b + a + c)
 \end{aligned}$$

$$\begin{aligned}
 (1) \quad & 3a^2 + 2b^2 + 6c^2 + 7ab + 11ac + 7bc \\
 &= 2b^2 + 7(a + c)b + (3a^2 + 11ac + 6c^2) \\
 &= 2b^2 + 7(a + c)b + (3a + 2c)(a + 3c) \\
 &= [2b + (a + 3c)][b + (3a + 2c)] \\
 &= \mathbf{(2b + a + 3c)(b + 3a + 2c)}
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad & 2a^2 + 2b^2 + 2c^2 - 4ab - 5ac + 5bc \\
 &= 2b^2 - (4a - 5c)b + (2a^2 - 5ac + 2c^2) \\
 &= 2b^2 - (4a - 5c)b + (2a - c)(a - 2c) \\
 &= [2b - (2a - c)][b - (a - 2c)] \\
 &= \mathbf{(2b - 2a + c)(b - a + 2c)}
 \end{aligned}$$

Note: Looking at the example on J42a and the example above, we can see that we get the same result by arranging in powers of a , b or c .

J 43b

2. Consider $3a^2 - 5ab + 2b^2 - a - b - 10$.

Arrange (1) in powers of a ,

(2) in powers of b ,

then factorise.

(1) In powers of a ,

$$\begin{aligned} & 3a^2 - 5ab + 2b^2 - a - b - 10 \\ &= 3a^2 - (5b + 1)a + (2b^2 - b - 10) \\ &= 3a^2 - (5b + 1)a + (2b - 5)(b + 2) \\ &= [3a - (2b - 5)][a - (b + 2)] \\ &= \mathbf{(3a - 2b + 5)(a - b - 2)} \end{aligned}$$

(2) In powers of b ,

$$\begin{aligned} & 3a^2 - 5ab + 2b^2 - a - b - 10 \\ &= 2b^2 - (5a + 1)b + (3a^2 - a - 10) \\ &= 2b^2 - (5a + 1)b + (3a + 5)(a - 2) \\ &= [2b - (3a + 5)][b - (a - 2)] \\ &= \mathbf{(2b - 3a - 5)(b - a + 2)} \end{aligned}$$

Factorisation IV

1. Factorise using the difference of two squares.

(i.e. rewrite as two terms $A^2 - B^2$)

Ex.

$$\begin{aligned}
 & 4a^2 + 4ab + b^2 - c^2 \\
 &= (4a^2 + 4ab + b^2) - c^2 \\
 &= (2a + b)^2 - c^2 \\
 &= [(2a + b) + c][(2a + b) - c] \\
 &= (2a + b + c)(2a + b - c)
 \end{aligned}$$

$$\begin{aligned}
 (1) \quad & a^2 + 6ab + 9b^2 - c^2 \\
 &= (a^2 + 6ab + 9b^2) - c^2 \\
 &= (a + 3b)^2 - c^2 \\
 &= [(a + 3b) + c][(a + 3b) - c] \\
 &= (\mathbf{a + 3b + c})(\mathbf{a + 3b - c})
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad & 4a^2 - 12ab + 9b^2 - 4c^2 \\
 &= (4a^2 - 12ab + 9b^2) - 4c^2 \\
 &= (2a - 3b)^2 - 4c^2 \\
 &= [(2a - 3b) + 2c][(2a - 3b) - 2c] \\
 &= (\mathbf{2a - 3b + 2c})(\mathbf{2a - 3b - 2c})
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad & 4a^2 - b^2 + 6bc - 9c^2 \\
 &= 4a^2 - (b^2 - 6bc + 9c^2) \\
 &= 4a^2 - (b - 3c)^2 \\
 &= [2a + (b - 3c)][2a - (b - 3c)] \\
 &= (\mathbf{2a + b - 3c})(\mathbf{2a - b + 3c})
 \end{aligned}$$

J 44b

2. Consider $a^2 + 2ab + b^2 - x^2 - 6x - 9$.

Arrange (1) in powers of a ,

(2) in powers of b ,

(3) as the difference of two squares,

then factorise.

(1) In powers of a ,

$$\begin{aligned} & a^2 + 2ab + b^2 - x^2 - 6x - 9 \\ &= a^2 + 2ba + [b^2 - (x^2 + 6x + 9)] \\ &= a^2 + 2ba + [b^2 - (x + 3)^2] \\ &= a^2 + 2ba + (b + x + 3)(b - x - 3) \\ &= [a + (b + x + 3)][a + (b - x - 3)] \\ &= \mathbf{(a + b + x + 3)(a + b - x - 3)} \end{aligned}$$

(2) In powers of b ,

$$\begin{aligned} & a^2 + 2ab + b^2 - x^2 - 6x - 9 \\ &= b^2 + 2ab + [a^2 - (x^2 + 6x + 9)] \\ &= b^2 + 2ab + [a^2 - (x + 3)^2] \\ &= b^2 + 2ab + (a + x + 3)(a - x - 3) \\ &= [b + (a + x + 3)][b + (a - x - 3)] \\ &= \mathbf{(b + a + x + 3)(b + a - x - 3)} \end{aligned}$$

(3) As the difference of two squares,

$$\begin{aligned} & a^2 + 2ab + b^2 - x^2 - 6x - 9 \\ &= (a^2 + 2ab + b^2) - (x^2 + 6x + 9) \\ &= (a + b)^2 - (x + 3)^2 \\ &= [(a + b) + (x + 3)][(a + b) - (x + 3)] \\ &= \mathbf{(a + b + x + 3)(a + b - x - 3)} \end{aligned}$$

J45a

KUMON

Factorisation IV

Factorise by any method.

$$\begin{aligned}(1) \quad & x^2 - y^2 - 4x + 4 \\ &= (x^2 - 4x + 4) - y^2 \\ &= (x - 2)^2 - y^2 \\ &= [(x - 2) + y][(x - 2) - y] \\ &= \mathbf{(x + y - 2)(x - y - 2)}\end{aligned}$$

$$\begin{aligned}(2) \quad & x^2 - y^2 - 6y - 9 \\ &= x^2 - (y^2 + 6y + 9) \\ &= x^2 - (y + 3)^2 \\ &= [x + (y + 3)][x - (y + 3)] \\ &= \mathbf{(x + y + 3)(x - y - 3)}\end{aligned}$$

$$\begin{aligned}(3) \quad & 4x^2 - 12xy + 9y^2 - 9 \\ &= (4x^2 - 12xy + 9y^2) - 9 \\ &= (2x - 3y)^2 - 9 \\ &= [(2x - 3y) + 3][(2x - 3y) - 3] \\ &= \mathbf{(2x - 3y + 3)(2x - 3y - 3)}\end{aligned}$$

J 45b

$$\begin{aligned}(4) \quad & x^2 - y^2 + 4ay - 4a^2 \\ &= x^2 - (y^2 - 4ay + 4a^2) \\ &= x^2 - (y - 2a)^2 \\ &= [x + (y - 2a)][x - (y - 2a)] \\ &= \mathbf{(x + y - 2a)(x - y + 2a)}\end{aligned}$$

$$\begin{aligned}(5) \quad & x^2 - 6ax - y^2 + 9a^2 \\ &= (x^2 - 6ax + 9a^2) - y^2 \\ &= (x - 3a)^2 - y^2 \\ &= [(x - 3a) + y][(x - 3a) - y] \\ &= \mathbf{(x - 3a + y)(x - 3a - y)}\end{aligned}$$

$$\begin{aligned}(6) \quad & 9x^2 + 4yz - 4y^2 - z^2 \\ &= 9x^2 - (4y^2 - 4yz + z^2) \\ &= 9x^2 - (2y - z)^2 \\ &= [3x + (2y - z)][3x - (2y - z)] \\ &= \mathbf{(3x + 2y - z)(3x - 2y + z)}\end{aligned}$$

$$\begin{aligned}(7) \quad & x^2 + 4y^2 - z^2 - 4 + 4xy - 4z \\ &= (x^2 + 4xy + 4y^2) - (z^2 + 4z + 4) \\ &= (x + 2y)^2 - (z + 2)^2 \\ &= [(x + 2y) + (z + 2)][(x + 2y) - (z + 2)] \\ &= \mathbf{(x + 2y + z + 2)(x + 2y - z - 2)}\end{aligned}$$

J46a

KUMON

Factorisation IV

Factorise by any method.

$$\begin{aligned}
 (1) \quad & a^2 - b^2 - c^2 + d^2 + 2ad + 2bc \\
 &= (a^2 + 2ad + d^2) - (b^2 - 2bc + c^2) \\
 &= (a+d)^2 - (b-c)^2 \\
 &= [(a+d) + (b-c)][(a+d) - (b-c)] \\
 &= \mathbf{(a+d+b-c)(a+d-b+c)}
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad & 2ad - 2bc - a^2 + b^2 + c^2 - d^2 \\
 &= (b^2 - 2bc + c^2) - (a^2 - 2ad + d^2) \\
 &= (b-c)^2 - (a-d)^2 \\
 &= [(b-c) + (a-d)][(b-c) - (a-d)] \\
 &= \mathbf{(b-c+a-d)(b-c-a+d)}
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad & a^2 + b^2 + 2bc - 2ca - 2ab \\
 &= (a^2 - 2ab + b^2) + 2bc - 2ca \\
 &= (a-b)^2 + 2c(b-a) \\
 &= (a-b)^2 - 2c(a-b) \\
 &= (a-b)[(a-b) - 2c] \\
 &= \mathbf{(a-b)(a-b-2c)}
 \end{aligned}$$

J 46b

$$\begin{aligned}(4) \quad & ax^2 - bx^2 - 2bx + 2ax - 3a + 3b \\ &= (a-b)x^2 + 2(a-b)x - 3(a-b) \\ &= (a-b)(x^2 + 2x - 3) \\ &= \mathbf{(a-b)(x+3)(x-1)}\end{aligned}$$

$$\begin{aligned}(5) \quad & 2x^2 + xy - y^2 + 3x - 3y - 2 \\ &= 2x^2 + (y+3)x - (y^2 + 3y + 2) \\ &= 2x^2 + (y+3)x - (y+2)(y+1) \\ &= [2x - (y+1)][x + (y+2)] \\ &= \mathbf{(2x - y - 1)(x + y + 2)}\end{aligned}$$

$$\begin{aligned}(6) \quad & 2x^2 + xy - y^2 + 3x + 1 \\ &= 2x^2 + (y+3)x - (y^2 - 1) \\ &= 2x^2 + (y+3)x - (y+1)(y-1) \\ &= [2x - (y-1)][x + (y+1)] \\ &= \mathbf{(2x - y + 1)(x + y + 1)}\end{aligned}$$

$$\begin{aligned}(7) \quad & x^2 - 4y^2 - x + 6y - 2 \\ &= x^2 - x - 2(2y^2 - 3y + 1) \\ &= x^2 - x - 2(2y-1)(y-1) \\ &= [x - (2y-1)][x + 2(y-1)] \\ &= \mathbf{(x - 2y + 1)(x + 2y - 2)}\end{aligned}$$

Factorisation IV

Formula

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

These formulas can be called the *sum of two cubes* and the *difference of two cubes*.

Factorise the following expressions.

$$(1) \quad x^3 + 8y^3 = (x + 2y)(x^2 - 2xy + 4y^2)$$

$$(2) \quad 8x^3 - 27y^3 = (2x - 3y)(4x^2 + 6xy + 9y^2)$$

$$(3) \quad a^3 - 8b^3 = (a - 2b)(a^2 + 2ab + 4b^2)$$

$$(4) \quad 8a^3 + 27b^3 = (2a + 3b)(4a^2 - 6ab + 9b^2)$$

$$(5) \quad a^3 - 64 = (a - 4)(a^2 + 4a + 16)$$

J47b

$$(6) \quad 27a^3 - 8b^3 = (3a - 2b)(9a^2 + 6ab + 4b^2)$$

$$(7) \quad 1 - x^3 = (1 - x)(1 + x + x^2)$$

$$(8) \quad a^3 + 8b^6 = (a + 2b^2)(a^2 - 2ab^2 + 4b^4)$$

$$(9) \quad a^3 - 27b^9 = (a - 3b^3)(a^2 + 3ab^3 + 9b^6)$$

$$(10) \quad x^6 + y^9 = (x^2 + y^3)(x^4 - x^2y^3 + y^6)$$

Factorisation IV

Formula

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

Factorise the following expressions.

$$(1) \quad 125a^3 + 8b^3 = (5a + 2b)(25a^2 - 10ab + 4b^2)$$

$$(2) \quad 27x^3 - 64 = (3x - 4)(9x^2 + 12x + 16)$$

$$(3) \quad 2x^3 + 16 = 2(x^3 + 8) = 2(x + 2)(x^2 - 2x + 4)$$

$$(4) \quad 64a^4 - 27a = a(64a^3 - 27) = a(4a - 3)(16a^2 + 12a + 9)$$

$$(5) \quad 64x - x^4 = x(64 - x^3) = x(4 - x)(16 + 4x + x^2)$$

J 48b

$$\begin{aligned}(6) \quad & (a+b)^3 - 8b^3 \\ &= [(a+b) - 2b][(a+b)^2 + (a+b) \cdot (2b) + (2b)^2] \\ &= (a-b)(a^2 + 2ab + b^2 + 2ab + 2b^2 + 4b^2) \\ &= \mathbf{(a-b)(a^2 + 4ab + 7b^2)}\end{aligned}$$

$$\begin{aligned}(7) \quad & 8(x+y)^3 - y^3 \\ &= [2(x+y) - y][4(x+y)^2 + 2(x+y)y + y^2] \\ &= (2x + 2y - y)(4x^2 + 8xy + 4y^2 + 2xy + 2y^2 + y^2) \\ &= \mathbf{(2x + y)(4x^2 + 10xy + 7y^2)}\end{aligned}$$

$$\begin{aligned}(8) \quad & (a+b)^3 - b^3 \\ &= [(a+b) - b][(a+b)^2 + (a+b)b + b^2] \\ &= a(a^2 + 2ab + b^2 + ab + b^2 + b^2) \\ &= \mathbf{a(a^2 + 3ab + 3b^2)}\end{aligned}$$

$$\begin{aligned}(9) \quad & (a+b)^3 - (b-c)^3 \\ &= [(a+b) - (b-c)][(a+b)^2 + (a+b)(b-c) + (b-c)^2] \\ &= (a+c)(a^2 + 2ab + b^2 + ab - ac + b^2 - bc + b^2 - 2bc + c^2) \\ &= \mathbf{(a+c)(a^2 + 3b^2 + c^2 + 3ab - 3bc - ac)}\end{aligned}$$

Factorisation IV

Factorise the following expressions.

$$\begin{aligned}
 (1) \quad & a^6 - b^6 \\
 &= (a^2 - b^2)(a^4 + a^2b^2 + b^4) \quad \curvearrowright \quad a^6 - b^6 = (a^2)^3 - (b^2)^3 \\
 &= (a + \boxed{b})(a - \boxed{b})[(a^2 + b^2)^2 - a^2b^2] \\
 &= (\mathbf{a + b})(\mathbf{a - b})[(a^2 + b^2) + ab][(a^2 + b^2) - ab] \\
 &= (\mathbf{a + b})(\mathbf{a - b})(\mathbf{a^2 + b^2 + ab})(\mathbf{a^2 + b^2 - ab})
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad & a^6 - b^6 \\
 &= (a^3 + \boxed{b^3})(a^3 - \boxed{b^3}) \quad \curvearrowright \quad a^6 - b^6 = (a^3)^2 - (b^3)^2 \\
 &= (\mathbf{a + b})(\mathbf{a^2 - ab + b^2})(\mathbf{a - b})(\mathbf{a^2 + ab + b^2})
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad & a^6 + b^6 \\
 &= (\mathbf{a^2 + b^2})(\mathbf{a^4 - a^2b^2 + b^4})
 \end{aligned}$$

$$\begin{aligned}
 (4) \quad & a^9 - b^9 \\
 &= (a^3 - b^3)(a^6 + a^3b^3 + b^6) \\
 &= (\mathbf{a - b})(\mathbf{a^2 + ab + b^2})(\mathbf{a^6 + a^3b^3 + b^6})
 \end{aligned}$$

$$\begin{aligned}
 (5) \quad & a^{12} + b^{12} \\
 &= (\mathbf{a^4 + b^4})(\mathbf{a^8 - a^4b^4 + b^8})
 \end{aligned}$$

Note: In (1) and (2), both methods give the same answer.

J49b

Ex.

$$\begin{aligned} & (x^3 + y^3) + xy(x + y) \\ &= (x + y)(x^2 - xy + y^2) + xy(x + y) \\ &= (x + y)(x^2 + y^2) \end{aligned}$$

$$\begin{aligned} (6) \quad & xy(x - y) + x^3 - y^3 \\ &= xy(x - y) + (x - y)(x^2 + xy + y^2) \\ &= (x - y)(x^2 + 2xy + y^2) \\ &= \mathbf{(x - y)(x + y)^2} \end{aligned}$$

$$\begin{aligned} (7) \quad & a^3 + b^3 + a + b \\ &= (a + b)(a^2 - ab + b^2) + (a + b) \\ &= \mathbf{(a + b)(a^2 - ab + b^2 + 1)} \end{aligned}$$

$$\begin{aligned} (8) \quad & a^3 - b^3 - 3ab(a - b) \\ &= (a - b)(a^2 + ab + b^2) - 3ab(a - b) \\ &= (a - b)(a^2 + ab + b^2 - 3ab) \\ &= (a - b)(a^2 - 2ab + b^2) \\ &= (a - b)(a - b)^2 \\ &= \mathbf{(a - b)^3} \end{aligned}$$

$$\begin{aligned} (9) \quad & 4 - x^2 + 4x^3 - x^5 \\ &= (4 - x^2) + x^3(4 - x^2) \\ &= (4 - x^2)(1 + x^3) \\ &= \mathbf{(2 + x)(2 - x)(1 + x)(1 - x + x^2)} \end{aligned}$$

J50a

KUMON

Factorisation IV

Factorise the following expressions.

$$\begin{aligned}
 (1) \quad & 2x^2 - 7xy + 3y^2 + 9x - 7y + 4 \\
 &= 2x^2 - (7y - 9)x + (3y^2 - 7y + 4) \\
 &= 2x^2 - (7y - 9)x + (3y - 4)(y - 1) \\
 &= [2x - (y - 1)][x - (3y - 4)] \\
 &= \mathbf{(2x - y + 1)(x - 3y + 4)}
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad & 3x^2 + 7xy + 2y^2 + 11x + 7y + 6 \\
 &= 3x^2 + (7y + 11)x + (2y^2 + 7y + 6) \\
 &= 3x^2 + (7y + 11)x + (2y + 3)(y + 2) \\
 &= [3x + (y + 2)][x + (2y + 3)] \\
 &= \mathbf{(3x + y + 2)(x + 2y + 3)}
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad & 2x^2 - 2y^2 - z^2 + 3yz + zx - 3xy \\
 &= 2x^2 - (3y - z)x - (2y^2 - 3yz + z^2) \\
 &= 2x^2 - (3y - z)x - (2y - z)(y - z) \\
 &= [2x + (y - z)][x - (2y - z)] \\
 &= \mathbf{(2x + y - z)(x - 2y + z)}
 \end{aligned}$$

$$\begin{aligned}
 (4) \quad & 2a^2 + 6b^2 - 18c^2 + 3bc - 7ab \\
 &= 2a^2 - 7ba + 3(2b^2 + bc - 6c^2) \\
 &= 2a^2 - 7ba + 3(2b - 3c)(b + 2c) \\
 &= [2a - 3(b + 2c)][a - (2b - 3c)] \\
 &= \mathbf{(2a - 3b - 6c)(a - 2b + 3c)}
 \end{aligned}$$

J 50b

$$\begin{aligned}(5) \quad & a^2 + 5ab + 5a - 6b^2 - 12b - 6 \\ &= a^2 + 5(b+1)a - 6(b^2 + 2b + 1) \\ &= a^2 + 5(b+1)a - 6(b+1)^2 \\ &= [a + 6(b+1)][a - (b+1)] \\ &= \mathbf{(a + 6b + 6)(a - b - 1)}\end{aligned}$$

$$\begin{aligned}(6) \quad & a^2 - 4b^2 - 3c^2 + 8bc - 2ac \\ &= a^2 - 2ca - (4b^2 - 8bc + 3c^2) \\ &= a^2 - 2ca - (2b - 3c)(2b - c) \\ &= [a + (2b - 3c)][a - (2b - c)] \\ &= \mathbf{(a + 2b - 3c)(a - 2b + c)}\end{aligned}$$

$$\begin{aligned}(7) \quad & a^2 - 4b^2 + 4bc - c^2 \\ &= a^2 - (4b^2 - 4bc + c^2) \\ &= a^2 - (2b - c)^2 \\ &= [a + (2b - c)][a - (2b - c)] \\ &= \mathbf{(a + 2b - c)(a - 2b + c)}\end{aligned}$$

$$\begin{aligned}(8) \quad & 1 - x^2 + x^3 - x^5 \\ &= (1 - x^2) + x^3(1 - x^2) \\ &= (1 - x^2)(1 + x^3) \\ &= (1 + x)(1 - x)(1 + x)(1 - x + x^2) \\ &= \mathbf{(1 + x)^2(1 - x)(1 - x + x^2)}\end{aligned}$$

Factorisation V

Factorise the following expressions.

Ex.

$$\begin{aligned}
 & x^4 - \underline{6x^2} + 1 \\
 = & x^4 - \underline{2x^2} + 1 - \underline{4x^2} && \begin{array}{l} \curvearrowright \text{Break up } -6x^2 \text{ into two terms, } -2x^2 \text{ and } -4x^2. \\ \curvearrowright \text{Factorise } x^4 - 2x^2 + 1 \text{ and rewrite } -4x^2 \text{ as } -(2x)^2. \end{array} \\
 = & (x^2 - 1)^2 - (2x)^2 \\
 = & (x^2 - 1 + 2x)(x^2 - 1 - 2x) && \curvearrowright \text{Factorise } (\quad)^2 - (\quad)^2. \\
 = & (x^2 + 2x - 1)(x^2 - 2x - 1)
 \end{aligned}$$

$$\begin{aligned}
 (1) \quad x^4 - 11x^2 + 1 &= (x^4 - 2x^2 + 1) - 9x^2 \\
 &= (x^2 - 1)^2 - (3x)^2 \\
 &= (x^2 - 1 + 3x)(x^2 - 1 - 3x) \\
 &= (\mathbf{x^2 + 3x - 1})(\mathbf{x^2 - 3x - 1})
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad x^4 - 27x^2 + 1 &= (x^4 - 2x^2 + 1) - 25x^2 \\
 &= (x^2 - 1)^2 - (5x)^2 \\
 &= (x^2 - 1 + 5x)(x^2 - 1 - 5x) \\
 &= (\mathbf{x^2 + 5x - 1})(\mathbf{x^2 - 5x - 1})
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad x^4 - 7x^2 + 1 &= (x^4 + 2x^2 + 1) - \boxed{9x^2} \\
 &= (x^2 + 1)^2 - (3x)^2 \\
 &= (x^2 + 1 + 3x)(x^2 + 1 - 3x) \\
 &= (\mathbf{x^2 + 3x + 1})(\mathbf{x^2 - 3x + 1})
 \end{aligned}$$

J5|b

$$\begin{aligned}(4) \quad x^4 - 23x^2 + 1 &= (x^4 + 2x^2 + 1) - 25x^2 \\ &= (x^2 + 1)^2 - (5x)^2 \\ &= (x^2 + 1 + 5x)(x^2 + 1 - 5x) \\ &= \mathbf{(x^2 + 5x + 1)(x^2 - 5x + 1)}\end{aligned}$$

$$\begin{aligned}(5) \quad a^4 - 13a^2 + 4 &= (a^4 - 4a^2 + 4) - 9a^2 \\ &= (a^2 - 2)^2 - (3a)^2 \\ &= (a^2 - 2 + 3a)(a^2 - 2 - 3a) \\ &= \mathbf{(a^2 + 3a - 2)(a^2 - 3a - 2)}\end{aligned}$$

$$\begin{aligned}(6) \quad x^4 + 2x^2 + 9 &= (x^4 + 6x^2 + 9) - 4x^2 \\ &= (x^2 + 3)^2 - (2x)^2 \\ &= (x^2 + 3 + 2x)(x^2 + 3 - 2x) \\ &= \mathbf{(x^2 + 2x + 3)(x^2 - 2x + 3)}\end{aligned}$$

$$\begin{aligned}(7) \quad x^4 + x^2 + 1 &= (x^4 + 2x^2 + 1) - x^2 \\ &= (x^2 + 1)^2 - x^2 \\ &= (x^2 + 1 + x)(x^2 + 1 - x) \\ &= \mathbf{(x^2 + x + 1)(x^2 - x + 1)}\end{aligned}$$

$$\begin{aligned}(8) \quad a^4 + a^2b^2 + b^4 &= (a^4 + 2a^2b^2 + b^4) - a^2b^2 \\ &= (a^2 + b^2)^2 - (ab)^2 \\ &= (a^2 + b^2 + ab)(a^2 + b^2 - ab) \\ &= \mathbf{(a^2 + ab + b^2)(a^2 - ab + b^2)}\end{aligned}$$

J52a

KUMON

Factorisation V

Factorise the following expressions.

$$\begin{aligned}
 (1) \quad x^4 - 18x^2 + 1 &= (x^4 - 2x^2 + 1) - 16x^2 \\
 &= (x^2 - 1)^2 - (4x)^2 \\
 &= [(x^2 - 1) + 4x][(x^2 - 1) - 4x] \\
 &= (\mathbf{x^2 + 4x - 1})(\mathbf{x^2 - 4x - 1})
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad x^4 - 14x^2 + 1 &= (x^4 + 2x^2 + 1) - 16x^2 \\
 &= (x^2 + 1)^2 - (4x)^2 \\
 &= [(x^2 + 1) + 4x][(x^2 + 1) - 4x] \\
 &= (\mathbf{x^2 + 4x + 1})(\mathbf{x^2 - 4x + 1})
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad x^4 - 38x^2 + 1 &= (x^4 - 2x^2 + 1) - 36x^2 \\
 &= (x^2 - 1)^2 - (6x)^2 \\
 &= [(x^2 - 1) + 6x][(x^2 - 1) - 6x] \\
 &= (\mathbf{x^2 + 6x - 1})(\mathbf{x^2 - 6x - 1})
 \end{aligned}$$

$$\begin{aligned}
 (4) \quad x^4 - 34x^2 + 1 &= (x^4 + 2x^2 + 1) - 36x^2 \\
 &= (x^2 + 1)^2 - (6x)^2 \\
 &= [(x^2 + 1) + 6x][(x^2 + 1) - 6x] \\
 &= (\mathbf{x^2 + 6x + 1})(\mathbf{x^2 - 6x + 1})
 \end{aligned}$$

J52b

$$\begin{aligned}
 (5) \quad 4x^4 + 3x^2 + 1 &= (4x^4 + 4x^2 + 1) - x^2 \\
 &= (2x^2 + 1)^2 - x^2 \\
 &= [(2x^2 + 1) + x][(2x^2 + 1) - x] \\
 &= \mathbf{(2x^2 + x + 1)(2x^2 - x + 1)}
 \end{aligned}$$

$$\begin{aligned}
 (6) \quad 4x^4 + 11x^2 + 9 &= (4x^4 + 12x^2 + 9) - x^2 \\
 &= (2x^2 + 3)^2 - x^2 \\
 &= [(2x^2 + 3) + x][(2x^2 + 3) - x] \\
 &= \mathbf{(2x^2 + x + 3)(2x^2 - x + 3)}
 \end{aligned}$$

$$\begin{aligned}
 (7)^* \quad x^4 - 10x^2 + 9 &= (x^4 - 6x^2 + 9) - 4x^2 \\
 &= (x^2 - 3)^2 - (2x)^2 \\
 &= [(x^2 - 3) + 2x][(x^2 - 3) - 2x] \\
 &= (x^2 + 2x - 3)(x^2 - 2x - 3) \\
 &= \mathbf{(x + 3)(x - 1)(x - 3)(x + 1)}
 \end{aligned}$$

The answer has
four factors.

<p>Alternative Solution</p> $ \begin{aligned} &x^4 - 10x^2 + 9 \\ &= (x^2 - 9)(x^2 - 1) \\ &= \mathbf{(x + 3)(x - 3)(x + 1)(x - 1)} \end{aligned} $

$$\begin{aligned}
 (8) \quad 9x^4 - 13x^2 + 4 &= (9x^4 - 12x^2 + 4) - x^2 \\
 &= (3x^2 - 2)^2 - x^2 \\
 &= [(3x^2 - 2) + x][(3x^2 - 2) - x] \\
 &= (3x^2 + x - 2)(3x^2 - x - 2) \\
 &= \mathbf{(3x - 2)(x + 1)(3x + 2)(x - 1)}
 \end{aligned}$$

J53a

Factorisation V

Factorise the following expressions.

Ex.

$$\begin{aligned}
 & (\underbrace{x^2+x-6})(\underbrace{x^2+x-2})+3 && \sim \text{ is repeated.} \\
 & = (A-6)(A-2)+3 \quad [\text{where } A=x^2+x] \\
 & = A^2-8A+15 \\
 & = (A-5)(A-3) \\
 & = (x^2+x-5)(x^2+x-3)
 \end{aligned}$$

Substitute A with x^2+x .

$$\begin{aligned}
 (1) \quad & (\underbrace{x^2+x-7})(\underbrace{x^2+x-5})-8 \\
 & = (A-7)(A-5)-8 \quad [\text{where } A=x^2+x] \\
 & = A^2-12A+27 \\
 & = (A-9)(A-3) \\
 & = (x^2+x-9)(x^2+x-3)
 \end{aligned}$$

Remember to express your answer in terms of x (or whatever variable is used in the original question).

$$\begin{aligned}
 (2) \quad & (\underbrace{x^2+2x-8})(\underbrace{x^2+2x+1})-10 \\
 & = (A-8)(A+1)-10 \quad [\text{where } A=x^2+2x] \\
 & = A^2-7A-18 \\
 & = (A-9)(A+2) \\
 & = (x^2+2x-9)(x^2+2x+2)
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad & (\underbrace{x^2+5x})(\underbrace{x^2+5x+6})-16 \\
 & = A(A+6)-16 \quad [\text{where } A=x^2+5x] \\
 & = A^2+6A-16 \\
 & = (A+8)(A-2) \\
 & = (x^2+5x+8)(x^2+5x-2)
 \end{aligned}$$

J53b

Ex.

$$\begin{aligned}
 & (\underline{x^2+x-5})(\underline{x^2+2x-5}) - 12x^2 \quad \rightsquigarrow \text{ is repeated.} \\
 & = (A+x)(A+2x) - 12x^2 \quad [\text{where } A=x^2-5] \\
 & = A^2 + 3xA - 10x^2 \\
 & = (A+5x)(A-2x) \\
 & = (x^2+5x-5)(x^2-2x-5)
 \end{aligned}$$

$$\begin{aligned}
 (4) \quad & (x^2-2x-7)(x^2+3x-7) - 6x^2 \\
 & = (A-2x)(A+3x) - 6x^2 \quad [\text{where } A=x^2-7] \\
 & = A^2 + xA - 12x^2 \\
 & = (A+4x)(A-3x) \\
 & = (x^2+4x-7)(x^2-3x-7)
 \end{aligned}$$

$$\begin{aligned}
 (5) \quad & (x^2+3x+3)(x^2-4x+3) + 6x^2 \\
 & = (A+3x)(A-4x) + 6x^2 \quad [\text{where } A=x^2+3] \\
 & = A^2 - xA - 6x^2 \\
 & = (A-3x)(A+2x) \\
 & = (x^2-3x+3)(x^2+2x+3)
 \end{aligned}$$

$$\begin{aligned}
 (6)^* \quad & (x^2-7x-18)(x^2+3x-18) + 24x^2 \\
 & = (A-7x)(A+3x) + 24x^2 \quad [\text{where } A=x^2-18] \\
 & = A^2 - 4xA + 3x^2 \\
 & = (A-3x)(A-x) \\
 & = (x^2-3x-18)(x^2-x-18) \\
 & = (x-6)(x+3)(x^2-x-18)
 \end{aligned}$$

Make sure your answer is factorised completely.

J54a

Factorisation V

Factorise the following expressions.

Ex.

$$\begin{aligned}
 & (x+1)(x+2)(x+3)(x+4) - 8 \\
 &= (x^2 + 5x + 6)(x^2 + 5x + 4) - 8 \\
 &= (x^2 + 5x)^2 + 10(x^2 + 5x) + 16 \\
 &= (x^2 + 5x + 8)(x^2 + 5x + 2)
 \end{aligned}$$



Multiply out $(x+2)(x+3)$
and $(x+1)(x+4)$.

Treat $x^2 + 5x$ as a single unit
(or substitute $x^2 + 5x = A$).

$$\begin{aligned}
 (1) \quad & (x-1)(x-2)(x-3)(x-4) - 15 \\
 &= (x-1)(x-4)(x-2)(x-3) - 15 \\
 &= (x^2 - 5x + 4)(x^2 - 5x + 6) - 15 \\
 &= (x^2 - 5x)^2 + 10(x^2 - 5x) + 9 \\
 &= (x^2 - 5x + 9)(x^2 - 5x + 1)
 \end{aligned}$$

Note:

In these questions, consider carefully which terms to multiply together. The aim is to form two brackets with the same unit at the start of each, e.g.

$$(\underline{x^2 - 5x + 4})(\underline{x^2 - 5x + 6}) - 15$$

or

$$(\underline{x^2 + 5x + 6})(\underline{x^2 + 5x + 4}) - 3$$

$$\begin{aligned}
 (2) \quad & (x+1)(x+2)(x+3)(x+4) - 3 \\
 &= (x+2)(x+3)(x+1)(x+4) - 3 \\
 &= (x^2 + 5x + 6)(x^2 + 5x + 4) - 3 \\
 &= (x^2 + 5x)^2 + 10(x^2 + 5x) + 21 \\
 &= (x^2 + 5x + 7)(x^2 + 5x + 3)
 \end{aligned}$$

J54b

$$\begin{aligned}(3) \quad & (x-1)(x-3)(x+2)(x+4)+24 \\ & = (x-1)(x+2)(x-3)(x+4)+24 \\ & = (x^2+x-2)(x^2+x-12)+24 \\ & = (x^2+x)^2-14(x^2+x)+48 \\ & = (x^2+x-8)(x^2+x-6) \\ & = \mathbf{(x^2+x-8)(x+3)(x-2)}\end{aligned}$$

Make sure your answer is factorised completely.

$$\begin{aligned}(4) \quad & x(x+1)(x+2)(x+3)-15 \\ & = x(x+3)(x+1)(x+2)-15 \\ & = (x^2+3x)(x^2+3x+2)-15 \\ & = (x^2+3x)^2+2(x^2+3x)-15 \\ & = \mathbf{(x^2+3x+5)(x^2+3x-3)}\end{aligned}$$

$$\begin{aligned}(5) \quad & (x+1)(x+2)(x+3)(x+4)+1 \\ & = (x+1)(x+4)(x+2)(x+3)+1 \\ & = (x^2+5x+4)(x^2+5x+6)+1 \\ & = (x^2+5x)^2+10(x^2+5x)+25 \\ & = \mathbf{(x^2+5x+5)^2}\end{aligned}$$

J55a

Factorisation V

Factorise the following expressions.

Ex.

$$\begin{aligned}
 & x^2(b-c) + \underline{b^2(c-x)} + \underline{c^2(x-b)} \\
 = & (b-c)x^2 + b^2c - b^2x + c^2x - c^2b \\
 = & (b-c)x^2 - (b^2 - c^2)x + bc(b-c) \\
 = & (b-c)x^2 - (b+c)(b-c)x + bc(b-c) \\
 = & (b-c)[x^2 - (b+c)x + bc] \\
 = & (b-c)(x-b)(x-c)
 \end{aligned}$$

Expand the underlined terms.

Arrange in powers of x .

Take out the common factor $(b-c)$.

$$\begin{aligned}
 (1) \quad & a^2(b-c) + b^2(c-a) + c^2(a-b) \\
 = & (b-c)a^2 + b^2c - b^2a + c^2a - c^2b \\
 = & (b-c)a^2 - (b^2 - c^2)a + bc(b-c) \\
 = & (b-c)a^2 - (b+c)(b-c)a + bc(b-c) \\
 = & (b-c)[a^2 - (b+c)a + bc] \\
 = & \mathbf{(b-c)(a-b)(a-c)}
 \end{aligned}$$

The key concept in these exercises is to expand the polynomial and arrange it in descending powers of a particular variable, in this case a , before factorising.

$$\begin{aligned}
 (2) \quad & x^2(y+z) + y^2(z+x) + z^2(x+y) + 2xyz \\
 = & (y+z)x^2 + y^2z + y^2x + z^2x + z^2y + 2xyz \\
 = & (y+z)x^2 + (y^2 + 2yz + z^2)x + yz(y+z) \\
 = & (y+z)x^2 + (y+z)^2x + yz(y+z) \\
 = & (y+z)[x^2 + (y+z)x + yz] \\
 = & \mathbf{(y+z)(x+y)(x+z)}
 \end{aligned}$$

J 55b

$$\begin{aligned}(3) \quad & bc(b-c) - ca(c+a) + ab(a+b) \\ &= bc(b-c) - c^2a - ca^2 + a^2b + ab^2 \\ &= (b-c)a^2 + (b^2 - c^2)a + bc(b-c) \\ &= (b-c)a^2 + (b+c)(b-c)a + bc(b-c) \\ &= (b-c)[a^2 + (b+c)a + bc] \\ &= \mathbf{(b-c)(a+b)(a+c)}\end{aligned}$$

$$\begin{aligned}(4) \quad & a^2(b+c) + b^2(c-a) + c^2(b-a) - 2abc \\ &= (b+c)a^2 + b^2c - b^2a + c^2b - c^2a - 2abc \\ &= (b+c)a^2 - (b^2 + 2bc + c^2)a + bc(b+c) \\ &= (b+c)a^2 - (b+c)^2a + bc(b+c) \\ &= (b+c)[a^2 - (b+c)a + bc] \\ &= \mathbf{(b+c)(a-b)(a-c)}\end{aligned}$$

Factorisation V

Factorise the following expressions.

$$\begin{aligned}
 (1) \quad & x^2(b-c) + b^2(c+x) - c^2(x+b) \\
 &= (b-c)x^2 + b^2c + b^2x - c^2x - c^2b \\
 &= (b-c)x^2 + (b^2 - c^2)x + bc(b-c) \\
 &= (b-c)x^2 + (b+c)(b-c)x + bc(b-c) \\
 &= (b-c)[x^2 + (b+c)x + bc] \\
 &= \mathbf{(b-c)(x+b)(x+c)}
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad & x(b^2 - c^2) + b(c^2 - x^2) + c(x^2 - b^2) \\
 &= x(b^2 - c^2) + bc^2 - bx^2 + cx^2 - cb^2 \\
 &= -(b-c)x^2 + (b^2 - c^2)x - bc(b-c) \\
 &= -(b-c)x^2 + (b+c)(b-c)x - bc(b-c) \\
 &= -(b-c)[x^2 - (b+c)x + bc] \\
 &= \mathbf{-(b-c)(x-b)(x-c)}
 \end{aligned}$$

Alternative Solution

$$\begin{aligned}
 & x(b^2 - c^2) + b(c^2 - x^2) + c(x^2 - b^2) \\
 &= x(b^2 - c^2) + bc^2 - bx^2 + cx^2 - cb^2 \\
 &= (c-b)x^2 - (c^2 - b^2)x + bc(c-b) \\
 &= (c-b)x^2 - (c+b)(c-b)x + bc(c-b) \\
 &= (c-b)[x^2 - (b+c)x + bc] \\
 &= \mathbf{(c-b)(x-b)(x-c)}
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad & x^2(y+z) + y^2(z-x) + z^2(y-x) - 2xyz \\
 &= (y+z)x^2 + y^2z - y^2x + z^2y - z^2x - 2xyz \\
 &= (y+z)x^2 - (y^2 + 2yz + z^2)x + yz(y+z) \\
 &= (y+z)[x^2 - (y+z)x + yz] \\
 &= \mathbf{(y+z)(x-y)(x-z)}
 \end{aligned}$$

J56b

$$\begin{aligned}(4) \quad & (x+y+z)(yz+zx+xy) - xyz \\ &= [x+(y+z)][(y+z)x+yz] - xyz \\ &= (y+z)x^2 + xyz + (y+z)^2x + yz(y+z) - xyz \\ &= (y+z)x^2 + (y+z)^2x + yz(y+z) \\ &= (y+z)[x^2 + (y+z)x + yz] \\ &= \mathbf{(y+z)(x+y)(x+z)}\end{aligned}$$

In this step,
treating $(y+z)$ as a
single unit
makes the expansion
much simpler.

$$\begin{aligned}(5) \quad & a^2b - a^2c - ac^2 - ab^2 - b^2c + bc^2 + 2abc \\ &= (b-c)a^2 - (b^2 - 2bc + c^2)a - bc(b-c) \\ &= (b-c)a^2 - (b-c)^2a - bc(b-c) \\ &= (b-c)[a^2 - (b-c)a - bc] \\ &= \mathbf{(b-c)(a-b)(a+c)}\end{aligned}$$

Factorisation V

Factorise the following expressions.

Ex.

$$x^2 - y^2 + 2yz - 2zx - 4x + 2y + 2z + 3$$

Arrange in powers of x .

[Sol]

$$\begin{aligned} & x^2 - y^2 + 2yz - 2zx - 4x + 2y + 2z + 3 \\ &= x^2 + (-2z - 4)x - [y^2 - (2z + 2)y - (2z + 3)] \quad \curvearrowright \text{Arrange in descending} \\ &= x^2 + (-2z - 4)x - (y + 1)[y - (2z + 3)] \quad \text{powers of } x. \\ &= [x - (y + 1)][x + (y - 2z - 3)] \\ &= (x - y - 1)(x + y - 2z - 3) \end{aligned}$$

$$(1) \quad x^2 - y^2 + 2zx + 2yz + 2y - 2z - 1$$

Arrange in powers of x .

$$\begin{aligned} \text{[Sol]} \quad & x^2 - y^2 + 2zx + 2yz + 2y - 2z - 1 \\ &= x^2 + 2zx - [y^2 - (2z + 2)y + (2z + 1)] \quad \curvearrowright \text{See foot note.} \\ &= x^2 + 2zx - (y - 1)[y - (2z + 1)] \\ &= [x + (y - 1)][x - (y - 2z - 1)] \\ &= (x + y - 1)(x - y + 2z + 1) \end{aligned}$$

Note: To factorise $y^2 - (2z + 2)y + (2z + 1)$, look for two terms that multiply to $(2z + 1)$ and add to $-(2z + 2)$. The terms are -1 and $-(2z + 1)$.

J57b

Ex.

$$x^2 - y^2 + 2yz - 2zx - 4x + 2y + 2z + 3$$

Arrange in powers of z .

[Sol]

$$\begin{aligned} & x^2 - y^2 + 2yz - 2zx - 4x + 2y + 2z + 3 \\ &= 2(y - x + 1)z + (x^2 - y^2 - 4x + 2y + 3) && \curvearrowright \text{Arrange in powers of } z. \\ &= 2(y - x + 1)z + [x^2 - 4x - (y^2 - 2y - 3)] \\ &= 2(y - x + 1)z + [x^2 - 4x - (y - 3)(y + 1)] \\ &= 2(y - x + 1)z + [x + (y - 3)][x - (y + 1)] \\ &= -2(x - y - 1)z + (x + y - 3)(x - y - 1) \\ &= (x - y - 1)(x + y - 2z - 3) && \curvearrowright \text{Take out the common} \\ & && \text{factor } (x - y - 1). \end{aligned}$$

(2) $x^2 - y^2 + 2zx + 2yz + 2y - 2z - 1$

Arrange in powers of z .

$$\begin{aligned} \text{[Sol]} \quad & x^2 - y^2 + 2zx + 2yz + 2y - 2z - 1 \\ &= 2(x + y - 1)z + (x^2 - y^2 + 2y - 1) \\ &= 2(x + y - 1)z + [x^2 - (y^2 - 2y + 1)] \\ &= 2(x + y - 1)z + [x^2 - (y - 1)^2] \\ &= 2(x + y - 1)z + [x + (y - 1)][x - (y - 1)] \\ &= 2(x + y - 1)z + (x + y - 1)(x - y + 1) \\ &= (x + y - 1)[2z + (x - y + 1)] \\ &= \mathbf{(x + y - 1)(x - y + 2z + 1)} \end{aligned}$$

J58a

KUMON

Factorisation V

1. $ab + 2ac + 3b^2 + 6bc - 5a - 13b + 4c - 10$

Arrange (1) in powers of a ,

(2) in powers of b ,

and factorise.

(1) In powers of a ,

$$\begin{aligned} & ab + 2ac + 3b^2 + 6bc - 5a - 13b + 4c - 10 \\ &= (b + 2c - 5)a + (3b^2 + 6bc - 13b + 4c - 10) \\ &= (b + 2c - 5)a + [3b^2 + (6c - 13)b + 2(2c - 5)] \\ &= (b + 2c - 5)a + (3b + 2)(b + 2c - 5) \\ &= (b + 2c - 5)[a + (3b + 2)] \\ &= \mathbf{(b + 2c - 5)(a + 3b + 2)} \end{aligned}$$

You will get a linear expression.

(2) In powers of b ,

$$\begin{aligned} & ab + 2ac + 3b^2 + 6bc - 5a - 13b + 4c - 10 \\ &= 3b^2 + (a + 6c - 13)b + (2ac - 5a + 4c - 10) \\ &= 3b^2 + (a + 6c - 13)b + a(2c - 5) + 2(2c - 5) \\ &= 3b^2 + (a + 6c - 13)b + (2c - 5)(a + 2) \\ &= [3b + (a + 2)][b + (2c - 5)] \\ &= (3b + a + 2)(b + 2c - 5) \\ &= \mathbf{(a + 3b + 2)(b + 2c - 5)} \end{aligned}$$

You will get a quadratic expression.

J58b

2. $a^2 + 3b^2 + 4ab + 2ac + 6bc - 4b + 4c - 4$

Arrange in powers of a , b or c , then factorise.

In powers of a ,

$$\begin{aligned} & a^2 + 3b^2 + 4ab + 2ac + 6bc - 4b + 4c - 4 \\ &= a^2 + 2(2b + c)a + [3b^2 + 2(3c - 2)b + 2 \cdot 2(c - 1)] \\ &= a^2 + 2(2b + c)a + (3b + 2)[b + 2(c - 1)] \\ &= a^2 + 2(2b + c)a + (3b + 2)(b + 2c - 2) \\ &= [a + (3b + 2)][a + (b + 2c - 2)] \\ &= \mathbf{(a + 3b + 2)(a + b + 2c - 2)} \end{aligned}$$

In powers of b ,

$$\begin{aligned} & a^2 + 3b^2 + 4ab + 2ac + 6bc - 4b + 4c - 4 \\ &= 3b^2 + 2(2a + 3c - 2)b + (a^2 + 2ac + 4c - 4) \\ &= 3b^2 + 2(2a + 3c - 2)b + (a + 2)(a - 2) + 2c(a + 2) \\ &= 3b^2 + 2(2a + 3c - 2)b + (a + 2)(a - 2 + 2c) \\ &= [3b + (a + 2)][b + (a - 2 + 2c)] \\ &= (3b + a + 2)(b + a - 2 + 2c) \\ &= \mathbf{(a + 3b + 2)(a + b + 2c - 2)} \end{aligned}$$

In powers of c ,

$$\begin{aligned} & a^2 + 3b^2 + 4ab + 2ac + 6bc - 4b + 4c - 4 \\ &= 2(a + 3b + 2)c + (a^2 + 3b^2 + 4ab - 4b - 4) \\ &= 2(a + 3b + 2)c + [a^2 + 4ba + (3b + 2)(b - 2)] \\ &= 2(a + 3b + 2)c + (a + 3b + 2)(a + b - 2) \\ &= (a + 3b + 2)(2c + a + b - 2) \\ &= \mathbf{(a + 3b + 2)(a + b + 2c - 2)} \end{aligned}$$

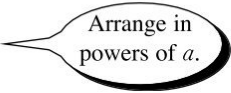
J59a

KUMON

Factorisation V

Factorise the following expressions.

$$\begin{aligned}
 (1) \quad & x^3 + (2a+1)x^2 + (a^2 + 2a - 1)x + (a^2 - 1) \\
 &= x^3 + 2ax^2 + x^2 + a^2x + 2ax - x + a^2 - 1 \\
 &= (x+1)a^2 + 2x(x+1)a + x^3 + x^2 - x - 1 \\
 &= (x+1)a^2 + 2x(x+1)a + x^2(x+1) - (x+1) \\
 &= (x+1)[a^2 + 2xa + (x^2 - 1)] \\
 &= (x+1)[a^2 + 2xa + (x+1)(x-1)] \\
 &= (x+1)[a + (x+1)][a + (x-1)] \\
 &= \mathbf{(x+1)(a+x+1)(a+x-1)} \\
 &[= \mathbf{(x+1)(x+a+1)(x+a-1)}]
 \end{aligned}$$


 Arrange in powers of a .

J59b

$$(2) \quad ax^2 - a^3 - a^2b + ab^2 + b^3 - bx^2$$

Arrange in powers of x .

$$= (a-b)x^2 - (a^3 - b^3) - ab(a-b)$$

$$= (a-b)x^2 - (a-b)(a^2 + ab + b^2) - ab(a-b)$$

Use the formula:
 $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

$$= (a-b)[x^2 - (a^2 + ab + b^2) - ab]$$

$$= (a-b)[x^2 - (a^2 + 2ab + b^2)]$$

$$= (a-b)[x^2 - (a+b)^2]$$

$$= \mathbf{(a-b)(x+a+b)(x-a-b)}$$

J60a

KUMON

Factorisation V

Factorise the following expressions.

$$\begin{aligned}
 (1) \quad & x^2 - y^2 + yz - zx - 4x + 2y + z + 3 \\
 & = x^2 - (z+4)x - [y^2 - (z+2)y - (z+3)] \\
 & = x^2 - (z+4)x - (y+1)(y-z-3) \\
 & = [x - (y+1)][x + (y-z-3)] \\
 & = \mathbf{(x-y-1)(x+y-z-3)}
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad & ac^2 - a^3 - a^2b + ab^2 + b^3 - bc^2 \\
 & = (a-b)c^2 - ab(a-b) - (a^3 - b^3) \\
 & = (a-b)c^2 - ab(a-b) - (a-b)(a^2 + ab + b^2) \\
 & = (a-b)[c^2 - ab - (a^2 + ab + b^2)] \\
 & = (a-b)[c^2 - (a^2 + 2ab + b^2)] \\
 & = (a-b)[c^2 - (a+b)^2] \\
 & = \mathbf{(a-b)(c+a+b)(c-a-b)}
 \end{aligned}$$

$$[= -(\mathbf{a-b})(\mathbf{a+b+c})(\mathbf{a+b-c})]$$

J 60b

$$\begin{aligned}(3) \quad & (a+b-c)(ab-bc-ca)+abc \\ & = [a+(b-c)][(b-c)a-bc]+abc \\ & = (b-c)a^2+[(b-c)^2-bc]a-(b-c)bc+abc \\ & = (b-c)a^2+[(b-c)^2-bc+bc]a-(b-c)bc \\ & = (b-c)a^2+(b-c)^2a-(b-c)bc \\ & = (b-c)[a^2+(b-c)a-bc] \\ & = \mathbf{(b-c)(a+b)(a-c)}\end{aligned}$$

$$\begin{aligned}(4) \quad & (xy-1)(x-1)(y+1)-xy \\ & = (y+1)[yx^2-(y+1)x+1]-xy \\ & = y(y+1)x^2-[(y+1)^2+y]x+(y+1) \\ & = [yx-(y+1)][(y+1)x-1] \\ & = \mathbf{(xy-y-1)(xy+x-1)}\end{aligned}$$

<p>Alternative Solution</p> $\begin{aligned}& (xy-1)(x-1)(y+1)-xy \\ & = (xy-1)[(xy-1)+(x-y)]-xy \\ & = (xy-1)^2+(x-y)(xy-1)-xy \\ & = [(xy-1)+x][(xy-1)-y] \\ & = (xy-1+x)(xy-1-y) \\ & = \mathbf{(xy+x-1)(xy-y-1)}\end{aligned}$
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J61a KUMON

Fractional Expressions

Reduce the following fractional expressions.

Ex.

$$\frac{2a^2}{3a^3} = \frac{2}{3a} \qquad \frac{2a^3b}{-4a^2b^5} = -\frac{a}{2b^4}$$

$$(1) \quad \frac{6ab^5}{9a^3b^2} = \frac{2b^3}{3a^2}$$

$$(2) \quad \frac{28a^3x^4}{35a^3x^5} = \frac{4}{5x}$$

$$(3) \quad \frac{ax}{a^2x^3} = \frac{1}{ax^2}$$

$$(4) \quad \frac{18a^3}{24a^5} = \frac{3}{4a^2}$$

$$(5) \quad \frac{2x^2y}{-6yz^2} = -\frac{x^2}{3z^2}$$

$$(6) \quad \frac{-6xz^2}{3xyz} = -\frac{2z}{y}$$

J61b

Ex.

$$\frac{x^2 - 25}{x^2 - 3x - 10} = \frac{(x+5)(x-5)}{(x-5)(x+2)} = \frac{x+5}{x+2} \quad \rightarrow \text{Cancel } (x-5).$$

$$\frac{x+1}{x^2 - x - 2} = \frac{x+1}{(x-2)(x+1)} = \frac{1}{x-2} \quad \rightarrow \text{Cancel } (x+1).$$

$$(7) \quad \frac{x^2 - 5x + 6}{x^2 - 7x + 12} = \frac{(x-3)(x-2)}{(x-4)(x-3)} = \frac{\mathbf{x-2}}{\mathbf{x-4}}$$

$$(8) \quad \frac{x^2 - x - 20}{2x^2 - 7x - 15} = \frac{(x-5)(x+4)}{(2x+3)(x-5)} = \frac{\mathbf{x+4}}{\mathbf{2x+3}}$$

$$(9) \quad \frac{x+2}{x^2 + x - 2} = \frac{x+2}{(x+2)(x-1)} = \frac{\mathbf{1}}{\mathbf{x-1}}$$

$$(10) \quad \frac{x^2 - 6x + 9}{x^2 - x - 6} = \frac{(x-3)^2}{(x-3)(x+2)} = \frac{\mathbf{x-3}}{\mathbf{x+2}}$$

$$(11) \quad \frac{3a^2 + 6a}{a^2 + 4a + 4} = \frac{3a(a+2)}{(a+2)^2} = \frac{\mathbf{3a}}{\mathbf{a+2}}$$

Fractional Expressions

Reduce the following fractional expressions.

Ex.

$$\frac{b-a}{(a-b)(x-3)} = \frac{-(a-b)}{(a-b)(x-3)} = -\frac{1}{x-3}$$

$$\frac{-x^2+5x-6}{x^2-7x+12} = \frac{-(x-3)(x-2)}{(x-4)(x-3)} = -\frac{x-2}{x-4}$$

$$(1) \quad \frac{x^2+5x-14}{4-x^2} = \frac{(x+7)(x-2)}{-(x+2)(x-2)} = -\frac{x+7}{x+2}$$

$$(2) \quad \frac{a^2b(x-a)}{ab^2(a-x)} = \frac{a^2b(x-a)}{-ab^2(x-a)} = -\frac{a}{b}$$

$$(3) \quad \frac{(x-5)^2}{(5+x)(5-x)} = \frac{(x-5)^2}{-(x+5)(x-5)} = -\frac{x-5}{x+5}$$

$$(4) \quad \frac{m-mx}{x^2-2x+1} = \frac{m(1-x)}{(x-1)^2} = \frac{-m(x-1)}{(x-1)^2} = -\frac{m}{x-1}$$

$$(5) \quad \frac{x^2-1}{(1+x)^2} = \frac{(x+1)(x-1)}{(x+1)^2} = \frac{x-1}{x+1}$$

J 62b

$$\begin{aligned} (6) \quad & \frac{x^2 - (y+z)^2}{z^2 - (x+y)^2} \\ &= \frac{(x+y+z)(x-y-z)}{(z+x+y)(z-x-y)} = \frac{(x+y+z)(x-y-z)}{-(x+y+z)(x+y-z)} = -\frac{\mathbf{x-y-z}}{\mathbf{x+y-z}} \end{aligned}$$

$$\begin{aligned} (7) \quad & \frac{(b-c)^2 - a^2}{(c+a)^2 - b^2} \\ &= \frac{(b-c+a)(b-c-a)}{(c+a+b)(c+a-b)} = \frac{-(a+b-c)(a-b+c)}{(a+b+c)(a-b+c)} = -\frac{\mathbf{a+b-c}}{\mathbf{a+b+c}} \end{aligned}$$

$$\begin{aligned} (8) \quad & \frac{x^3 + 3x^2y + 3xy^2 + y^3}{x^3 + y^3} \\ &= \frac{(x+y)^3}{(x+y)(x^2 - xy + y^2)} = \frac{(x+y)(x+y)^2}{(x+y)(x^2 - xy + y^2)} = \frac{\mathbf{(x+y)^2}}{\mathbf{x^2 - xy + y^2}} \end{aligned}$$

$$\begin{aligned} (9) \quad & \frac{x^3 - 3x^2y + 3xy^2 - y^3}{x^3 - y^3} \\ &= \frac{(x-y)^3}{(x-y)(x^2 + xy + y^2)} = \frac{(x-y)(x-y)^2}{(x-y)(x^2 + xy + y^2)} = \frac{\mathbf{(x-y)^2}}{\mathbf{x^2 + xy + y^2}} \end{aligned}$$

$$\begin{aligned} (10) \quad & \frac{a^2 - ab + b^2}{a^3 + b^3} \\ &= \frac{a^2 - ab + b^2}{(a+b)(a^2 - ab + b^2)} = \frac{\mathbf{1}}{\mathbf{a+b}} \end{aligned}$$

J63a KUMON

Fractional Expressions

Reduce the following fractional expressions.

$$\begin{aligned}
 (1) \quad & \frac{ax+ay+bx+by}{ax-by-bx+ay} \\
 &= \frac{a(x+y)+b(x+y)}{a(x+y)-b(x+y)} = \frac{(x+y)(a+b)}{(x+y)(a-b)} = \frac{\mathbf{a+b}}{\mathbf{a-b}}
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad & \frac{x^2-xy-cy+cx}{x^2+bx-xy-by} \\
 &= \frac{x(x-y)+c(x-y)}{x(x-y)+b(x-y)} = \frac{(x-y)(x+c)}{(x-y)(x+b)} = \frac{\mathbf{x+c}}{\mathbf{x+b}}
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad & \frac{x^2+7x+12}{x^3+4x^2-9x-36} \\
 &= \frac{(x+4)(x+3)}{x^2(x+4)-9(x+4)} = \frac{(x+4)(x+3)}{(x+4)(x^2-9)} \\
 &= \frac{(x+4)(x+3)}{(x+4)(x+3)(x-3)} = \frac{\mathbf{1}}{\mathbf{x-3}}
 \end{aligned}$$

$$\begin{aligned}
 (4) \quad & \frac{a^2-ab+ab^2}{a^2-b^2+ab^2+b^3} \\
 &= \frac{a(a-b+b^2)}{(a+b)(a-b)+b^2(a+b)} = \frac{a(a-b+b^2)}{(a+b)(a-b+b^2)} = \frac{\mathbf{a}}{\mathbf{a+b}}
 \end{aligned}$$

J 63b

$$\begin{aligned}(5) \quad & \frac{a^2 - ab + ac - bc}{a^3 - b^3} \\ &= \frac{a(a-b) + c(a-b)}{(a-b)(a^2 + ab + b^2)} = \frac{(a-b)(a+c)}{(a-b)(a^2 + ab + b^2)} = \frac{\mathbf{a+c}}{\mathbf{a^2+ab+b^2}}\end{aligned}$$

$$\begin{aligned}(6) \quad & \frac{x^3 - y^3}{x^4 + x^2y^2 + y^4} \\ &= \frac{(x-y)(x^2 + xy + y^2)}{(x^2 + y^2)^2 - x^2y^2} = \frac{(x-y)(x^2 + xy + y^2)}{(x^2 + xy + y^2)(x^2 - xy + y^2)} \\ &= \frac{\mathbf{x-y}}{\mathbf{x^2-xy+y^2}}\end{aligned}$$

$$\begin{aligned}(7) \quad & \frac{x^3 - 2x^2y + xy^2}{x^2 - 3xy + 2y^2} \\ &= \frac{x(x^2 - 2xy + y^2)}{(x-2y)(x-y)} = \frac{x(x-y)^2}{(x-2y)(x-y)} = \frac{\mathbf{x(x-y)}}{\mathbf{x-2y}}\end{aligned}$$

J64a KUMON

Fractional Expressions

Simplify the following fractional expressions.

Ex.

$$\frac{3ax^2}{b^4} \times \frac{2a^3b^2}{9xy} = \frac{2a^4x}{3b^2y}$$

$$(1) \quad \frac{2a^2}{3bc} \times \frac{5b^3c^3}{8a^3x} = \frac{5b^2c^2}{12ax}$$

$$(2) \quad \frac{x^2}{yz} \times \frac{y^2}{zx} \times \frac{z^2}{xy} = 1$$

$$(3) \quad \frac{4a^2b}{5x^2y} \div \frac{2ab^2}{15xy^2} = \frac{4a^2b}{5x^2y} \times \frac{15xy^2}{2ab^2} = \frac{6ay}{bx}$$

$$(4) \quad \frac{2a^2}{x^2y^2} \div \frac{-a}{xy} \times \frac{x^3}{2a^3} = -\frac{2a^2}{x^2y^2} \times \frac{xy}{a} \times \frac{x^3}{2a^3} = -\frac{x^2}{a^2y}$$

$$(5) \quad \frac{b^3}{a} \times \frac{b^2}{2a^4} \div \left(-\frac{4a^6}{b^6}\right) = -\frac{b^3}{a} \times \frac{b^2}{2a^4} \times \frac{b^6}{4a^6} = -\frac{b^{11}}{8a^{11}}$$

J 64b

$$(6) \quad \frac{xy}{z^2} \div \frac{x^4}{y^2 z^2} \div \frac{y^6}{z^3 x^3}$$
$$= \frac{xy}{z^2} \times \frac{y^2 z^2}{x^4} \times \frac{z^3 x^3}{y^6} = \frac{\mathbf{z^3}}{\mathbf{y^3}}$$

$$(7) \quad \frac{(-4a^2b)^2}{21x^3y^2} \times \frac{3x^2y}{(2a)^2b}$$
$$= \frac{16a^4b^2}{21x^3y^2} \times \frac{3x^2y}{4a^2b} = \frac{\mathbf{4a^2b}}{\mathbf{7xy}}$$

$$(8) \quad \frac{(7a^2b)^2}{21x^3y^2} \div \frac{3x^2y}{35(-ab)^3}$$
$$= \frac{49a^4b^2}{21x^3y^2} \times \left(-\frac{35a^3b^3}{3x^2y} \right) = -\frac{\mathbf{245a^7b^5}}{\mathbf{9x^5y^3}}$$

$$(9) \quad \frac{(-3x^4y)^2}{4a^2b^3} \times \frac{8a^3b^2}{(-3xy^2)^3} \div \frac{(2a^2x^3)^2}{9b^3y^5}$$
$$= \frac{9x^8y^2}{4a^2b^3} \times \left(-\frac{8a^3b^2}{27x^3y^6} \right) \times \frac{9b^3y^5}{4a^4x^6} = -\frac{\mathbf{3b^2y}}{\mathbf{2a^3x}}$$

J65a KUMON

Fractional Expressions

Simplify the following fractional expressions.

$$\begin{aligned}(1) \quad & \frac{x^2-1}{3} \times \frac{6a}{x-1} \\ & = \frac{(x+1)(x-1)}{3} \times \frac{6a}{x-1} = \mathbf{2a(x+1)}\end{aligned}$$

$$\begin{aligned}(2) \quad & \frac{x^2+2x}{x^2-9} \times \frac{x^2-3x}{x^2-4} \\ & = \frac{x(x+2)}{(x+3)(x-3)} \times \frac{x(x-3)}{(x+2)(x-2)} = \frac{\mathbf{x^2}}{\mathbf{(x+3)(x-2)}}\end{aligned}$$

$$\begin{aligned}(3) \quad & \frac{(x-1)^2}{y^3} \times \frac{xy^2+y^2}{x^2-1} \\ & = \frac{(x-1)^2}{y^3} \times \frac{y^2(x+1)}{(x+1)(x-1)} = \frac{\mathbf{x-1}}{\mathbf{y}}\end{aligned}$$

$$\begin{aligned}(4) \quad & \frac{(a+b)^2}{a-b} \div \frac{b+a}{(b-a)^2} \\ & = \frac{(a+b)^2}{a-b} \times \frac{(a-b)^2}{a+b} = \mathbf{(a+b)(a-b)}\end{aligned}$$

J 65b

$$\begin{aligned}(5) \quad & \frac{a-b}{a^2+ab} \div \frac{ab-b^2}{a+b} \\ &= \frac{a-b}{a^2+ab} \times \frac{a+b}{ab-b^2} = \frac{a-b}{a(a+b)} \times \frac{a+b}{b(a-b)} = \frac{\mathbf{1}}{\mathbf{ab}}\end{aligned}$$

$$\begin{aligned}(6) \quad & \frac{a(a-b)^2}{b(a^2-b^2)} \times \frac{ab+b^2}{a-b} \\ &= \frac{a(a-b)^2}{b(a+b)(a-b)} \times \frac{b(a+b)}{a-b} = \mathbf{a}\end{aligned}$$

$$\begin{aligned}(7) \quad & \frac{x+1}{x^2+4x+4} \times \frac{x+2}{x^2+6x+9} \div \frac{(x+1)^2}{x+3} \\ &= \frac{x+1}{(x+2)^2} \times \frac{x+2}{(x+3)^2} \times \frac{x+3}{(x+1)^2} = \frac{\mathbf{1}}{\mathbf{(x+1)(x+2)(x+3)}}\end{aligned}$$

$$\begin{aligned}(8) \quad & \frac{x+a}{(x-a)^2} \times \frac{x^2-a^2}{x^2+a^2} \times \frac{x^4-a^4}{(x+a)^3} \\ &= \frac{x+a}{(x-a)^2} \times \frac{(x+a)(x-a)}{x^2+a^2} \times \frac{(x^2+a^2)(x^2-a^2)}{(x+a)^3} \\ &= \frac{x+a}{(x-a)^2} \times \frac{(x+a)(x-a)}{x^2+a^2} \times \frac{(x^2+a^2)(x+a)(x-a)}{(x+a)^3} = \mathbf{1}\end{aligned}$$

J66a KUMON

Fractional Expressions

Simplify the following fractional expressions.

Ex.

$$\frac{1}{2x} + \frac{1}{3x} = \frac{3}{6x} + \frac{2}{6x} = \frac{5}{6x} \quad \text{☞ First find the common denominator.}$$

$$(1) \quad \frac{1}{4x} + \frac{1}{6x} = \frac{3}{12x} + \frac{2}{12x} = \frac{5}{12x}$$

$$(2) \quad \frac{1}{2x} - \frac{1}{3x} = \frac{3}{6x} - \frac{2}{6x} = \frac{1}{6x}$$

$$(3) \quad \frac{1}{2x} - \frac{3}{4x} + \frac{5}{6x} = \frac{6}{12x} - \frac{9}{12x} + \frac{10}{12x} = \frac{7}{12x}$$

$$(4) \quad \frac{2b}{3a} - \frac{3b}{4a} + \frac{5b}{6a} = \frac{8b}{12a} - \frac{9b}{12a} + \frac{10b}{12a} = \frac{9b}{12a} = \frac{3b}{4a}$$

$$(5) \quad \frac{2c}{3ab} + \frac{3c}{4ab} + \frac{c}{2ab} = \frac{8c}{12ab} + \frac{9c}{12ab} + \frac{6c}{12ab} = \frac{23c}{12ab}$$

J66b

Ex.

$$\frac{1}{a} + \frac{1}{b} = \frac{b}{ab} + \frac{a}{ab} = \frac{a+b}{ab}$$

$$(6) \quad \frac{3}{ax} + \frac{1}{bx} = \frac{3b}{abx} + \frac{a}{abx} = \frac{\mathbf{a+3b}}{\mathbf{abx}}$$

$$(7) \quad \frac{5}{x} + \frac{3x+1}{x^2} = \frac{5x}{x^2} + \frac{3x+1}{x^2} = \frac{\mathbf{8x+1}}{\mathbf{x^2}}$$

$$(8) \quad \frac{3}{x} - \frac{3x-1}{x^2} = \frac{3x}{x^2} - \frac{3x-1}{x^2} = \frac{\mathbf{1}}{\mathbf{x^2}}$$

$$(9) \quad \frac{2}{x} + \frac{3}{x+3} = \frac{2(x+3)}{x(x+3)} + \frac{3x}{x(x+3)} = \frac{\mathbf{5x+6}}{\mathbf{x(x+3)}}$$

$$(10) \quad \frac{2}{x-4} + \frac{2}{x+4} = \frac{2(x+4)}{(x-4)(x+4)} + \frac{2(x-4)}{(x-4)(x+4)} = \frac{\mathbf{4x}}{\mathbf{(x-4)(x+4)}}$$

Fractional Expressions

Simplify the following fractional expressions.

Ex.

$$\frac{x-5}{x+5} + \frac{x+5}{x-5} = \frac{(x-5)^2}{(x+5)(x-5)} + \frac{(x+5)^2}{(x+5)(x-5)} = \frac{2(x^2+25)}{(x+5)(x-5)}$$

$$\begin{aligned} (1) \quad \frac{x-5}{x+5} - \frac{x+5}{x-5} &= \frac{(x-5)^2}{(x+5)(x-5)} - \frac{(x+5)^2}{(x+5)(x-5)} \\ &= \frac{x^2 - 10x + 25 - x^2 - 10x - 25}{(x+5)(x-5)} = -\frac{20x}{(x+5)(x-5)} \end{aligned}$$

$$\begin{aligned} (2) \quad \frac{x+1}{x^2} + \frac{2}{x} \\ &= \frac{x+1}{x^2} + \frac{2x}{x^2} = \frac{3x+1}{x^2} \end{aligned}$$

$$\begin{aligned} (3) \quad \frac{x+1}{(x+2)^2} - \frac{1}{x+2} \\ &= \frac{x+1}{(x+2)^2} - \frac{x+2}{(x+2)^2} = -\frac{1}{(x+2)^2} \end{aligned}$$

$$\begin{aligned} (4) \quad \frac{3}{ab} + \frac{2}{bc} \\ &= \frac{3c}{abc} + \frac{2a}{abc} = \frac{2a+3c}{abc} \end{aligned}$$

$$\begin{aligned} (5) \quad \frac{3}{(x+1)(x+2)} + \frac{2}{(x+2)(x+3)} \\ &= \frac{3(x+3)}{(x+1)(x+2)(x+3)} + \frac{2(x+1)}{(x+1)(x+2)(x+3)} = \frac{5x+11}{(x+1)(x+2)(x+3)} \end{aligned}$$

J67b

Ex.

$$x + \frac{x}{x-1} = \frac{x(x-1)}{x-1} + \frac{x}{x-1} = \frac{x^2}{x-1}$$

$$\begin{aligned} (6) \quad x+1 + \frac{1}{x-1} \\ = \frac{(x+1)(x-1)}{x-1} + \frac{1}{x-1} = \frac{x^2-1}{x-1} + \frac{1}{x-1} = \frac{x^2}{x-1} \end{aligned}$$

$$\begin{aligned} (7) \quad 1 - \frac{4}{x-1} + \frac{12}{x-3} \\ = \frac{(x-1)(x-3)}{(x-1)(x-3)} - \frac{4(x-3)}{(x-1)(x-3)} + \frac{12(x-1)}{(x-1)(x-3)} \\ = \frac{x^2-4x+3-4x+12+12x-12}{(x-1)(x-3)} = \frac{x^2+4x+3}{(x-1)(x-3)} \\ \left[= \frac{(x+3)(x+1)}{(x-1)(x-3)} \right] \end{aligned}$$

$$\begin{aligned} (8) \quad 1 - \frac{a}{1+a} + \frac{a}{1-a} \\ = \frac{(1+a)(1-a)}{(1+a)(1-a)} - \frac{a(1-a)}{(1+a)(1-a)} + \frac{a(1+a)}{(1+a)(1-a)} \\ = \frac{1-a^2-a+a^2+a+a^2}{(1+a)(1-a)} \\ = \frac{a^2+1}{(1+a)(1-a)} \left[= -\frac{a^2+1}{(a+1)(a-1)} \right] \end{aligned}$$

$$(9)^* \quad \frac{1}{a-b} + \frac{1}{b-a} = \frac{1}{a-b} - \frac{1}{a-b} = 0$$

J68a


KUMON

Fractional Expressions

Simplify the following fractional expressions.

Ex.

$$\begin{aligned}
 & \frac{1}{x+7} - \frac{1}{x-1} + \frac{1}{x-7} - \frac{1}{x+1} \\
 = & \frac{(x-7)+(x+7)}{(x+7)(x-7)} - \frac{(x+1)+(x-1)}{(x-1)(x+1)} \\
 = & \frac{2x}{(x+7)(x-7)} - \frac{2x}{(x-1)(x+1)} \\
 = & \frac{2x[(x-1)(x+1) - (x+7)(x-7)]}{(x+7)(x-7)(x-1)(x+1)} \\
 = & \frac{2x[(x^2-1) - (x^2-49)]}{(x+7)(x-7)(x-1)(x+1)} = \frac{96x}{(x+7)(x-7)(x-1)(x+1)}
 \end{aligned}$$


 Rearrange terms into matching pairs.

$$\begin{aligned}
 (1) \quad & \frac{1}{x-3} - \frac{1}{x+3} - \frac{1}{x-1} + \frac{1}{x+1} \\
 = & \frac{(x+3) - (x-3)}{(x-3)(x+3)} - \frac{(x+1) - (x-1)}{(x-1)(x+1)} \\
 = & \frac{6}{(x-3)(x+3)} - \frac{2}{(x-1)(x+1)} \\
 = & \frac{6(x-1)(x+1) - 2(x-3)(x+3)}{(x-3)(x+3)(x-1)(x+1)} \\
 = & \frac{6(x^2-1) - 2(x^2-9)}{(x-3)(x+3)(x-1)(x+1)} \\
 = & \frac{4x^2+12}{(x-3)(x+3)(x-1)(x+1)} \\
 \left[= & \frac{4(x^2+3)}{(x-3)(x+3)(x-1)(x+1)} \right]
 \end{aligned}$$

J68b

$$\begin{aligned}
 (2) \quad & \frac{1}{a+1} - \frac{1}{a+3} - \frac{1}{a+2} + \frac{1}{a+4} \\
 &= \frac{(a+2)-(a+1)}{(a+1)(a+2)} - \frac{(a+4)-(a+3)}{(a+3)(a+4)} \\
 &= \frac{1}{(a+1)(a+2)} - \frac{1}{(a+3)(a+4)} \\
 &= \frac{(a+3)(a+4) - (a+1)(a+2)}{(a+1)(a+2)(a+3)(a+4)} \\
 &= \frac{(a^2+7a+12) - (a^2+3a+2)}{(a+1)(a+2)(a+3)(a+4)} \\
 &= \frac{4a+10}{(a+1)(a+2)(a+3)(a+4)} \\
 & \left[= \frac{2(2a+5)}{(a+1)(a+2)(a+3)(a+4)} \right]
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad & \frac{1}{a-1} - \frac{1}{a+1} - \frac{2}{a^2+1} - \frac{4}{a^4+1} \\
 &= \frac{(a+1)-(a-1)}{(a-1)(a+1)} - \frac{2}{a^2+1} - \frac{4}{a^4+1} \\
 &= \frac{2}{a^2-1} - \frac{2}{a^2+1} - \frac{4}{a^4+1} \\
 &= \frac{2(a^2+1) - 2(a^2-1)}{(a^2-1)(a^2+1)} - \frac{4}{a^4+1} \\
 &= \frac{4}{a^4-1} - \frac{4}{a^4+1} \\
 &= \frac{4(a^4+1) - 4(a^4-1)}{(a^4-1)(a^4+1)} \\
 &= \frac{8}{(a^4-1)(a^4+1)}
 \end{aligned}$$

First find the common denominator of

$$\frac{1}{a-1} \text{ and } \frac{1}{a+1}.$$

$$\left[= \frac{8}{a^8-1} \right]$$

Fractional Expressions

Simplify the following fractional expressions.

Ex.

$$\frac{\frac{3}{5}}{6} = \frac{3}{5} \div 6 = \frac{1}{10}$$

$$(1) \quad \frac{\frac{3}{4}}{9} = \frac{3}{4} \div 9 = \frac{3}{4} \times \frac{1}{9} = \frac{1}{12}$$

$$(2) \quad \frac{\frac{3}{5}}{6} = 3 \div \frac{5}{6} = 3 \times \frac{6}{5} = \frac{18}{5}$$

Ex.

$$\frac{1 + \frac{1}{a}}{1 - \frac{1}{a}} = \left(1 + \frac{1}{a}\right) \div \left(1 - \frac{1}{a}\right) = \frac{a+1}{a} \times \frac{a}{a-1} = \frac{a+1}{a-1}$$

$$(3) \quad \frac{1}{1 + \frac{1}{x}} = 1 \div \left(1 + \frac{1}{x}\right) = 1 \times \frac{x}{x+1} = \frac{x}{x+1}$$

$$(4) \quad \frac{1 + \frac{1}{x}}{x+1} = \left(1 + \frac{1}{x}\right) \div (x+1) = \frac{x+1}{x} \times \frac{1}{x+1} = \frac{1}{x}$$

J69b

Ex.

$$\frac{1 + \frac{1}{a}}{1 - \frac{1}{a}} = \frac{\left(1 + \frac{1}{a}\right)a}{\left(1 - \frac{1}{a}\right)a} = \frac{a+1}{a-1} \quad \text{☞ Multiply numerator and denominator by } a.$$

$$(5) \quad \frac{1}{1 + \frac{1}{x}} = \frac{1 \cdot x}{\left(1 + \frac{1}{x}\right)x} = \frac{x}{x+1}$$

$$(6) \quad \frac{1}{1 + \frac{1}{x+2}} = \frac{1 \cdot (x+2)}{\left(1 + \frac{1}{x+2}\right)(x+2)} = \frac{x+2}{x+3}$$

$$(7) \quad \frac{1}{1 - \frac{1}{1+x}} = \frac{1+x}{\left(1 - \frac{1}{1+x}\right)(1+x)} = \frac{x+1}{x}$$

$$(8) \quad \frac{1 - \frac{2}{x-5}}{1 + \frac{3}{x-5}} = \frac{\left(1 - \frac{2}{x-5}\right)(x-5)}{\left(1 + \frac{3}{x-5}\right)(x-5)} = \frac{x-7}{x-2}$$

Fractional Expressions

Simplify the following fractional expressions.

$$\begin{aligned}
 (1) \quad & \frac{x^2 + 5x + 6}{x^2 + 4x + 4} \\
 &= \frac{(x+3)(x+2)}{(x+2)^2} \\
 &= \frac{\mathbf{x+3}}{\mathbf{x+2}}
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad & \frac{(a+b)^2 - (c+d)^2}{(a+c)^2 - (b+d)^2} \\
 &= \frac{(a+b+c+d)(a+b-c-d)}{(a+c+b+d)(a+c-b-d)} \\
 &= \frac{\mathbf{a+b-c-d}}{\mathbf{a-b+c-d}}
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad & \frac{12a^2 - 6a}{2a^2 - a - 1} \times \frac{4a^2 - 1}{12a^3 + 18a^2 - 12a} \\
 &= \frac{6a(2a-1)}{(2a+1)(a-1)} \times \frac{(2a+1)(2a-1)}{6a(2a^2+3a-2)} \\
 &= \frac{6a(2a-1)}{(2a+1)(a-1)} \times \frac{(2a+1)(2a-1)}{6a(2a-1)(a+2)} \\
 &= \frac{\mathbf{2a-1}}{\mathbf{(a-1)(a+2)}}
 \end{aligned}$$

J70b

$$\begin{aligned}(4) \quad & \frac{a-b}{ab} + \frac{b-c}{bc} + \frac{c-a}{ca} \\ &= \frac{c(a-b) + a(b-c) + b(c-a)}{abc} \\ &= \frac{ac - bc + ab - ac + bc - ab}{abc} \\ &= \mathbf{0}\end{aligned}$$

$$\begin{aligned}(5) \quad & \frac{2}{x-2} - \frac{1}{x+2} + \frac{6-x}{x^2-4} \\ &= \frac{2(x+2) - (x-2)}{(x-2)(x+2)} + \frac{6-x}{x^2-4} \\ &= \frac{x+6+6-x}{x^2-4} \\ &= \frac{\mathbf{12}}{\mathbf{x^2-4}} \\ & \left[= \frac{\mathbf{12}}{\mathbf{(x+2)(x-2)}} \right]\end{aligned}$$

$$\begin{aligned}(6) \quad & \frac{3}{(x+1)(x+2)} - \frac{3}{(x+2)(x+3)} \\ &= \frac{3(x+3) - 3(x+1)}{(x+1)(x+2)(x+3)} \\ &= \frac{3x+9-3x-3}{(x+1)(x+2)(x+3)} \\ &= \frac{\mathbf{6}}{\mathbf{(x+1)(x+2)(x+3)}}\end{aligned}$$

J71a KUMON

Irrational Numbers I

Expand the following expressions.

Ex.

$$\sqrt{3}(2\sqrt{6} + \sqrt{3}) = 2\sqrt{18} + 3 = 6\sqrt{2} + 3$$

$$(2\sqrt{3} + \sqrt{2})(\sqrt{3} + 3\sqrt{2}) = 6 + 6\sqrt{6} + \sqrt{6} + 6 = 12 + 7\sqrt{6}$$

$$(1) \quad \sqrt{3}(2\sqrt{6} - \sqrt{12}) = 2\sqrt{18} - \sqrt{36} = 6\sqrt{2} - 6$$

$$(2) \quad (3\sqrt{18} - 5\sqrt{8})\sqrt{3} = 3\sqrt{54} - 5\sqrt{24} = 9\sqrt{6} - 10\sqrt{6} = -\sqrt{6}$$

$$(3) \quad (\sqrt{5} + 5\sqrt{2})(2\sqrt{5} - \sqrt{2}) = 10 + 9\sqrt{10} - 10 = 9\sqrt{10}$$

$$(4) \quad (\sqrt{5} + 4\sqrt{2})(2\sqrt{5} - 2\sqrt{2}) = 10 + 6\sqrt{10} - 16 = -6 + 6\sqrt{10}$$

$$(5) \quad (3\sqrt{2} + \sqrt{3})(\sqrt{2} - 2\sqrt{3}) = 6 - 5\sqrt{6} - 6 = -5\sqrt{6}$$

Definitions:

Real Numbers	{	Rational numbers	: All numbers that can be expressed as fractions or integers, e.g. 4, 0, -3, $\frac{3}{10}$, $-\frac{18}{5}$, $\frac{23}{6}$.
		Irrational numbers	: Numbers that cannot be expressed as fractions, e.g. $\sqrt{2}$, $-\sqrt{5}$, π .

J71b

Ex.

$$(\sqrt{6} + \sqrt{3})^2 = (\sqrt{6})^2 + 2\sqrt{6} \times \sqrt{3} + (\sqrt{3})^2 = 9 + 2\sqrt{18} = 9 + 6\sqrt{2}$$

$$(6) \quad (\sqrt{2} + \sqrt{6})^2 = 2 + 2\sqrt{12} + 6 = \mathbf{8 + 4\sqrt{3}}$$

$$(7) \quad (\sqrt{6} - \sqrt{3})^2 = 6 - 2\sqrt{18} + 3 = \mathbf{9 - 6\sqrt{2}}$$

$$(8) \quad (3\sqrt{2} - \sqrt{6})^2 = 18 - 6\sqrt{12} + 6 = \mathbf{24 - 12\sqrt{3}}$$

$$(9) \quad (2\sqrt{3} + \sqrt{6})^2 = 12 + 4\sqrt{18} + 6 = \mathbf{18 + 12\sqrt{2}}$$

$$(10) \quad (2\sqrt{6} - 3\sqrt{2})^2 = 24 - 12\sqrt{12} + 18 = \mathbf{42 - 24\sqrt{3}}$$

J72a

KUMON

Irrational Numbers I

Expand the following expressions.

Ex.

$$(\sqrt{7} + 2\sqrt{3})(\sqrt{7} - 2\sqrt{3}) = (\sqrt{7})^2 - (2\sqrt{3})^2 = 7 - 12 = -5$$

$$(1) \quad (\sqrt{7} + 5)(\sqrt{7} - 5) \\ = 7 - 25 = -18$$

$$(2) \quad (\sqrt{5} + \sqrt{3})(\sqrt{5} - \sqrt{3}) \\ = 5 - 3 = 2$$

$$(3) \quad (3\sqrt{7} + 3\sqrt{5})(3\sqrt{7} - 3\sqrt{5}) \\ = 63 - 45 = 18$$

$$(4) \quad (3\sqrt{2} + 2\sqrt{3})(2\sqrt{3} - 3\sqrt{2}) \\ = (2\sqrt{3} + 3\sqrt{2})(2\sqrt{3} - 3\sqrt{2}) \\ = 12 - 18 = -6$$

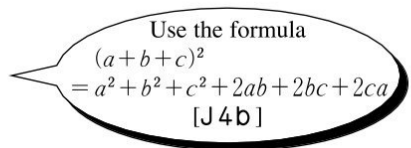
$$(5) \quad (5 - \sqrt{5})(-\sqrt{5} - 5) \\ = -(5 - \sqrt{5})(5 + \sqrt{5}) \\ = -(25 - 5) = -20$$

J72b

$$\begin{aligned}(6) \quad & (\sqrt{10} - \sqrt{2})^2 \\ & = 10 - 2\sqrt{20} + 2 \\ & = \mathbf{12 - 4\sqrt{5}}\end{aligned}$$

$$\begin{aligned}(7) \quad & (\sqrt{2} + \sqrt{3})(\sqrt{2} - \sqrt{3} + 1) \\ & = 2 - \sqrt{6} + \sqrt{2} + \sqrt{6} - 3 + \sqrt{3} \\ & = \mathbf{-1 + \sqrt{2} + \sqrt{3}}\end{aligned}$$

$$\begin{aligned}(8) \quad & (\sqrt{3} + \sqrt{2} + \sqrt{6})^2 \\ & = 3 + 2 + 6 + 2\sqrt{6} + 2\sqrt{12} + 2\sqrt{18} \\ & = \mathbf{11 + 2\sqrt{6} + 4\sqrt{3} + 6\sqrt{2}}\end{aligned}$$



$$\begin{aligned}(9) \quad & (\sqrt{3} - \sqrt{2} + \sqrt{6})^2 \\ & = 3 + 2 + 6 - 2\sqrt{6} - 2\sqrt{12} + 2\sqrt{18} \\ & = \mathbf{11 - 2\sqrt{6} - 4\sqrt{3} + 6\sqrt{2}}\end{aligned}$$

J73a

KUMON

Irrational Numbers I

Expand the following expressions.

Ex.

$$\begin{aligned}
 & (\sqrt{3} + \sqrt{2} + \sqrt{5})(\sqrt{3} + \sqrt{2} - \sqrt{5}) \\
 &= [(\sqrt{3} + \sqrt{2}) + \sqrt{5}][(\sqrt{3} + \sqrt{2}) - \sqrt{5}] \\
 &= (\sqrt{3} + \sqrt{2})^2 - 5 \\
 &= 3 + 2\sqrt{6} + 2 - 5 \\
 &= 2\sqrt{6}
 \end{aligned}$$

$$\begin{aligned}
 (1) \quad & (\sqrt{3} - \sqrt{2} - \sqrt{5})(\sqrt{3} - \sqrt{2} + \sqrt{5}) \\
 &= [(\sqrt{3} - \sqrt{2}) - \sqrt{5}][(\sqrt{3} - \sqrt{2}) + \sqrt{5}] \\
 &= (\sqrt{3} - \sqrt{2})^2 - 5 \\
 &= 3 - 2\sqrt{6} + 2 - 5 \\
 &= -2\sqrt{6}
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad & (\sqrt{3} + \sqrt{2} - \sqrt{5})(\sqrt{3} - \sqrt{2} + \sqrt{5}) \\
 &= [\sqrt{3} + (\sqrt{2} - \sqrt{5})][\sqrt{3} - (\sqrt{2} - \sqrt{5})] \\
 &= 3 - (\sqrt{2} - \sqrt{5})^2 \\
 &= 3 - (2 - 2\sqrt{10} + 5) \\
 &= 3 - 2 + 2\sqrt{10} - 5 \\
 &= -4 + 2\sqrt{10}
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad & (\sqrt{2} - \sqrt{3} + \sqrt{6})(\sqrt{2} + \sqrt{3} - \sqrt{6}) \\
 &= [\sqrt{2} - (\sqrt{3} - \sqrt{6})][\sqrt{2} + (\sqrt{3} - \sqrt{6})] \\
 &= 2 - (\sqrt{3} - \sqrt{6})^2 \\
 &= 2 - (3 - 2\sqrt{18} + 6) \\
 &= 2 - 3 + 2\sqrt{18} - 6 \\
 &= -7 + 6\sqrt{2}
 \end{aligned}$$

J73b

$$\begin{aligned}(4) \quad & (\sqrt{2} - \sqrt{3} + \sqrt{5})(\sqrt{2} - \sqrt{3} - \sqrt{5}) \\ & = [(\sqrt{2} - \sqrt{3}) + \sqrt{5}][(\sqrt{2} - \sqrt{3}) - \sqrt{5}] \\ & = (\sqrt{2} - \sqrt{3})^2 - 5 \\ & = 2 - 2\sqrt{6} + 3 - 5 \\ & = \mathbf{-2\sqrt{6}}\end{aligned}$$

$$\begin{aligned}(5) \quad & (\sqrt{2} - \sqrt{3} + \sqrt{5})(\sqrt{2} + \sqrt{3} - \sqrt{5}) \\ & = [\sqrt{2} - (\sqrt{3} - \sqrt{5})][\sqrt{2} + (\sqrt{3} - \sqrt{5})] \\ & = 2 - (\sqrt{3} - \sqrt{5})^2 \\ & = 2 - (3 - 2\sqrt{15} + 5) \\ & = 2 - 3 + 2\sqrt{15} - 5 \\ & = \mathbf{-6 + 2\sqrt{15}}\end{aligned}$$

$$\begin{aligned}(6) \quad & (\sqrt{3} + \sqrt{5} + \sqrt{7} + 1)(\sqrt{3} + \sqrt{5} - \sqrt{7} - 1) \\ & = [(\sqrt{3} + \sqrt{5}) + (\sqrt{7} + 1)][(\sqrt{3} + \sqrt{5}) - (\sqrt{7} + 1)] \\ & = (\sqrt{3} + \sqrt{5})^2 - (\sqrt{7} + 1)^2 \\ & = (3 + 2\sqrt{15} + 5) - (7 + 2\sqrt{7} + 1) \\ & = 3 + 2\sqrt{15} + 5 - 7 - 2\sqrt{7} - 1 \\ & = \mathbf{2\sqrt{15} - 2\sqrt{7}}\end{aligned}$$

Irrational Numbers I

1. Rationalise the denominators.

(Eliminate the radical sign $\sqrt{\quad}$ from the denominator.)

Ex.

$$\frac{\sqrt{3}}{\sqrt{5}} = \frac{\sqrt{3} \times \sqrt{5}}{\sqrt{5} \times \sqrt{5}} = \frac{\sqrt{15}}{5}$$

$$(1) \quad \frac{\sqrt{5}}{\sqrt{7}} = \frac{\sqrt{5} \times \sqrt{7}}{\sqrt{7} \times \sqrt{7}} = \frac{\sqrt{35}}{7}$$

$$(2) \quad \sqrt{\frac{4}{5}} = \frac{\sqrt{4}}{\sqrt{5}} = \frac{2}{\sqrt{5}} = \frac{2 \times \sqrt{5}}{\sqrt{5} \times \sqrt{5}} = \frac{2\sqrt{5}}{5}$$

$$(3) \quad \frac{\sqrt{2}}{2\sqrt{3}} = \frac{\sqrt{2} \times \sqrt{3}}{2\sqrt{3} \times \sqrt{3}} = \frac{\sqrt{6}}{6}$$

$$(4) \quad \frac{\sqrt{6}}{3\sqrt{2}} = \frac{\sqrt{6} \times \sqrt{2}}{3\sqrt{2} \times \sqrt{2}} = \frac{2\sqrt{3}}{6} = \frac{\sqrt{3}}{3}$$

$$(5) \quad \sqrt{\frac{1}{20}} = \frac{\sqrt{1}}{\sqrt{20}} = \frac{1}{\sqrt{20}} = \frac{1}{2\sqrt{5}} = \frac{1 \times \sqrt{5}}{2\sqrt{5} \times \sqrt{5}} = \frac{\sqrt{5}}{10}$$

$$(6) \quad \frac{5\sqrt{2}}{2\sqrt{6}} = \frac{5\sqrt{2} \times \sqrt{6}}{2\sqrt{6} \times \sqrt{6}} = \frac{5\sqrt{12}}{12} = \frac{10\sqrt{3}}{12} = \frac{5\sqrt{3}}{6}$$

J74b

2. Rationalise the denominators, then evaluate.

Ex.

$$\sqrt{3} + \frac{2}{\sqrt{3}} = \sqrt{3} + \frac{2\sqrt{3}}{3} = \frac{5\sqrt{3}}{3}$$

$$(1) \quad \frac{5}{2\sqrt{3}} - \sqrt{3} = \frac{5\sqrt{3}}{6} - \sqrt{3} = -\frac{\sqrt{3}}{6}$$

$$(2) \quad \frac{2\sqrt{3}}{\sqrt{2}} + 2\sqrt{6} = \frac{2\sqrt{6}}{2} + 2\sqrt{6} = 3\sqrt{6}$$

$$(3) \quad \frac{3\sqrt{8}}{2\sqrt{3}} + \frac{\sqrt{6}}{3} = \frac{3\sqrt{24}}{6} + \frac{\sqrt{6}}{3} = \sqrt{6} + \frac{\sqrt{6}}{3} = \frac{4\sqrt{6}}{3}$$

$$(4) \quad 3\sqrt{50} - 4\sqrt{32} + \sqrt{\frac{1}{2}} = 15\sqrt{2} - 16\sqrt{2} + \frac{\sqrt{2}}{2} = -\frac{\sqrt{2}}{2}$$

$$(5) \quad \frac{6-2\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}(6-2\sqrt{3})}{3} = \frac{6\sqrt{3}-6}{3} = 2\sqrt{3}-2$$

$$(6) \quad \sqrt{\frac{8}{3}} - \sqrt{\frac{3}{8}} = \frac{\sqrt{24}}{3} - \frac{\sqrt{24}}{8} = \frac{2\sqrt{6}}{3} - \frac{\sqrt{6}}{4} = \frac{8\sqrt{6}}{12} - \frac{3\sqrt{6}}{12} = \frac{5\sqrt{6}}{12}$$

J75a

KUMON

Irrational Numbers I

Rationalise the denominators.

Ex.

$$\frac{1}{\sqrt{3}+1} = \frac{\sqrt{3}-1}{(\sqrt{3}+1)(\sqrt{3}-1)} = \frac{\sqrt{3}-1}{2}$$

$$(1) \quad \frac{1}{\sqrt{5}+1} = \frac{\sqrt{5}-1}{(\sqrt{5}+1)(\sqrt{5}-1)} = \frac{\sqrt{5}-1}{4}$$

$$(2) \quad \frac{4}{\sqrt{3}+1} = \frac{4(\sqrt{3}-1)}{(\sqrt{3}+1)(\sqrt{3}-1)} = \frac{4(\sqrt{3}-1)}{2} = 2\sqrt{3}-2$$

$$(3) \quad \frac{4}{\sqrt{5}+1} = \frac{4(\sqrt{5}-1)}{(\sqrt{5}+1)(\sqrt{5}-1)} = \frac{4(\sqrt{5}-1)}{4} = \sqrt{5}-1$$

$$(4) \quad \frac{1}{\sqrt{3}+\sqrt{2}} = \frac{\sqrt{3}-\sqrt{2}}{(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})} = \frac{\sqrt{3}-\sqrt{2}}{1} = \sqrt{3}-\sqrt{2}$$

$$(5) \quad \frac{1}{\sqrt{3}-\sqrt{2}} = \frac{\sqrt{3}+\sqrt{2}}{(\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2})} = \frac{\sqrt{3}+\sqrt{2}}{1} = \sqrt{3}+\sqrt{2}$$

J75b

Ex.

$$\frac{\sqrt{2}+1}{\sqrt{2}-1} = \frac{(\sqrt{2}+1)^2}{(\sqrt{2}-1)(\sqrt{2}+1)} = 3+2\sqrt{2}$$

$$(6) \quad \frac{\sqrt{2}-1}{\sqrt{2}+1} = \frac{(\sqrt{2}-1)^2}{(\sqrt{2}+1)(\sqrt{2}-1)} = \frac{2-2\sqrt{2}+1}{1} = 3-2\sqrt{2}$$

$$(7) \quad \frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}} = \frac{(\sqrt{5}+\sqrt{3})^2}{(\sqrt{5}-\sqrt{3})(\sqrt{5}+\sqrt{3})} = \frac{8+2\sqrt{15}}{2} = 4+\sqrt{15}$$

$$(8) \quad \frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}} = \frac{(\sqrt{5}-\sqrt{3})^2}{(\sqrt{5}+\sqrt{3})(\sqrt{5}-\sqrt{3})} = \frac{8-2\sqrt{15}}{2} = 4-\sqrt{15}$$

$$(9) \quad \frac{\sqrt{5}+2\sqrt{3}}{\sqrt{5}-\sqrt{3}} = \frac{(\sqrt{5}+2\sqrt{3})(\sqrt{5}+\sqrt{3})}{(\sqrt{5}-\sqrt{3})(\sqrt{5}+\sqrt{3})} = \frac{11+3\sqrt{15}}{2}$$

$$(10) \quad \frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+2\sqrt{3}} = \frac{(\sqrt{5}-\sqrt{3})(\sqrt{5}-2\sqrt{3})}{(\sqrt{5}+2\sqrt{3})(\sqrt{5}-2\sqrt{3})} = \frac{11-3\sqrt{15}}{-7} = -\frac{11-3\sqrt{15}}{7}$$

J76a KUMON

Irrational Numbers I

1. Rationalise the denominators.

$$\begin{aligned}
 (1) \quad & \frac{\sqrt{2}}{\sqrt{3}-\sqrt{2}} \\
 &= \frac{\sqrt{2}(\sqrt{3}+\sqrt{2})}{(\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2})} = \frac{\sqrt{6}+2}{1} = \sqrt{6}+2
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad & \frac{\sqrt{6}-2\sqrt{3}}{\sqrt{6}+2\sqrt{3}} \\
 &= \frac{(\sqrt{6}-2\sqrt{3})^2}{(\sqrt{6}+2\sqrt{3})(\sqrt{6}-2\sqrt{3})} = \frac{18-12\sqrt{2}}{-6} = -3+2\sqrt{2}
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad & \frac{2\sqrt{3}-4\sqrt{2}}{\sqrt{3}-\sqrt{2}} \\
 &= \frac{(2\sqrt{3}-4\sqrt{2})(\sqrt{3}+\sqrt{2})}{(\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2})} = -2-2\sqrt{6}
 \end{aligned}$$

$$\begin{aligned}
 (4) \quad & \frac{2\sqrt{3}+8}{3+\sqrt{3}} \\
 &= \frac{(2\sqrt{3}+8)(3-\sqrt{3})}{(3+\sqrt{3})(3-\sqrt{3})} = \frac{-2\sqrt{3}+18}{6} = \frac{-\sqrt{3}+9}{3}
 \end{aligned}$$

$$\begin{aligned}
 (5) \quad & \frac{3\sqrt{2}+2\sqrt{3}}{2\sqrt{3}-3\sqrt{2}} \\
 &= \frac{(3\sqrt{2}+2\sqrt{3})(2\sqrt{3}+3\sqrt{2})}{(2\sqrt{3}-3\sqrt{2})(2\sqrt{3}+3\sqrt{2})} = \frac{12\sqrt{6}+30}{-6} = -2\sqrt{6}-5
 \end{aligned}$$

J76b

2. Simplify the following expressions.

$$\begin{aligned} (1) \quad & \frac{1}{(\sqrt{3} + \sqrt{2})^2} \\ &= \frac{(\sqrt{3} - \sqrt{2})^2}{(\sqrt{3} + \sqrt{2})^2(\sqrt{3} - \sqrt{2})^2} = \frac{5 - 2\sqrt{6}}{(3 - 2)^2} = \mathbf{5 - 2\sqrt{6}} \end{aligned}$$

$$\begin{aligned} (2) \quad & \frac{1}{(\sqrt{10} - \sqrt{2})^2} \\ &= \frac{(\sqrt{10} + \sqrt{2})^2}{(\sqrt{10} - \sqrt{2})^2(\sqrt{10} + \sqrt{2})^2} = \frac{12 + 4\sqrt{5}}{(10 - 2)^2} = \frac{12 + 4\sqrt{5}}{64} = \mathbf{\frac{3 + \sqrt{5}}{16}} \end{aligned}$$

$$\begin{aligned} (3) \quad & \frac{3\sqrt{2}}{\sqrt{6} - \sqrt{3}} - \frac{3}{3 - \sqrt{6}} \\ &= \frac{3\sqrt{2}(\sqrt{6} + \sqrt{3})}{(\sqrt{6} - \sqrt{3})(\sqrt{6} + \sqrt{3})} - \frac{3(3 + \sqrt{6})}{(3 - \sqrt{6})(3 + \sqrt{6})} \\ &= \frac{6\sqrt{3} + 3\sqrt{6}}{3} - \frac{9 + 3\sqrt{6}}{3} \\ &= 2\sqrt{3} + \sqrt{6} - 3 - \sqrt{6} = \mathbf{2\sqrt{3} - 3} \end{aligned}$$

$$\begin{aligned} (4) \quad & \frac{1}{(\sqrt{3} - \sqrt{2})^2} + \frac{1}{(\sqrt{3} + \sqrt{2})^2} \\ &= \frac{(\sqrt{3} + \sqrt{2})^2 + (\sqrt{3} - \sqrt{2})^2}{(\sqrt{3} - \sqrt{2})^2(\sqrt{3} + \sqrt{2})^2} = \frac{5 + 2\sqrt{6} + 5 - 2\sqrt{6}}{(3 - 2)^2} = \mathbf{10} \end{aligned}$$

J77a

KUMON

Irrational Numbers I

1. Simplify the following expressions.

$$(1) \quad (\sqrt{3} + \sqrt{2})^2 = 3 + 2\sqrt{6} + 2 = \mathbf{5 + 2\sqrt{6}}$$

$$(2) \quad (\sqrt{5} + \sqrt{2})^2 = 5 + 2\sqrt{10} + 2 = \mathbf{7 + 2\sqrt{10}}$$

$$(3) \quad (\sqrt{7} + \sqrt{2})^2 = 7 + 2\sqrt{14} + 2 = \mathbf{9 + 2\sqrt{14}}$$

$$(4) \quad (\sqrt{2} + 1)^2 = 2 + 2\sqrt{2} + 1 = \mathbf{3 + 2\sqrt{2}}$$

$$(5) \quad (\sqrt{5} - \sqrt{3})^2 = 5 - 2\sqrt{15} + 3 = \mathbf{8 - 2\sqrt{15}}$$

$$(6) \quad (\sqrt{5} - \sqrt{2})^2 = 5 - 2\sqrt{10} + 2 = \mathbf{7 - 2\sqrt{10}}$$

J77b

From $(\sqrt{3} + \sqrt{2})^2 = 5 + 2\sqrt{6}$, we can deduce:

$$\sqrt{5 + 2\sqrt{6}} = \sqrt{(\sqrt{3} + \sqrt{2})^2} = \sqrt{3} + \sqrt{2} \quad (\text{see Note below})$$

(A term with a radical sign under a radical sign is called a **double radical**.)

2. Simplify the following expressions.

Ex.

$$\sqrt{7 + 2\sqrt{10}} = \sqrt{5} + \sqrt{2}$$

$$\begin{array}{cc} \uparrow & \uparrow \\ 5+2 & 5 \times 2 \end{array}$$



Look for two numbers that multiply to 10 and add to 7.

$$(1) \quad \sqrt{9 + 2\sqrt{14}} = \sqrt{7} + \sqrt{2}$$

$$(2) \quad \sqrt{3 + 2\sqrt{2}} = \sqrt{2} + \sqrt{1} = \sqrt{2} + 1$$

$$(3) \quad \sqrt{6 + 2\sqrt{5}} = \sqrt{5} + \sqrt{1} = \sqrt{5} + 1$$

$$(4) \quad \sqrt{6 + 2\sqrt{8}} = \sqrt{4} + \sqrt{2} = 2 + \sqrt{2}$$

$$(5) \quad \sqrt{10 + 2\sqrt{21}} = \sqrt{7} + \sqrt{3}$$

Note: $\sqrt{5 + 2\sqrt{6}} = -\sqrt{3} - \sqrt{2}$ is **not correct**.

($\sqrt{5 + 2\sqrt{6}}$ should be positive, but $-\sqrt{3} - \sqrt{2}$ is negative.)

J78a

KUMON

Irrational Numbers I

Simplify the following expressions.

Ex.

$$\sqrt{8-2\sqrt{15}} = \sqrt{5}-\sqrt{3} \quad (\text{The answer is not } \sqrt{3}-\sqrt{5}, \text{ see Note below.})$$

$$\begin{array}{cc} \uparrow & \uparrow \\ 5+3 & 5 \times 3 \end{array}$$



Look for two numbers that multiply to 15 and add to 8.

$$(1) \quad \sqrt{7-2\sqrt{10}} = \sqrt{5}-\sqrt{2}$$

$$(2) \quad \sqrt{10-2\sqrt{21}} = \sqrt{7}-\sqrt{3}$$

$$(3) \quad \sqrt{15-2\sqrt{50}} = \sqrt{10}-\sqrt{5}$$

$$(4) \quad \sqrt{6-2\sqrt{5}} = \sqrt{5}-\sqrt{1} = \sqrt{5}-1$$

$$(5) \quad \sqrt{5-2\sqrt{6}} = \sqrt{3}-\sqrt{2}$$

Note: $\sqrt{8-2\sqrt{15}} = \sqrt{3}-\sqrt{5}$ is **not correct**.

($\sqrt{8-2\sqrt{15}}$ should be positive, but $\sqrt{3}-\sqrt{5}$ is negative.)

J78b

Ex.

$$\sqrt{4+\sqrt{12}} = \sqrt{4+\underline{2}\sqrt{3}} = \sqrt{3} + \sqrt{1} = \sqrt{3} + 1$$



Rewrite to give 2 in the position shown.

$$\sqrt{7+4\sqrt{3}} = \sqrt{7+\underline{2}\sqrt{12}} = \sqrt{4} + \sqrt{3} = 2 + \sqrt{3}$$

$$(6) \quad \sqrt{6-\sqrt{20}} = \sqrt{6-2\sqrt{5}} = \sqrt{5} - \sqrt{1} = \sqrt{5} - 1$$

$$(7) \quad \sqrt{7-\sqrt{48}} = \sqrt{7-2\sqrt{12}} = \sqrt{4} - \sqrt{3} = 2 - \sqrt{3}$$

$$(8) \quad \sqrt{8+4\sqrt{3}} = \sqrt{8+2\sqrt{12}} = \sqrt{6} + \sqrt{2}$$

$$(9) \quad \sqrt{12-6\sqrt{3}} = \sqrt{12-2\sqrt{27}} = \sqrt{9} - \sqrt{3} = 3 - \sqrt{3}$$

$$(10) \quad \sqrt{19+8\sqrt{3}} = \sqrt{19+2\sqrt{48}} = \sqrt{16} + \sqrt{3} = 4 + \sqrt{3}$$

J79a

KUMON

Irrational Numbers I

Simplify the following expressions.

Ex.

$$\sqrt{2+\sqrt{3}} = \sqrt{\frac{4+2\sqrt{3}}{2}} = \frac{\sqrt{4+2\sqrt{3}}}{\sqrt{2}} = \frac{\sqrt{3}+1}{\sqrt{2}} = \frac{\sqrt{6}+\sqrt{2}}{2}$$

Note:

$$\sqrt{2+\sqrt{3}} = \sqrt{\frac{2(2+\sqrt{3})}{2}}$$

$$(1) \quad \sqrt{3+\sqrt{5}} = \sqrt{\frac{6+2\sqrt{5}}{2}} = \frac{\sqrt{6+2\sqrt{5}}}{\sqrt{2}} = \frac{\sqrt{5}+1}{\sqrt{2}} = \frac{\sqrt{10}+\sqrt{2}}{2}$$

$$(2) \quad \sqrt{4-\sqrt{7}} = \sqrt{\frac{8-2\sqrt{7}}{2}} = \frac{\sqrt{8-2\sqrt{7}}}{\sqrt{2}} = \frac{\sqrt{7}-1}{\sqrt{2}} = \frac{\sqrt{14}-\sqrt{2}}{2}$$

$$(3) \quad \sqrt{5-\sqrt{21}} = \sqrt{\frac{10-2\sqrt{21}}{2}} = \frac{\sqrt{10-2\sqrt{21}}}{\sqrt{2}} = \frac{\sqrt{7}-\sqrt{3}}{\sqrt{2}} = \frac{\sqrt{14}-\sqrt{6}}{2}$$

$$(4) \quad \sqrt{7+\sqrt{45}} = \sqrt{\frac{14+2\sqrt{45}}{2}} = \frac{\sqrt{14+2\sqrt{45}}}{\sqrt{2}} = \frac{3+\sqrt{5}}{\sqrt{2}} = \frac{3\sqrt{2}+\sqrt{10}}{2}$$

J79b

Ex.

$$\sqrt{9+3\sqrt{8}} = \sqrt{9+\underline{6\sqrt{2}}} = \sqrt{9+\underline{2\sqrt{18}}} = \sqrt{6} + \sqrt{3} \quad \text{☞}$$

Pay close attention to the underlined terms.

$$\begin{aligned} (5) \quad & \sqrt{28-5\sqrt{12}} \\ & = \sqrt{28-10\sqrt{3}} = \sqrt{28-2\sqrt{75}} = \sqrt{25}-\sqrt{3} = \mathbf{5-\sqrt{3}} \end{aligned}$$

$$\begin{aligned} (6) \quad & \sqrt{14+3\sqrt{20}} \\ & = \sqrt{14+6\sqrt{5}} = \sqrt{14+2\sqrt{45}} = \sqrt{9} + \sqrt{5} = \mathbf{3+\sqrt{5}} \end{aligned}$$

$$\begin{aligned} (7) \quad & \sqrt{11+2\sqrt{30}} - \sqrt{11-2\sqrt{30}} \\ & = (\sqrt{6} + \sqrt{5}) - (\sqrt{6} - \sqrt{5}) = \mathbf{2\sqrt{5}} \end{aligned}$$

$$\begin{aligned} (8) \quad & \frac{1}{\sqrt{11-4\sqrt{7}}} \\ & = \frac{1}{\sqrt{11-2\sqrt{28}}} = \frac{1}{\sqrt{7}-\sqrt{4}} = \frac{1}{\sqrt{7}-2} \\ & = \frac{\sqrt{7}+2}{(\sqrt{7}-2)(\sqrt{7}+2)} = \frac{\sqrt{7}+2}{\mathbf{3}} \end{aligned}$$

J80a KUMON

Irrational Numbers I

1. Simplify the following expressions.

$$(1) \quad \sqrt{4+2\sqrt{3}} = \sqrt{3} + 1$$

$$(2) \quad \sqrt{8-2\sqrt{15}} = \sqrt{5} - \sqrt{3}$$

$$(3) \quad \sqrt{2+\sqrt{3}} = \frac{\sqrt{4+2\sqrt{3}}}{\sqrt{2}} = \frac{\sqrt{3}+1}{\sqrt{2}} = \frac{\sqrt{6}+\sqrt{2}}{2}$$

2. Simplify the following expressions.

$$\begin{aligned} (1) \quad & \frac{1}{(2-\sqrt{3})^2} + \frac{1}{(2+\sqrt{3})^2} \\ &= \frac{(2+\sqrt{3})^2 + (2-\sqrt{3})^2}{(2-\sqrt{3})^2(2+\sqrt{3})^2} \\ &= \frac{7+4\sqrt{3}+7-4\sqrt{3}}{(4-3)^2} \\ &= 14 \end{aligned}$$

$$\begin{aligned} (2) \quad & \frac{1}{\sqrt{12+2\sqrt{35}}} + \frac{1}{\sqrt{12-2\sqrt{35}}} \\ &= \frac{1}{\sqrt{7}+\sqrt{5}} + \frac{1}{\sqrt{7}-\sqrt{5}} \\ &= \frac{(\sqrt{7}-\sqrt{5})+(\sqrt{7}+\sqrt{5})}{(\sqrt{7}+\sqrt{5})(\sqrt{7}-\sqrt{5})} \\ &= \frac{2\sqrt{7}}{2} = \sqrt{7} \end{aligned}$$

J 80b

3. Simplify the following expressions.

$$\begin{aligned}(1) \quad & (3\sqrt{2} + 2\sqrt{3} + \sqrt{6})(3\sqrt{2} - 2\sqrt{3} + \sqrt{6}) \\ &= [(3\sqrt{2} + \sqrt{6}) + 2\sqrt{3}][(3\sqrt{2} + \sqrt{6}) - 2\sqrt{3}] \\ &= (3\sqrt{2} + \sqrt{6})^2 - (2\sqrt{3})^2 \\ &= 24 + 6\sqrt{12} - 12 \\ &= \mathbf{12 + 12\sqrt{3}}\end{aligned}$$

$$\begin{aligned}(2) \quad & \sqrt{5} - \sqrt{\frac{5}{4}} + \sqrt{\frac{4}{5}} \\ &= \sqrt{5} - \frac{\sqrt{5}}{2} + \frac{2}{\sqrt{5}} = \sqrt{5} - \frac{\sqrt{5}}{2} + \frac{2\sqrt{5}}{5} = \frac{10\sqrt{5} - 5\sqrt{5} + 4\sqrt{5}}{10} = \mathbf{\frac{9\sqrt{5}}{10}}\end{aligned}$$

$$\begin{aligned}(3) \quad & \frac{3\sqrt{2}}{\sqrt{6} - \sqrt{3}} - \frac{3}{3 - \sqrt{6}} \\ &= \frac{3\sqrt{2}(\sqrt{6} + \sqrt{3})}{(\sqrt{6} - \sqrt{3})(\sqrt{6} + \sqrt{3})} - \frac{3(3 + \sqrt{6})}{(3 - \sqrt{6})(3 + \sqrt{6})} \\ &= \frac{6\sqrt{3} + 3\sqrt{6} - 9 - 3\sqrt{6}}{3} \\ &= \frac{6\sqrt{3} - 9}{3} \\ &= \mathbf{2\sqrt{3} - 3}\end{aligned}$$

$$\begin{aligned}(4) \quad & \frac{3\sqrt{2}}{\sqrt{3} + \sqrt{6}} - \frac{4\sqrt{3}}{\sqrt{6} + \sqrt{2}} + \frac{\sqrt{6}}{\sqrt{3} + \sqrt{2}} \\ &= \frac{3\sqrt{2}(\sqrt{3} - \sqrt{6})}{(\sqrt{3} + \sqrt{6})(\sqrt{3} - \sqrt{6})} - \frac{4\sqrt{3}(\sqrt{6} - \sqrt{2})}{(\sqrt{6} + \sqrt{2})(\sqrt{6} - \sqrt{2})} + \frac{\sqrt{6}(\sqrt{3} - \sqrt{2})}{(\sqrt{3} + \sqrt{2})(\sqrt{3} - \sqrt{2})} \\ &= \frac{3\sqrt{6} - 3\sqrt{12}}{3 - 6} - \frac{4\sqrt{18} - 4\sqrt{6}}{6 - 2} + \frac{\sqrt{18} - \sqrt{12}}{3 - 2} \\ &= \frac{3\sqrt{6} - 6\sqrt{3}}{-3} - \frac{12\sqrt{2} - 4\sqrt{6}}{4} + \frac{3\sqrt{2} - 2\sqrt{3}}{1} \\ &= -\sqrt{6} + 2\sqrt{3} - 3\sqrt{2} + \sqrt{6} + 3\sqrt{2} - 2\sqrt{3} \\ &= \mathbf{0}\end{aligned}$$

J81a KUMON

Irrational Numbers II

1. If $x = 5 + \sqrt{3}$ and $y = 5 - \sqrt{3}$, evaluate the following expressions.

$$(1) \quad x + y = (5 + \sqrt{3}) + (5 - \sqrt{3}) = \mathbf{10}$$

$$(2) \quad xy = (5 + \sqrt{3})(5 - \sqrt{3}) = 25 - 3 = \mathbf{22}$$

2. If $x = \sqrt{5} + \sqrt{2}$ and $y = \sqrt{5} - \sqrt{2}$, evaluate the following expressions.

$$(1) \quad x + y = (\sqrt{5} + \sqrt{2}) + (\sqrt{5} - \sqrt{2}) = \mathbf{2\sqrt{5}}$$

$$(2) \quad xy = (\sqrt{5} + \sqrt{2})(\sqrt{5} - \sqrt{2}) = \mathbf{3}$$

3. If $x = \frac{3 + \sqrt{5}}{4}$ and $y = \frac{3 - \sqrt{5}}{4}$, evaluate the following expressions.

$$(1) \quad x + y = \left(\frac{3 + \sqrt{5}}{4}\right) + \left(\frac{3 - \sqrt{5}}{4}\right) = \frac{6}{4} = \mathbf{\frac{3}{2}}$$

$$(2) \quad xy = \left(\frac{3 + \sqrt{5}}{4}\right)\left(\frac{3 - \sqrt{5}}{4}\right) = \frac{4}{16} = \mathbf{\frac{1}{4}}$$

J8|b

4. If $x = \frac{1}{\sqrt{3}+1}$ and $y = \frac{1}{\sqrt{3}-1}$, evaluate the following expressions.

$$\begin{aligned}(1) \quad x+y &= \left(\frac{1}{\sqrt{3}+1}\right) + \left(\frac{1}{\sqrt{3}-1}\right) \\ &= \frac{\sqrt{3}-1}{2} + \frac{\sqrt{3}+1}{2} \\ &= \sqrt{3}\end{aligned}$$

$$(2) \quad xy = \left(\frac{1}{\sqrt{3}+1}\right)\left(\frac{1}{\sqrt{3}-1}\right) = \frac{1}{2}$$

5. If $x = \frac{\sqrt{7}-\sqrt{5}}{\sqrt{7}+\sqrt{5}}$ and $y = \frac{\sqrt{7}+\sqrt{5}}{\sqrt{7}-\sqrt{5}}$, evaluate the following expressions.

$$\begin{aligned}(1) \quad x+y &= \left(\frac{\sqrt{7}-\sqrt{5}}{\sqrt{7}+\sqrt{5}}\right) + \left(\frac{\sqrt{7}+\sqrt{5}}{\sqrt{7}-\sqrt{5}}\right) \\ &= \frac{(\sqrt{7}-\sqrt{5})^2}{2} + \frac{(\sqrt{7}+\sqrt{5})^2}{2} \\ &= \frac{12-2\sqrt{35}+12+2\sqrt{35}}{2} \\ &= \mathbf{12}\end{aligned}$$

$$(2) \quad xy = \left(\frac{\sqrt{7}-\sqrt{5}}{\sqrt{7}+\sqrt{5}}\right)\left(\frac{\sqrt{7}+\sqrt{5}}{\sqrt{7}-\sqrt{5}}\right) = \mathbf{1}$$

J82a

KUMON

Irrational Numbers II

1. If $x + y = a$ and $xy = b$, express the following in terms of a and b .

$$(1) \quad x^2 + y^2 = (x + y)^2 - 2xy = \mathbf{a^2 - 2b}$$

$$\begin{aligned} (2) \quad x^4 + y^4 &= (x^2 + y^2)^2 - 2x^2y^2 \\ &= (a^2 - 2b)^2 - 2b^2 \\ &= a^4 - 4a^2b + 4b^2 - 2b^2 \\ &= \mathbf{a^4 - 4a^2b + 2b^2} \end{aligned}$$

Use your answer
from (1).

$$\begin{aligned} (3) \quad (x - y)^2 &= x^2 - 2xy + y^2 \\ &= x^2 + y^2 - 2xy \\ &= a^2 - 2b - 2b \\ &= \mathbf{a^2 - 4b} \end{aligned}$$

Use your answer
from (1).

$$\left[\begin{array}{l} \text{Alternative Solution} \\ (x - y)^2 = (x + y)^2 - 4xy \\ = \mathbf{a^2 - 4b} \end{array} \right]$$

$$\begin{aligned} (4) \quad x^3 + y^3 &= (x + y)^3 - \boxed{3}xy(x + y) \\ &= a^3 - 3ba \\ &= \mathbf{a^3 - 3ab} \end{aligned}$$

J 82b

2. If $x = 2 + \sqrt{3}$ and $y = 2 - \sqrt{3}$, evaluate the following expressions.

$$(1) \quad x + y = (2 + \sqrt{3}) + (2 - \sqrt{3}) \\ = 4$$

$$(2) \quad xy = (2 + \sqrt{3})(2 - \sqrt{3}) \\ = 1$$

$$(3) \quad x^2 + y^2 = (x + y)^2 - 2xy \\ = 16 - 2 \\ = 14$$

Use your answers from
(1) and (2).

$$(4) \quad x^4 + y^4 = (x^2 + y^2)^2 - 2x^2y^2 \\ = 14^2 - 2 \\ = 196 - 2 \\ = 194$$

Use your answers from
(2) and (3).

$$(5) \quad (x - y)^2 = x^2 - 2xy + y^2 \\ = x^2 + y^2 - 2xy \\ = 14 - 2 \\ = 12$$

Use your answers from
(2) and (3).

$$\left[\begin{array}{l} \text{Alternative Solution} \\ (x - y)^2 = [(2 + \sqrt{3}) - (2 - \sqrt{3})]^2 \\ = (2\sqrt{3})^2 \\ = 12 \end{array} \right]$$

$$(6) \quad x^3 + y^3 = (x + y)^3 - 3xy(x + y) \\ = 64 - 3 \times 1 \times 4 \\ = 64 - 12 \\ = 52$$

Use your answers from
(1) and (2).

J83a

KUMON

Irrational Numbers II

1. If $x = \sqrt{5} + \sqrt{3}$ and $y = \sqrt{5} - \sqrt{3}$, evaluate the following expressions.

$$\begin{aligned} (1) \quad x + y &= (\sqrt{5} + \sqrt{3}) + (\sqrt{5} - \sqrt{3}) \\ &= 2\sqrt{5} \end{aligned}$$

$$\begin{aligned} (2) \quad xy &= (\sqrt{5} + \sqrt{3})(\sqrt{5} - \sqrt{3}) \\ &= 5 - 3 \\ &= 2 \end{aligned}$$

$$\begin{aligned} (3) \quad x^2 + y^2 &= (x + y)^2 - 2xy \\ &= (2\sqrt{5})^2 - 2 \times 2 \\ &= 20 - 4 \\ &= 16 \end{aligned}$$

$$\begin{aligned} (4) \quad x^4 + y^4 &= (x^2 + y^2)^2 - 2x^2y^2 \\ &= 16^2 - 2 \times 4 \\ &= 256 - 8 \\ &= 248 \end{aligned}$$

$$\begin{aligned} (5) \quad (x - y)^2 &= x^2 - 2xy + y^2 \\ &= 16 - 2 \times 2 \\ &= 12 \end{aligned}$$

$$\left[\begin{array}{l} \text{Alternative Solution} \\ (x - y)^2 = [(\sqrt{5} + \sqrt{3}) - (\sqrt{5} - \sqrt{3})]^2 \\ \quad = (2\sqrt{3})^2 \\ \quad = 12 \end{array} \right]$$

$$\begin{aligned} (6) \quad x^3 + y^3 &= (x + y)^3 - 3xy(x + y) \\ &= (2\sqrt{5})^3 - 3 \times 2 \times 2\sqrt{5} \\ &= 40\sqrt{5} - 12\sqrt{5} \\ &= 28\sqrt{5} \end{aligned}$$

J 83b

2. If $x = 4 + 2\sqrt{3}$ and $y = 4 - 2\sqrt{3}$, evaluate the following expressions.

$$\begin{aligned}(1) \quad x + y &= (4 + 2\sqrt{3}) + (4 - 2\sqrt{3}) \\ &= \mathbf{8}\end{aligned}$$

$$\begin{aligned}(2) \quad xy &= (4 + 2\sqrt{3})(4 - 2\sqrt{3}) \\ &= 16 - 12 \\ &= \mathbf{4}\end{aligned}$$

$$\begin{aligned}(3) \quad x^2 + y^2 &= (x + y)^2 - 2xy \\ &= 64 - 8 \\ &= \mathbf{56}\end{aligned}$$

$$\begin{aligned}(4) \quad x^3 + y^3 &= (x + y)^3 - 3xy(x + y) \\ &= 8^3 - 3 \times 4 \times 8 \\ &= 512 - 96 \\ &= \mathbf{416}\end{aligned}$$

$$\begin{aligned}(5) \quad \frac{y}{x} + \frac{x}{y} &= \frac{y^2 + x^2}{xy} \\ &= \frac{56}{4} \\ &= \mathbf{14}\end{aligned}$$

$$\begin{aligned}(6) \quad \frac{1}{x^2} + \frac{1}{y^2} &= \frac{y^2 + x^2}{x^2 y^2} \\ &= \frac{56}{16} \\ &= \frac{\mathbf{7}}{\mathbf{2}}\end{aligned}$$

J84a KUMON

Irrational Numbers II

1. If $x = \frac{1}{2+\sqrt{3}}$ and $y = \frac{1}{2-\sqrt{3}}$, evaluate the following expressions.

$$\begin{aligned}(1) \quad x+y &= \left(\frac{1}{2+\sqrt{3}}\right) + \left(\frac{1}{2-\sqrt{3}}\right) \\ &= 2-\sqrt{3}+2+\sqrt{3} \\ &= \mathbf{4}\end{aligned}$$

$$(2) \quad xy = \left(\frac{1}{2+\sqrt{3}}\right)\left(\frac{1}{2-\sqrt{3}}\right) = \mathbf{1}$$

$$\begin{aligned}(3) \quad x^2+y^2 &= (x+y)^2-2xy \\ &= 16-2 \\ &= \mathbf{14}\end{aligned}$$

$$\begin{aligned}(4) \quad x^3+y^3 &= (x+y)^3-3xy(x+y) \\ &= 4^3-3\times 1\times 4 \\ &= 64-12 \\ &= \mathbf{52}\end{aligned}$$

J84b

2. If $x = \frac{\sqrt{5} - \sqrt{3}}{\sqrt{5} + \sqrt{3}}$ and $y = \frac{\sqrt{5} + \sqrt{3}}{\sqrt{5} - \sqrt{3}}$, evaluate the following expressions.

$$\begin{aligned}(1) \quad x + y &= \left(\frac{\sqrt{5} - \sqrt{3}}{\sqrt{5} + \sqrt{3}} \right) + \left(\frac{\sqrt{5} + \sqrt{3}}{\sqrt{5} - \sqrt{3}} \right) \\ &= \frac{(\sqrt{5} - \sqrt{3})^2}{2} + \frac{(\sqrt{5} + \sqrt{3})^2}{2} \\ &= \frac{8 - 2\sqrt{15} + 8 + 2\sqrt{15}}{2} \\ &= \mathbf{8}\end{aligned}$$

$$\begin{aligned}(2) \quad xy &= \left(\frac{\sqrt{5} - \sqrt{3}}{\sqrt{5} + \sqrt{3}} \right) \left(\frac{\sqrt{5} + \sqrt{3}}{\sqrt{5} - \sqrt{3}} \right) \\ &= \mathbf{1}\end{aligned}$$

$$\begin{aligned}(3) \quad (x - y)^2 &= (x + y)^2 - 4xy \\ &= 64 - 4 \\ &= \mathbf{60}\end{aligned}$$

$$\begin{aligned}(4) \quad \frac{x}{y^2} + \frac{y}{x^2} &= \frac{x^3 + y^3}{x^2 y^2} \\ &= \frac{(x + y)^3 - 3xy(x + y)}{x^2 y^2} \\ &= 8^3 - 3 \times 1 \times 8 \\ &= \mathbf{488}\end{aligned}$$

J85a KUMON

Irrational Numbers II

1. Evaluate the following expressions.

Ex.

If $x + y = \sqrt{2}$ and $x^2 + y^2 = 6$

$$xy = \frac{(x+y)^2 - (x^2 + y^2)}{2} = \frac{2-6}{2} = -2$$

(1) If $x + y = \sqrt{6}$ and $x^2 + y^2 = 4$

$$xy = \frac{(x+y)^2 - (x^2 + y^2)}{2} = \frac{6-4}{2} = 1$$

(2) If $x - y = \sqrt{6}$ and $x^2 + y^2 = 4$

$$xy = -\frac{(x-y)^2 - (x^2 + y^2)}{2} = -\frac{6-4}{2} = -1$$

(3) If $x - y = \sqrt{3}$ and $x^2 + y^2 = 5$

$$xy = -\frac{(x-y)^2 - (x^2 + y^2)}{2} = -\frac{3-5}{2} = 1$$

(4) If $x + y = \sqrt{10}$ and $(x - y)^2 = 2$

$$xy = \frac{(x+y)^2 - (x-y)^2}{4} = \frac{10-2}{4} = 2$$

J 85b

2. If $x + y = \sqrt{3}$ and $x^2 + y^2 = 5$, evaluate the following expressions.

(1) xy

$$= \frac{(x+y)^2 - (x^2 + y^2)}{2} = \frac{3 - 5}{2} = -1$$

(2) $x^3 + y^3$

$$\begin{aligned} &= (x+y)^3 - 3xy(x+y) \\ &= (\sqrt{3})^3 - 3 \times (-1) \times \sqrt{3} \\ &= 3\sqrt{3} + 3\sqrt{3} \\ &= \mathbf{6\sqrt{3}} \end{aligned}$$

(3) $\frac{y}{x} + \frac{x}{y}$

$$\begin{aligned} &= \frac{y^2 + x^2}{xy} \\ &= \frac{5}{-1} \\ &= \mathbf{-5} \end{aligned}$$

(4) $\frac{y}{x^2} + \frac{x}{y^2}$

$$\begin{aligned} &= \frac{y^3 + x^3}{x^2 y^2} \\ &= \mathbf{6\sqrt{3}} \end{aligned}$$

J86a KUMON

Irrational Numbers II

Any positive number a has two *square roots*.
 (For example, 9 has two square roots, 3 and -3 .)

1. Complete the following exercises.

(1) The square roots of 16 are: $\boxed{4}$ and $\boxed{-4}$

(2) The square roots of 25 are: $\boxed{5}$ and $\boxed{-5}$

(3) The square roots of 5^2 are: $\boxed{5}$ and $\boxed{-5}$

(4) The square roots of $(-5)^2$ are: $\boxed{5}$ and $\boxed{-5}$

The *square roots* of a are the positive root, \sqrt{a} ,
 and the negative root, $-\sqrt{a}$.
 (For example, $\sqrt{9} = 3$ and $-\sqrt{9} = -3$.)

(5) $\sqrt{36} = \boxed{6}$

(6) $-\sqrt{36} = \boxed{-6}$

(7) $\sqrt{(-5)^2} = \boxed{5}$

(8) $-\sqrt{(-5)^2} = \boxed{-5}$

Note: The symbol $\sqrt{\quad}$ refers to the positive square root.
 \sqrt{a} is called the principal square root of a .

J 86b

2. Determine if the underlined are true or false. If true, write “True”. If false, write “False” and the correct answer.

(1) $\sqrt{5^2} = \underline{5}$ and $\underline{-5}$ False/correct answer is 5

(2) $\sqrt{(-5)^2} = \underline{-5}$ False/correct answer is 5

(3) $\sqrt{9} = \underline{\pm 3}$ False/correct answer is 3

(4) $-\sqrt{16} = \underline{-4}$ True

(5) $\sqrt{(3-1)^2} = \underline{-2}$ False/correct answer is 2

(6) $\sqrt{(1-3)^2} = \underline{2}$ True

J87a KUMON

Irrational Numbers II

1. Evaluate the following expressions.

(1) When $a = -2$, $\sqrt{a^2} = \sqrt{(-2)^2} = \mathbf{2}$

(2) When $a = -1$, $\sqrt{a^2} = \sqrt{(-1)^2} = \mathbf{1}$

(3) When $a = 0$, $\sqrt{a^2} = \sqrt{0^2} = \mathbf{0}$

(4) When $a = 1$, $\sqrt{a^2} = \sqrt{1^2} = \mathbf{1}$

(5) When $a = 2$, $\sqrt{a^2} = \sqrt{2^2} = \mathbf{2}$

(6) When $a = 3$, $\sqrt{a^2} = \sqrt{3^2} = \mathbf{3}$

(7) When $a = -3$, $\sqrt{a^2} = \sqrt{(-3)^2} = \mathbf{3}$

2. Fill in each blank with either a , $-a$, a^2 , or $-a^2$.

(1) When $a \geq 0$, (a is 0 or a positive number), $\sqrt{a^2} = \boxed{\mathbf{a}}$

(2) When $a < 0$, (a is a negative number), $\sqrt{a^2} = \boxed{\mathbf{-a}}$

J 87b

3. Evaluate the following expressions.

(1) When $x = 5$, $\sqrt{(x-3)^2} = \sqrt{(5-3)^2} = 2$

(2) When $x = 4$, $\sqrt{(x-3)^2} = \sqrt{(4-3)^2} = 1$

(3) When $x = 3$, $\sqrt{(x-3)^2} = \sqrt{(3-3)^2} = 0$



(4) When $x = 2$, $\sqrt{(x-3)^2} = \sqrt{(2-3)^2} = 1$

(5) When $x = 1$, $\sqrt{(x-3)^2} = \sqrt{(1-3)^2} = 2$

(6) When $x = 0$, $\sqrt{(x-3)^2} = \sqrt{(0-3)^2} = 3$

(7) When $x = -1$, $\sqrt{(x-3)^2} = \sqrt{(-1-3)^2} = 4$

Note Summary

- When $a \geq b$, $\sqrt{(a-b)^2} = a-b$  When the term in brackets is positive or zero.
- When $a < b$, $\sqrt{(a-b)^2} = -(a-b)$  When the term in brackets is negative.

Irrational Numbers II

Rewrite the following square roots.

Ex.

$$\sqrt{(x-3)^2} = \begin{cases} x-3 & (\text{when } x \geq 3) \quad \text{From } x-3 \geq 0. \\ -(x-3) & (\text{when } x < 3) \quad \text{From } x-3 < 0. \end{cases}$$

$$(1) \quad \sqrt{(x-2)^2} = \begin{cases} x-2 & (\text{when } x \geq 2) \\ -(x-2) & (\text{when } x < 2) \end{cases}$$

$$(2) \quad \sqrt{(x-1)^2} = \begin{cases} x-1 & (\text{when } x \geq 1) \\ -(x-1) & (\text{when } x < 1) \end{cases}$$

$$(3) \quad \sqrt{(x+1)^2} = \begin{cases} x+1 & (\text{when } x \geq -1) \\ -(x+1) & (\text{when } x < -1) \end{cases}$$

$$(4) \quad \sqrt{x^2} = \begin{cases} x & (\text{when } x \geq 0) \\ -x & (\text{when } x < 0) \end{cases}$$

J 88b

$$(5) \quad \sqrt{(x-4)^2} = \begin{cases} x-4 & (\text{when } x \geq 4) \\ -(x-4) & (\text{when } x < 4) \end{cases}$$

$$(6) \quad \sqrt{(x-a)^2} = \begin{cases} x-a & (\text{when } x \geq a) \\ -(x-a) & (\text{when } x < a) \end{cases}$$

$$(7) \quad \sqrt{(x+3)^2} = \begin{cases} x+3 & (\text{when } x \geq -3) \\ -(x+3) & (\text{when } x < -3) \end{cases}$$

$$(8) \quad \sqrt{(x+a)^2} = \begin{cases} x+a & (\text{when } x \geq -a) \\ -(x+a) & (\text{when } x < -a) \end{cases}$$

$$(9) \quad \sqrt{(x-2a)^2} = \begin{cases} x-2a & (\text{when } x \geq 2a) \\ -(x-2a) & (\text{when } x < 2a) \end{cases}$$

Irrational Numbers II

Ex.Rewrite the following square roots when $x > 3$.

$$\sqrt{(x-1)^2} = x-1 \quad \text{☞} \quad \text{If } x > 3, \text{ then } x-1 > 0 \text{ is always true.}$$

1. Rewrite the following square roots when $x > 5$.

$$(1) \quad \sqrt{(x-2)^2} = \mathbf{x-2}$$

$$(2) \quad \sqrt{(x-4)^2} = \mathbf{x-4}$$

Ex.Rewrite the following square roots when $x > 3$.

$$\sqrt{(x-5)^2} = \begin{cases} x-5 & (\text{when } x \geq 5) & \text{☞} \quad \text{From } x-5 \geq 0. \\ -(x-5) & (\text{when } \underline{3 < x < 5}) & \text{☞} \quad \text{From } x-5 < 0 \\ & & \text{and } \underline{x > 3.} \end{cases}$$

2. Rewrite the following square roots when $x > 2$.

$$(1) \quad \sqrt{(x-3)^2} = \begin{cases} \mathbf{x-3} & (\text{when } \mathbf{x \geq 3}) \\ \mathbf{-(x-3)} & (\text{when } \mathbf{2 < x < 3}) \end{cases}$$

$$(2) \quad \sqrt{(x-4)^2} = \begin{cases} \mathbf{x-4} & (\text{when } \mathbf{x \geq 4}) \\ \mathbf{-(x-4)} & (\text{when } \mathbf{2 < x < 4}) \end{cases}$$

J 89b

3. Rewrite the following square roots when $x > 4$.

$$(1) \quad \sqrt{(x-2)^2} = \mathbf{x-2}$$

$$(2) \quad \sqrt{(x-4)^2} = \mathbf{x-4}$$

$$(3) \quad \sqrt{(x-6)^2} = \begin{cases} \mathbf{x-6} & \text{(when } x \geq 6 \text{)} \\ \mathbf{-(x-6)} & \text{(when } 4 < x < 6 \text{)} \end{cases}$$

4. Rewrite the following square roots when $x < -3$.

$$(1) \quad \sqrt{(x-2)^2} = \mathbf{-(x-2)}$$

$$(2) \quad \sqrt{(x+2)^2} = \mathbf{-(x+2)}$$

$$(3) \quad \sqrt{(x+5)^2} = \begin{cases} \mathbf{x+5} & \text{(when } -5 \leq x < -3 \text{)} \\ \mathbf{-(x+5)} & \text{(when } x < -5 \text{)} \end{cases}$$

J90a

KUMON

Irrational Numbers II

1. If $x = \frac{\sqrt{7} + \sqrt{3}}{\sqrt{7} - \sqrt{3}}$ and $y = \frac{\sqrt{7} - \sqrt{3}}{\sqrt{7} + \sqrt{3}}$, evaluate the following expressions.

$$\begin{aligned}
 (1) \quad x + y &= \left(\frac{\sqrt{7} + \sqrt{3}}{\sqrt{7} - \sqrt{3}} \right) + \left(\frac{\sqrt{7} - \sqrt{3}}{\sqrt{7} + \sqrt{3}} \right) \\
 &= \frac{(\sqrt{7} + \sqrt{3})^2}{4} + \frac{(\sqrt{7} - \sqrt{3})^2}{4} \\
 &= \frac{10 + 2\sqrt{21} + 10 - 2\sqrt{21}}{4} \\
 &= \mathbf{5}
 \end{aligned}$$

$$(2) \quad xy = \left(\frac{\sqrt{7} + \sqrt{3}}{\sqrt{7} - \sqrt{3}} \right) \left(\frac{\sqrt{7} - \sqrt{3}}{\sqrt{7} + \sqrt{3}} \right) = \mathbf{1}$$

$$\begin{aligned}
 (3) \quad \frac{y}{x} + \frac{x}{y} &= \frac{y^2 + x^2}{xy} \\
 &= \frac{(x + y)^2 - 2xy}{xy} \\
 &= 25 - 2 = \mathbf{23}
 \end{aligned}$$

$$\begin{aligned}
 (4) \quad \frac{x^2}{y} + \frac{y^2}{x} &= \frac{x^3 + y^3}{xy} \\
 &= \frac{(x + y)^3 - 3xy(x + y)}{xy} \\
 &= 5^3 - 3 \times 1 \times 5 \\
 &= \mathbf{110}
 \end{aligned}$$

J90b

2. Rewrite the following square roots.

$$(1) \quad \sqrt{(x-2)^2} = \begin{cases} x-2 & (\text{when } x \geq 2) \\ -(x-2) & (\text{when } x < 2) \end{cases}$$

$$(2) \quad \sqrt{(x+5)^2} = \begin{cases} x+5 & (\text{when } x \geq -5) \\ -(x+5) & (\text{when } x < -5) \end{cases}$$

$$(3) \quad \sqrt{(x-a)^2} = \begin{cases} x-a & (\text{when } x \geq a) \\ -(x-a) & (\text{when } x < a) \end{cases}$$

$$(4) \quad \sqrt{(x+2a)^2} = \begin{cases} x+2a & (\text{when } x \geq -2a) \\ -(x+2a) & (\text{when } x < -2a) \end{cases}$$


Consider this!

Rewrite the following square roots.

$$(1) \quad \sqrt{a^4}$$

[Sol] $\sqrt{a^4} = \sqrt{(a^2)^2} = \boxed{a^2}$  Whether $a \geq 0$ or $a < 0$, then $a^2 \geq 0$ in either case.

$$(2) \quad \sqrt{a^6}$$

[Sol] $\sqrt{a^6} = \sqrt{(a^3)^2}$  The sign of a^3 depends on the sign of a .

$$\sqrt{a^6} = \begin{cases} \boxed{a^3} & (\text{when } a \geq 0) \\ \boxed{-a^3} & (\text{when } a < 0) \end{cases}$$

Quadratic Equations I

Solve the following quadratic equations.

Ex. 1

$$x^2 + 3x - 10 = 0$$

$$\text{[Sol]} (x+5)(x-2) = 0$$

$$x = -5, 2$$

Ex. 2

$$6x^2 - x = 12$$

$$\text{[Sol]} 6x^2 - x - 12 = 0$$

$$(3x+4)(2x-3) = 0$$

$$x = -\frac{4}{3}, \frac{3}{2}$$

$$(1) \quad x^2 + 5x + 6 = 0$$

$$(x+3)(x+2) = 0$$

$$\mathbf{x = -3, -2}$$

$$(3) \quad x^2 - 6x = 16$$

$$x^2 - 6x - 16 = 0$$

$$(x-8)(x+2) = 0$$

$$\mathbf{x = 8, -2}$$

$$(2) \quad x^2 - 12x + 35 = 0$$

$$(x-7)(x-5) = 0$$

$$\mathbf{x = 7, 5}$$

$$(4) \quad x^2 - 2 = x$$

$$x^2 - x - 2 = 0$$

$$(x-2)(x+1) = 0$$

$$\mathbf{x = 2, -1}$$

J91b

$$(5) \quad (x-2)(x-1) = 0$$

$$\mathbf{x = 2, 1}$$

$$(9) \quad x^2 - 2x - 35 = 0$$

$$(x-7)(x+5) = 0$$

$$\mathbf{x = 7, -5}$$

$$(6) \quad (x-3)(x+2) = 0$$

$$\mathbf{x = 3, -2}$$

$$(10) \quad x^2 + 2x - 99 = 0$$

$$(x+11)(x-9) = 0$$

$$\mathbf{x = -11, 9}$$

$$(7) \quad (3x+2)(5x-6) = 0$$

$$\mathbf{x = -\frac{2}{3}, \frac{6}{5}}$$

$$(11) \quad 2x^2 - 17x + 30 = 0$$

$$(2x-5)(x-6) = 0$$

$$\mathbf{x = \frac{5}{2}, 6}$$

$$(8) \quad x(x-6) = 40$$

$$x^2 - 6x - 40 = 0$$

$$(x-10)(x+4) = 0$$

$$\mathbf{x = 10, -4}$$

$$(12) \quad 3x^2 = 8 - 10x$$

$$3x^2 + 10x - 8 = 0$$

$$(3x-2)(x+4) = 0$$

$$\mathbf{x = \frac{2}{3}, -4}$$

J92a KUMON

Quadratic Equations I

Solve the following quadratic equations.

$$(1) \quad x^2 - 8x + 15 = 0$$

$$(x - 5)(x - 3) = 0$$

$$\mathbf{x = 5, 3}$$

$$(4) \quad x^2 - 3x = 0$$

$$x(x - 3) = 0$$

$$\mathbf{x = 0, 3}$$

$$(2) \quad m^2 - 3m - 10 = 0$$

$$(m - 5)(m + 2) = 0$$

$$\mathbf{m = 5, -2}$$

$$(5) \quad 3x^2 - x = 0$$

$$x(3x - 1) = 0$$

$$\mathbf{x = 0, \frac{1}{3}}$$

$$(3) \quad 2a^2 - 13a + 6 = 0$$

$$(2a - 1)(a - 6) = 0$$

$$\mathbf{a = \frac{1}{2}, 6}$$

$$(6) \quad 2x^2 = 7x$$

$$2x^2 - 7x = 0$$

$$x(2x - 7) = 0$$

$$\mathbf{x = 0, \frac{7}{2}}$$

J92b

$$(7) \quad 2x^2 - 5x + 2 = 0$$

$$(2x - 1)(x - 2) = 0$$

$$\mathbf{x = \frac{1}{2}, 2}$$

$$(10) \quad 4x^2 - x - 18 = 0$$

$$(4x - 9)(x + 2) = 0$$

$$\mathbf{x = \frac{9}{4}, -2}$$

$$(8) \quad x^2 + x - 12 = 0$$

$$(x + 4)(x - 3) = 0$$

$$\mathbf{x = -4, 3}$$

$$(11) \quad x^2 - 3ax - 28a^2 = 0$$

$$(x - 7a)(x + 4a) = 0$$

$$\mathbf{x = 7a, -4a}$$

$$(9) \quad x^2 + x - 420 = 0$$

$$(x + 21)(x - 20) = 0$$

$$\mathbf{x = -21, 20}$$

$$(12) \quad 2x^2 - ax - a^2 = 0$$

$$(2x + a)(x - a) = 0$$

$$\mathbf{x = -\frac{a}{2}, a}$$

Quadratic Equations I

Solve the following quadratic equations.

Ex.

① $4x^2 - 25 = 0$

[Sol 1] $(2x+5)(2x-5) = 0$ [Sol 2] $4x^2 = 25$

$$x = -\frac{5}{2}, \frac{5}{2}$$

$$x^2 = \frac{25}{4}$$

$$x = \pm \frac{5}{2}$$

② $x^2 = 18$

[Sol] $x = \pm \sqrt{18}$
 $= \pm 3\sqrt{2}$

③ $2x^2 = 3$

[Sol] $x^2 = \frac{3}{2}$

$$x = \pm \sqrt{\frac{3}{2}} = \pm \frac{\sqrt{6}}{2}$$

(1) $4x^2 - 9 = 0$

$$(2x+3)(2x-3) = 0$$

$$x = -\frac{3}{2}, \frac{3}{2}$$

$$\left[x = \pm \frac{3}{2} \right]$$

(3) $9x^2 = 144$

$$x^2 = 16$$

$$x = \pm 4$$

(2) $4x^2 = 81$

$$x^2 = \frac{81}{4}$$

$$x = \pm \frac{9}{2}$$

(4) $x^2 = 12$

$$x = \pm \sqrt{12}$$

$$= \pm 2\sqrt{3}$$

J93b

$$(5) \quad x^2 = 50$$

$$\begin{aligned}x &= \pm\sqrt{50} \\ &= \pm 5\sqrt{2}\end{aligned}$$

$$(8) \quad 2(x+5)(x-5)+21=7$$

$$\begin{aligned}2(x^2-25) &= -14 \\ x^2-25 &= -7 \\ x^2 &= 18 \\ x &= \pm\sqrt{18} \\ &= \pm 3\sqrt{2}\end{aligned}$$

$$(6) \quad 2x^2-5=0$$

$$\begin{aligned}2x^2 &= 5 \\ x^2 &= \frac{5}{2} \\ x &= \pm\sqrt{\frac{5}{2}} \\ &= \pm\frac{\sqrt{10}}{2}\end{aligned}$$

$$(9) \quad (2+3x)^2+(2-3x)^2=26$$

$$\begin{aligned}4+12x+9x^2+4-12x+9x^2 &= 26 \\ 18x^2+8 &= 26 \\ 18x^2 &= 18 \\ x^2 &= 1 \\ x &= \pm 1\end{aligned}$$

$$(7) \quad (x+8)(x-8)=225$$

$$\begin{aligned}x^2-64 &= 225 \\ x^2 &= 289 \\ x &= \pm 17\end{aligned}$$

$$(10) \quad \left(\frac{x+1}{2}\right)^2 - \frac{x}{2} = 2$$

$$\begin{aligned}\frac{x^2+2x+1}{4} - \frac{x}{2} &= 2 \\ x^2+2x+1-2x &= 8 \\ x^2 &= 7 \\ x &= \pm\sqrt{7}\end{aligned}$$

Quadratic Equations I

Solve the following quadratic equations.

Ex. 1

$$(x-2)^2 = 5$$

$$[\text{Sol}] \quad x-2 = \pm\sqrt{5}$$

$$x = 2 \pm \sqrt{5}$$

Ex. 2

$$(x-3)^2 = 25$$

$$[\text{Sol}] \quad x-3 = \pm 5$$

$$x = 3 \pm 5$$

$$x = 8, -2$$

$$(1) \quad (x-1)^2 = 7$$

$$x-1 = \pm\sqrt{7}$$

$$x = 1 \pm \sqrt{7}$$

$$(4) \quad (x+3)^2 = 4$$

$$x+3 = \pm 2$$

$$x = -3 \pm 2$$

$$x = -1, -5$$

$$(2) \quad (x+3)^2 = 5$$

$$x+3 = \pm\sqrt{5}$$

$$x = -3 \pm \sqrt{5}$$

$$(5) \quad (x-5)^2 = 25$$

$$x-5 = \pm 5$$

$$x = 5 \pm 5$$

$$x = 10, 0$$

$$(3) \quad x^2 - 6x + 9 = 2$$

$$(x-3)^2 = 2$$

$$x-3 = \pm\sqrt{2}$$

$$x = 3 \pm \sqrt{2}$$

$$(6) \quad (x-3)^2 - 9 = 0$$

$$x-3 = \pm 3$$

$$x = 3 \pm 3$$

$$x = 6, 0$$

J94b

$$(7) \quad (2x-3)^2 = 25$$

$$2x-3 = \pm 5$$

$$2x = 3 \pm 5$$

$$x = \frac{3 \pm 5}{2}$$

$$\mathbf{x = 4, -1}$$

$$(10)^* \quad (2x-3)^2 = a \quad (a \geq 0)$$

$$2x-3 = \pm \sqrt{a}$$

$$2x = 3 \pm \sqrt{a}$$

$$\mathbf{x = \frac{3 \pm \sqrt{a}}{2}}$$

$$(8) \quad (3x+5)^2 = 4$$

$$3x+5 = \pm 2$$

$$3x = -5 \pm 2$$

$$x = \frac{-5 \pm 2}{3}$$

$$\mathbf{x = -1, -\frac{7}{3}}$$

$$(11) \quad (ax+5)^2 = b \quad (a \neq 0, b \geq 0)$$

$$ax+5 = \pm \sqrt{b}$$

$$ax = -5 \pm \sqrt{b}$$

$$\mathbf{x = \frac{-5 \pm \sqrt{b}}{a}}$$

$$(9) \quad (2x-3)^2 - 4 = 0$$

$$(2x-3)^2 = 4$$

$$2x-3 = \pm 2$$

$$2x = 3 \pm 2$$

$$x = \frac{3 \pm 2}{2}$$

$$\mathbf{x = \frac{5}{2}, \frac{1}{2}}$$

$$(12) \quad (ax-b)^2 = c^2 \quad (a \neq 0)$$

$$ax-b = \pm c$$

$$ax = b \pm c$$

$$\mathbf{x = \frac{b \pm c}{a}}$$

Quadratic Equations I

Solve the following equations by completing the square.

Ex.

$$x^2 - 6x - 1 = 0$$


$$[\text{Sol}] \quad x^2 - 6x = 1$$

$$x^2 - 6x + 9 = 1 + 9$$

$$(x - 3)^2 = 10$$

$$x - 3 = \pm\sqrt{10}$$

$$x = 3 \pm \sqrt{10}$$


 Add $3^2 = 9$ to both sides.
 This is called
completing the square.

$$(1) \quad x^2 - 6x - 3 = 0$$

$$\begin{aligned}
 x^2 - 6x &= 3 \\
 x^2 - 6x + 9 &= 3 + 9 \\
 (x - 3)^2 &= 12 \\
 x - 3 &= \pm\sqrt{12} \\
 &= \pm 2\sqrt{3} \\
 \mathbf{x} &= \mathbf{3 \pm 2\sqrt{3}}
 \end{aligned}$$

$$(3) \quad x^2 + 6x + 6 = 0$$

$$\begin{aligned}
 x^2 + 6x &= -6 \\
 x^2 + 6x + 9 &= -6 + 9 \\
 (x + 3)^2 &= 3 \\
 x + 3 &= \pm\sqrt{3} \\
 \mathbf{x} &= \mathbf{-3 \pm \sqrt{3}}
 \end{aligned}$$

It may help to write
 this step as
 $x^2 - 2 \cdot 3 \cdot x + 3^2 = 3 + 3^2$.

$$(2) \quad x^2 - 8x + 5 = 0$$

$$\begin{aligned}
 x^2 - 8x &= -5 \\
 x^2 - 8x + 16 &= -5 + 16 \\
 (x - 4)^2 &= 11 \\
 x - 4 &= \pm\sqrt{11} \\
 \mathbf{x} &= \mathbf{4 \pm \sqrt{11}}
 \end{aligned}$$

$$(4) \quad x^2 - 2ax + 1 = 0$$

$$\begin{aligned}
 [\text{Sol}] \quad x^2 - 2ax &= -1 \\
 x^2 - 2ax + \boxed{a^2} &= -1 + \boxed{a^2} \\
 (\mathbf{x - a})^2 &= a^2 - 1 \\
 x - a &= \pm\sqrt{\boxed{a^2 - 1}} \\
 \mathbf{x} &= \mathbf{a \pm \sqrt{a^2 - 1}}
 \end{aligned}$$

J95b

Ex.

$$x^2 - 5x - 7 = 0$$

$$[\text{Sol}] \quad x^2 - 5x = 7$$

➤ Add $\left(\frac{5}{2}\right)^2$ to both sides.

$$x^2 - 5x + \left(\frac{5}{2}\right)^2 = 7 + \left(\frac{5}{2}\right)^2$$

$$\left(x - \frac{5}{2}\right)^2 = \frac{53}{4}$$

$$x - \frac{5}{2} = \pm \frac{\sqrt{53}}{2}$$

$$x = \frac{5 \pm \sqrt{53}}{2}$$

$$(5) \quad x^2 - 3x - 7 = 0$$

$$x^2 - 3x = 7$$

$$x^2 - 3x + \left(\frac{3}{2}\right)^2 = 7 + \left(\frac{3}{2}\right)^2$$

$$\left(x - \frac{3}{2}\right)^2 = \frac{37}{4}$$

$$x - \frac{3}{2} = \pm \frac{\sqrt{37}}{2}$$

$$x = \frac{3 \pm \sqrt{37}}{2}$$

$$(6) \quad x^2 + x - 1 = 0$$

$$x^2 + x = 1$$

$$x^2 + x + \left(\frac{1}{2}\right)^2 = 1 + \left(\frac{1}{2}\right)^2$$

$$\left(x + \frac{1}{2}\right)^2 = \frac{5}{4}$$

$$x + \frac{1}{2} = \pm \frac{\sqrt{5}}{2}$$

$$x = \frac{-1 \pm \sqrt{5}}{2}$$

$$(7) \quad x^2 + \frac{1}{4}x - \frac{1}{2} = 0$$

$$x^2 + \frac{1}{4}x = \frac{1}{2}$$

$$x^2 + \frac{1}{4}x + \left(\frac{1}{8}\right)^2 = \frac{1}{2} + \left(\frac{1}{8}\right)^2$$

$$\left(x + \frac{1}{8}\right)^2 = \frac{33}{64}$$

$$x + \frac{1}{8} = \pm \frac{\sqrt{33}}{8}$$

$$x = \frac{-1 \pm \sqrt{33}}{8}$$

$$(8) \quad x^2 + ax - 4 = 0$$

$$[\text{Sol}] \quad x^2 + ax = 4$$

$$x^2 + ax + \left(\frac{a}{2}\right)^2 = 4 + \left(\frac{a}{2}\right)^2$$

$$\left(x + \frac{a}{2}\right)^2 = \frac{16 + a^2}{4}$$

$$x + \frac{a}{2} = \pm \frac{\sqrt{16 + a^2}}{2}$$

$$x = \frac{-a \pm \sqrt{16 + a^2}}{2}$$

Quadratic Equations I

Solve the following equations by completing the square.

Ex.

$$2x^2 - 3x - 1 = 0$$

$$[\text{Sol}] \quad x^2 - \frac{3}{2}x = \frac{1}{2}$$

$$x^2 - \frac{3}{2}x + \left(\frac{3}{4}\right)^2 = \frac{1}{2} + \left(\frac{3}{4}\right)^2$$

$$\left(x - \frac{3}{4}\right)^2 = \frac{17}{16}$$

$$x - \frac{3}{4} = \pm \frac{\sqrt{17}}{4}$$

$$x = \frac{3 \pm \sqrt{17}}{4}$$

$$(2) \quad 4x^2 + 3x - 2 = 0$$

$$x^2 + \frac{3}{4}x = \frac{1}{2}$$

$$x^2 + \frac{3}{4}x + \left(\frac{3}{8}\right)^2 = \frac{1}{2} + \left(\frac{3}{8}\right)^2$$

$$\left(x + \frac{3}{8}\right)^2 = \frac{41}{64}$$

$$x + \frac{3}{8} = \pm \frac{\sqrt{41}}{8}$$

$$x = \frac{-3 \pm \sqrt{41}}{8}$$

$$(1) \quad 3x^2 + 5x - 3 = 0$$

$$x^2 + \frac{5}{3}x = 1$$

$$x^2 + \frac{5}{3}x + \left(\frac{5}{6}\right)^2 = 1 + \left(\frac{5}{6}\right)^2$$

$$\left(x + \frac{5}{6}\right)^2 = \frac{61}{36}$$

$$x + \frac{5}{6} = \pm \frac{\sqrt{61}}{6}$$

$$x = \frac{-5 \pm \sqrt{61}}{6}$$

$$(3) \quad 4x^2 - 5x - 3 = 0$$

$$x^2 - \frac{5}{4}x = \frac{3}{4}$$

$$x^2 - \frac{5}{4}x + \left(\frac{5}{8}\right)^2 = \frac{3}{4} + \left(\frac{5}{8}\right)^2$$

$$\left(x - \frac{5}{8}\right)^2 = \frac{73}{64}$$

$$x - \frac{5}{8} = \pm \frac{\sqrt{73}}{8}$$

$$x = \frac{5 \pm \sqrt{73}}{8}$$

J96b

Ex.

$$3x^2 - 5x + 2 = 0$$

$$[\text{Sol}] \quad x^2 - \frac{5}{3}x = -\frac{2}{3}$$

$$x^2 - \frac{5}{3}x + \left(\frac{5}{6}\right)^2 = -\frac{2}{3} + \left(\frac{5}{6}\right)^2$$

$$\left(x - \frac{5}{6}\right)^2 = \frac{1}{36}$$

$$x - \frac{5}{6} = \pm \frac{1}{6}$$

$$x = \frac{5}{6} \pm \frac{1}{6}$$

$$x = 1, \frac{2}{3}$$

$$(5) \quad 4x^2 - 4x - 3 = 0$$

$$x^2 - x = \frac{3}{4}$$

$$x^2 - x + \left(\frac{1}{2}\right)^2 = \frac{3}{4} + \left(\frac{1}{2}\right)^2$$

$$\left(x - \frac{1}{2}\right)^2 = 1$$

$$x - \frac{1}{2} = \pm 1$$

$$x = \frac{1}{2} \pm 1$$

$$\mathbf{x = \frac{3}{2}, -\frac{1}{2}}$$

$$(4) \quad 3x^2 - 2x - 1 = 0$$

$$x^2 - \frac{2}{3}x = \frac{1}{3}$$

$$x^2 - \frac{2}{3}x + \left(\frac{1}{3}\right)^2 = \frac{1}{3} + \left(\frac{1}{3}\right)^2$$

$$\left(x - \frac{1}{3}\right)^2 = \frac{4}{9}$$

$$x - \frac{1}{3} = \pm \frac{2}{3}$$

$$x = \frac{1}{3} \pm \frac{2}{3}$$

$$\mathbf{x = 1, -\frac{1}{3}}$$

$$(6) \quad x^2 - 2ax = b$$

$$x^2 - 2ax + a^2 = b + a^2$$

$$(x - a)^2 = b + a^2$$

$$x - a = \pm \sqrt{b + a^2}$$

$$\mathbf{x = a \pm \sqrt{b + a^2}}$$

Quadratic Equations I

1. Solve the following equations by completing the square.

$$(1) \quad 3x^2 - 7x + 2 = 0$$

$$x^2 - \frac{7}{3}x = -\frac{2}{3}$$

$$x^2 - \frac{7}{3}x + \left(\frac{7}{6}\right)^2 = -\frac{2}{3} + \left(\frac{7}{6}\right)^2$$

$$\left(x - \frac{7}{6}\right)^2 = \frac{25}{36}$$

$$x - \frac{7}{6} = \pm \frac{5}{6}$$

$$x = \frac{7}{6} \pm \frac{5}{6}$$

$$\mathbf{x = 2, \frac{1}{3}}$$

$$(3)^* \quad ax^2 + bx + c = 0 \quad (a \neq 0)$$

$$[\text{Sol}] \quad x^2 + \frac{b}{a}x = -\frac{c}{a}$$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

$$\left(x + \frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$\mathbf{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

$$(2) \quad 3x^2 - 7x + 1 = 0$$

$$x^2 - \frac{7}{3}x = -\frac{1}{3}$$

$$x^2 - \frac{7}{3}x + \left(\frac{7}{6}\right)^2 = -\frac{1}{3} + \left(\frac{7}{6}\right)^2$$

$$\left(x - \frac{7}{6}\right)^2 = \frac{37}{36}$$

$$x - \frac{7}{6} = \pm \frac{\sqrt{37}}{6}$$

$$\mathbf{x = \frac{7 \pm \sqrt{37}}{6}}$$

J97b

From J97a, question 1.(3):

Quadratic Formula

$$\text{When } ax^2 + bx + c = 0, \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

2. Solve the following equations using the quadratic formula.

Ex.

$$3x^2 + 9x + 2 = 0$$

[Sol] $a = 3, b = 9, c = 2$

$$\begin{aligned} x &= \frac{-9 \pm \sqrt{9^2 - 4 \times 3 \times 2}}{2 \times 3} \\ &= \frac{-9 \pm \sqrt{81 - 24}}{6} \\ &= \frac{-9 \pm \sqrt{57}}{6} \end{aligned}$$

$$(2) \quad 2x^2 + 7x + 1 = 0$$

$$a = 2, b = 7, c = 1$$

$$\begin{aligned} x &= \frac{-7 \pm \sqrt{7^2 - 4 \times 2 \times 1}}{2 \times 2} \\ &= \frac{-7 \pm \sqrt{49 - 8}}{4} \\ &= \frac{-7 \pm \sqrt{41}}{4} \end{aligned}$$

$$(1) \quad 3x^2 - 9x + 2 = 0$$

[Sol] $a = 3, b = \boxed{-9}, c = 2$

$$\begin{aligned} x &= \frac{9 \pm \sqrt{(-9)^2 - 4 \times 3 \times 2}}{2 \times 3} \\ &= \frac{9 \pm \sqrt{81 - 24}}{6} \\ &= \frac{9 \pm \sqrt{57}}{6} \end{aligned}$$

$$(3) \quad 2x^2 - 7x - 1 = 0$$

$$a = 2, b = -7, c = -1$$

$$\begin{aligned} x &= \frac{7 \pm \sqrt{(-7)^2 - 4 \times 2 \times (-1)}}{2 \times 2} \\ &= \frac{7 \pm \sqrt{49 + 8}}{4} \\ &= \frac{7 \pm \sqrt{57}}{4} \end{aligned}$$

Quadratic Equations I

Quadratic Formula

$$\text{When } ax^2 + bx + c = 0, \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Solve the following equations using the quadratic formula.

Ex.

$$3x^2 + 7x + 2 = 0$$

$$[\text{Sol}] \quad x = \frac{-7 \pm \sqrt{7^2 - 4 \times 3 \times 2}}{2 \times 3} = \frac{-7 \pm \sqrt{25}}{6} = \frac{-7 \pm 5}{6}$$

$$x = -\frac{1}{3}, -2$$

$$(1) \quad 3x^2 - 7x + 2 = 0$$

$$x = \frac{7 \pm \sqrt{(-7)^2 - 4 \times 3 \times 2}}{2 \times 3}$$

$$= \frac{7 \pm \sqrt{25}}{6}$$

$$= \frac{7 \pm 5}{6}$$

$$x = 2, \frac{1}{3}$$

$$(3) \quad 3x^2 - 5x - 8 = 0$$

$$x = \frac{5 \pm \sqrt{(-5)^2 - 4 \times 3 \times (-8)}}{2 \times 3}$$

$$= \frac{5 \pm \sqrt{121}}{6}$$

$$= \frac{5 \pm 11}{6}$$

$$x = \frac{8}{3}, -1$$

$$(2) \quad 3x^2 + 5x - 8 = 0$$

$$x = \frac{-5 \pm \sqrt{5^2 - 4 \times 3 \times (-8)}}{2 \times 3}$$

$$= \frac{-5 \pm \sqrt{121}}{6}$$

$$= \frac{-5 \pm 11}{6}$$

$$x = 1, -\frac{8}{3}$$

$$(4) \quad 2x^2 + 7x + 3 = 0$$

$$x = \frac{-7 \pm \sqrt{7^2 - 4 \times 2 \times 3}}{2 \times 2}$$

$$= \frac{-7 \pm \sqrt{25}}{4}$$

$$= \frac{-7 \pm 5}{4}$$

$$x = -\frac{1}{2}, -3$$

J98b

$$(5) \quad 2x^2 + 5x + 3 = 0$$

$$x = \frac{-5 \pm \sqrt{5^2 - 4 \times 2 \times 3}}{2 \times 2}$$
$$= \frac{-5 \pm 1}{4}$$

$$x = -1, -\frac{3}{2}$$

$$(8) \quad 2x^2 - 7x + 1 = 0$$

$$x = \frac{7 \pm \sqrt{(-7)^2 - 4 \times 2 \times 1}}{2 \times 2}$$
$$= \frac{7 \pm \sqrt{41}}{4}$$

$$(6) \quad 2x^2 - 5x + 3 = 0$$

$$x = \frac{5 \pm \sqrt{(-5)^2 - 4 \times 2 \times 3}}{2 \times 2}$$
$$= \frac{5 \pm 1}{4}$$

$$x = \frac{3}{2}, 1$$

$$(9) \quad 4x^2 - 9x + 3 = 0$$

$$x = \frac{9 \pm \sqrt{(-9)^2 - 4 \times 4 \times 3}}{2 \times 4}$$
$$= \frac{9 \pm \sqrt{33}}{8}$$

$$(7) \quad 2x^2 + 7x - 1 = 0$$

$$x = \frac{-7 \pm \sqrt{7^2 - 4 \times 2 \times (-1)}}{2 \times 2}$$
$$x = \frac{-7 \pm \sqrt{57}}{4}$$

$$(10) \quad 3x^2 + 7x - 6 = 0$$

$$x = \frac{-7 \pm \sqrt{7^2 - 4 \times 3 \times (-6)}}{2 \times 3}$$
$$= \frac{-7 \pm \sqrt{121}}{6}$$
$$= \frac{-7 \pm 11}{6}$$

$$x = \frac{2}{3}, -3$$

Quadratic Equations I


Quadratic Formula

$$\text{When } ax^2 + bx + c = 0, \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Solve the following equations using the quadratic formula.

Ex.

$$-2x^2 - 7x + 3 = 0$$

[Sol] $2x^2 + 7x - 3 = 0$  Multiply both sides by -1 .

$$x = \frac{-7 \pm \sqrt{73}}{4}$$

$$(1) \quad -2x^2 - 7x - 3 = 0$$

$$2x^2 + 7x + 3 = 0$$

$$x = \frac{-7 \pm \sqrt{25}}{4}$$

$$= \frac{-7 \pm 5}{4}$$

$$x = -\frac{1}{2}, -3$$

$$(2) \quad -2x^2 + 5x - 2 = 0$$

$$2x^2 - 5x + 2 = 0$$

$$x = \frac{5 \pm \sqrt{9}}{4}$$

$$= \frac{5 \pm 3}{4}$$

$$x = 2, \frac{1}{2}$$

$$(3) \quad -4x^2 + 9x + 2 = 0$$

$$4x^2 - 9x - 2 = 0$$

$$x = \frac{9 \pm \sqrt{113}}{8}$$

$$(4) \quad -2x^2 - 7x - 1 = 0$$

$$2x^2 + 7x + 1 = 0$$

$$x = \frac{-7 \pm \sqrt{41}}{4}$$

J99b

$$(5) \quad 3x^2 + 5x + 1 = 0$$

$$x = \frac{-5 \pm \sqrt{13}}{6}$$

$$(8) \quad -12x^2 + 11x - 2 = 0$$

$$12x^2 - 11x + 2 = 0$$

$$x = \frac{11 \pm \sqrt{25}}{24}$$

$$= \frac{11 \pm 5}{24}$$

$$x = \frac{2}{3}, \frac{1}{4}$$

$$(6) \quad 3x^2 + 5x + 2 = 0$$

$$x = \frac{-5 \pm 1}{6}$$

$$x = -\frac{2}{3}, -1$$

$$(9) \quad -3x^2 + 7x - 2 = 0$$

$$3x^2 - 7x + 2 = 0$$

$$x = \frac{7 \pm \sqrt{25}}{6}$$

$$= \frac{7 \pm 5}{6}$$

$$x = 2, \frac{1}{3}$$

$$(7) \quad 2x^2 + 5x - 1 = 0$$

$$x = \frac{-5 \pm \sqrt{33}}{4}$$

$$(10) \quad -3x^2 - 7x + 5 = 0$$

$$3x^2 + 7x - 5 = 0$$

$$x = \frac{-7 \pm \sqrt{109}}{6}$$

J100a KUMON

Quadratic Equations I

Solve the following equations using the quadratic formula.

$$(1) \quad 3x^2 + 9x + 5 = 0$$

$$x = \frac{-9 \pm \sqrt{21}}{6}$$

$$(4) \quad 3x^2 - 9x + 5 = 0$$

$$x = \frac{9 \pm \sqrt{21}}{6}$$

$$(2) \quad 2x^2 + 7x + 3 = 0$$

$$x = \frac{-7 \pm \sqrt{25}}{4}$$

$$= \frac{-7 \pm 5}{4}$$

$$x = -\frac{1}{2}, -3$$

$$(5) \quad 8x = 3x^2 + 5$$

$$3x^2 - 8x + 5 = 0$$

$$x = \frac{8 \pm \sqrt{4}}{6}$$

$$= \frac{8 \pm 2}{6}$$

$$x = \frac{5}{3}, 1$$

$$(3) \quad -x^2 - 7x + 3 = 0$$

$$x^2 + 7x - 3 = 0$$

$$x = \frac{-7 \pm \sqrt{61}}{2}$$

J 100b

$$(6) \quad 9x^2 - 13x + 4 = 0$$

$$x = \frac{13 \pm \sqrt{25}}{18}$$

$$= \frac{13 \pm 5}{18}$$

$$x = 1, \frac{4}{9}$$

$$(9) \quad x^2 - 7x + 1 = 0$$

$$x = \frac{7 \pm \sqrt{45}}{2}$$

$$= \frac{7 \pm 3\sqrt{5}}{2}$$

$$(7) \quad -x^2 - 5x + 3 = 0$$

$$x^2 + 5x - 3 = 0$$

$$x = \frac{-5 \pm \sqrt{37}}{2}$$

$$(10) \quad 3x^2 - 11x + 7 = 0$$

$$x = \frac{11 \pm \sqrt{37}}{6}$$

$$(8) \quad -3x^2 - 8x - 5 = 0$$

$$3x^2 + 8x + 5 = 0$$

$$x = \frac{-8 \pm \sqrt{4}}{6}$$

$$= \frac{-8 \pm 2}{6}$$

$$x = -1, -\frac{5}{3}$$

J101a KUMON

Quadratic Equations II

Formula

$$\text{When } ax^2 + bx + c = 0, \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Solve the following equations using the quadratic formula.

$$(1) \quad 3x^2 + 9x + 2 = 0$$

$$\begin{aligned} x &= \frac{-9 \pm \sqrt{9^2 - 4 \times 3 \times 2}}{2 \times 3} \\ &= \frac{-9 \pm \sqrt{57}}{6} \end{aligned}$$

$$(4) \quad 3x^2 - 9x + 2 = 0$$

$$\begin{aligned} x &= \frac{9 \pm \sqrt{(-9)^2 - 4 \times 3 \times 2}}{2 \times 3} \\ &= \frac{9 \pm \sqrt{57}}{6} \end{aligned}$$

$$(2) \quad 2x^2 + 7x + 1 = 0$$

$$\begin{aligned} x &= \frac{-7 \pm \sqrt{7^2 - 4 \times 2 \times 1}}{2 \times 2} \\ &= \frac{-7 \pm \sqrt{41}}{4} \end{aligned}$$

$$(5) \quad 2x^2 + 7x - 1 = 0$$

$$\begin{aligned} x &= \frac{-7 \pm \sqrt{7^2 - 4 \times 2 \times (-1)}}{2 \times 2} \\ &= \frac{-7 \pm \sqrt{57}}{4} \end{aligned}$$

$$(3) \quad 4x^2 + 9x + 3 = 0$$

$$\begin{aligned} x &= \frac{-9 \pm \sqrt{9^2 - 4 \times 4 \times 3}}{2 \times 4} \\ &= \frac{-9 \pm \sqrt{33}}{8} \end{aligned}$$

J101b

$$(6) \quad 3x^2 + 7x + 2 = 0$$

$$x = \frac{-7 \pm \sqrt{7^2 - 4 \times 3 \times 2}}{2 \times 3}$$

$$= \frac{-7 \pm \sqrt{25}}{6}$$

$$= \frac{-7 \pm 5}{6}$$

$$x = -\frac{1}{3}, -2$$

$$(9) \quad 3x^2 + 7x - 6 = 0$$

$$x = \frac{-7 \pm \sqrt{7^2 - 4 \times 3 \times (-6)}}{2 \times 3}$$

$$= \frac{-7 \pm \sqrt{121}}{6}$$

$$= \frac{-7 \pm 11}{6}$$

$$x = \frac{2}{3}, -3$$

$$(7) \quad 3x^2 - 7x + 2 = 0$$

$$x = \frac{7 \pm \sqrt{(-7)^2 - 4 \times 3 \times 2}}{2 \times 3}$$

$$= \frac{7 \pm \sqrt{25}}{6}$$

$$= \frac{7 \pm 5}{6}$$

$$x = 2, \frac{1}{3}$$

$$(10)^* \quad ax^2 + 2b'x + c = 0 \quad (a \neq 0)$$

(Substitute a , $2b'$, c into the quadratic formula.)

$$[\text{Sol}] \quad x = \frac{-2b' \pm \sqrt{(2b')^2 - 4ac}}{2a}$$

$$= \frac{-2b' \pm 2\sqrt{\mathbf{b'^2 - ac}}}{2a}$$

$$= \frac{-\mathbf{b'} \pm \sqrt{\mathbf{b'^2 - ac}}}{\mathbf{a}}$$

$$(8) \quad 3x^2 + 5x - 8 = 0$$

$$x = \frac{-5 \pm \sqrt{5^2 - 4 \times 3 \times (-8)}}{2 \times 3}$$

$$= \frac{-5 \pm \sqrt{121}}{6}$$

$$= \frac{-5 \pm 11}{6}$$

$$x = 1, -\frac{8}{3}$$

J102a KUMON

Quadratic Equations II

From the result of J101b question (10), we can derive the Quadratic Formula II, which can be used when the coefficient of x is an even number.

Quadratic Formula II

$$\text{When } ax^2 + 2b'x + c = 0, \quad x = \frac{-b' \pm \sqrt{b'^2 - ac}}{a}$$

Ex.

$$3x^2 + 10x + 1 = 0$$

$$[\text{Sol}] \quad x = \frac{-5 \pm \sqrt{5^2 - 3 \times 1}}{3} \quad \text{From } b' = 5.$$

$$= \frac{-5 \pm \sqrt{22}}{3}$$

1. Solve the following equations using the quadratic formula II.

(1) $3x^2 + 14x + 5 = 0$

$$\begin{aligned} x &= \frac{-7 \pm \sqrt{7^2 - 3 \times 5}}{3} \\ &= \frac{-7 \pm \sqrt{34}}{3} \end{aligned}$$

(3) $x^2 - 8x + 5 = 0$

$$\begin{aligned} x &= \frac{4 \pm \sqrt{(-4)^2 - 1 \times 5}}{1} \\ &= 4 \pm \sqrt{11} \end{aligned}$$

(2) $3x^2 + 8x + 5 = 0$

$$\begin{aligned} x &= \frac{-4 \pm \sqrt{4^2 - 3 \times 5}}{3} \\ &= \frac{-4 \pm 1}{3} \end{aligned}$$

$$x = -1, -\frac{5}{3}$$

(4) $x^2 + 6x - 3 = 0$

$$\begin{aligned} x &= \frac{-3 \pm \sqrt{3^2 - 1 \times (-3)}}{1} \\ &= -3 \pm 2\sqrt{3} \end{aligned}$$

Quadratic Formula IWhen $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Quadratic Formula IIWhen $ax^2 + 2b'x + c = 0$,

$$x = \frac{-b' \pm \sqrt{b'^2 - ac}}{a}$$

2. Solve the following equations using either quadratic formula.

(1) $4x^2 + 9x + 1 = 0$

$$\begin{aligned} x &= \frac{-9 \pm \sqrt{9^2 - 4 \times 4 \times 1}}{2 \times 4} \\ &= \frac{-9 \pm \sqrt{65}}{8} \end{aligned}$$

(2) $3x^2 - 10x + 3 = 0$

$$\begin{aligned} x &= \frac{5 \pm \sqrt{(-5)^2 - 3 \times 3}}{3} \\ &= \frac{5 \pm 4}{3} \\ x &= 3, \frac{1}{3} \end{aligned}$$

(3) $x^2 + 8x - 3 = 0$

$$\begin{aligned} x &= \frac{-4 \pm \sqrt{4^2 - 1 \times (-3)}}{1} \\ &= -4 \pm \sqrt{19} \end{aligned}$$

(4) $-2x^2 - 7x + 3 = 0$

$$\begin{aligned} 2x^2 + 7x - 3 &= 0 \\ x &= \frac{-7 \pm \sqrt{7^2 - 4 \times 2 \times (-3)}}{2 \times 2} \\ &= \frac{-7 \pm \sqrt{73}}{4} \end{aligned}$$

(5) $-2x^2 - 6x + 7 = 0$

$$\begin{aligned} 2x^2 + 6x - 7 &= 0 \\ x &= \frac{-3 \pm \sqrt{3^2 - 2 \times (-7)}}{2} \\ &= \frac{-3 \pm \sqrt{23}}{2} \end{aligned}$$

(6) $-2x^2 - 7x - 3 = 0$

$$\begin{aligned} 2x^2 + 7x + 3 &= 0 \\ x &= \frac{-7 \pm \sqrt{7^2 - 4 \times 2 \times 3}}{2 \times 2} \\ &= \frac{-7 \pm 5}{4} \\ x &= -\frac{1}{2}, -3 \end{aligned}$$

Quadratic Equations II

Quadratic Formula I

When $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Quadratic Formula II

When $ax^2 + 2b'x + c = 0$,

$$x = \frac{-b' \pm \sqrt{b'^2 - ac}}{a}$$

Solve the following equations using either quadratic formula.

(1) $3x^2 + 14x + 5 = 0$

$$\begin{aligned} x &= \frac{-7 \pm \sqrt{7^2 - 3 \times 5}}{3} \\ &= \frac{-7 \pm \sqrt{34}}{3} \end{aligned}$$

(3) $-x^2 - 8x + 3 = 0$

$$\begin{aligned} x^2 + 8x - 3 &= 0 \\ x &= \frac{-4 \pm \sqrt{4^2 - 1 \times (-3)}}{1} \\ &= -4 \pm \sqrt{19} \end{aligned}$$

(2) $2x^2 + 6x + 3 = 0$

$$\begin{aligned} x &= \frac{-3 \pm \sqrt{3^2 - 2 \times 3}}{2} \\ &= \frac{-3 \pm \sqrt{3}}{2} \end{aligned}$$

(4) $3x = 3x^2 - 6$

Take out the common factor 3.

$$\begin{aligned} 3x^2 - 3x - 6 &= 0 \\ x^2 - x - 2 &= 0 \\ x &= \frac{1 \pm \sqrt{(-1)^2 - 4 \times 1 \times (-2)}}{2 \times 1} \\ &= \frac{1 \pm 3}{2} \\ x &= 2, -1 \end{aligned}$$

J 103b

$$(5) \quad 4x + 5 = 3x^2$$

$$3x^2 - 4x - 5 = 0$$

$$\begin{aligned} x &= \frac{2 \pm \sqrt{(-2)^2 - 3 \times (-5)}}{3} \\ &= \frac{2 \pm \sqrt{19}}{3} \end{aligned}$$

$$(8) \quad x^2 - 10x + 1 = 0$$

$$\begin{aligned} x &= \frac{5 \pm \sqrt{(-5)^2 - 1 \times 1}}{1} \\ &= 5 \pm \sqrt{24} \\ &= 5 \pm 2\sqrt{6} \end{aligned}$$

$$(6) \quad 3x^2 - 8x - 4 = 0$$

$$\begin{aligned} x &= \frac{4 \pm \sqrt{(-4)^2 - 3 \times (-4)}}{3} \\ &= \frac{4 \pm \sqrt{28}}{3} \\ &= \frac{4 \pm 2\sqrt{7}}{3} \end{aligned}$$

$$(9) \quad -3x^2 - 8x - 5 = 0$$

$$\begin{aligned} 3x^2 + 8x + 5 &= 0 \\ x &= \frac{-4 \pm \sqrt{4^2 - 3 \times 5}}{3} \\ &= \frac{-4 \pm 1}{3} \\ x &= -1, -\frac{5}{3} \end{aligned}$$

$$(7) \quad 9x^2 - 12x + 4 = 0$$

$$\begin{aligned} x &= \frac{6 \pm \sqrt{(-6)^2 - 9 \times 4}}{9} \\ &= \frac{6 \pm 0}{9} \\ &= \frac{2}{3} \end{aligned}$$

$$(10) \quad 3x^2 - 9x + 4 = 0$$

$$\begin{aligned} x &= \frac{9 \pm \sqrt{(-9)^2 - 4 \times 3 \times 4}}{2 \times 3} \\ &= \frac{9 \pm \sqrt{33}}{6} \end{aligned}$$

J104a KUMON

Quadratic Equations II

Solve the following equations either by factorisation or using the quadratic formula.

$$(1) \quad x^2 - 8x + 12 = 0$$

$$(x-6)(x-2) = 0$$

$$\mathbf{x = 6, 2}$$

$$(4) \quad 5(x^2 + 2x - 1) = -5$$

$$x^2 + 2x - 1 = -1$$

$$x^2 + 2x = 0$$

$$x(x+2) = 0$$

$$\mathbf{x = 0, -2}$$

$$(2) \quad x^2 + 9 = 6x$$

$$x^2 - 6x + 9 = 0$$

$$(x-3)^2 = 0$$

$$\mathbf{x = 3}$$

$$(5) \quad 3x^2 + 5x = 8$$

$$3x^2 + 5x - 8 = 0$$

$$(3x+8)(x-1) = 0$$

$$\mathbf{x = -\frac{8}{3}, 1}$$

$$(3) \quad -9x^2 + 6x - 1 = 0$$

$$9x^2 - 6x + 1 = 0$$

$$(3x-1)^2 = 0$$

$$\mathbf{x = \frac{1}{3}}$$

J 104b

$$(6) \quad 3x^2 = 4 - 5x$$

$$3x^2 + 5x - 4 = 0$$

$$\begin{aligned} x &= \frac{-5 \pm \sqrt{5^2 - 4 \times 3 \times (-4)}}{2 \times 3} \\ &= \frac{-5 \pm \sqrt{73}}{6} \end{aligned}$$

$$(8) \quad 3x^2 - 4x - 1 = 0$$

$$\begin{aligned} x &= \frac{2 \pm \sqrt{(-2)^2 - 3 \times (-1)}}{3} \\ &= \frac{2 \pm \sqrt{7}}{3} \end{aligned}$$

$$(7) \quad 3x^2 - 4x + 7 = 5x^2 - 7x + 8 \quad (9) \quad (x+2)^2 - 3(x+2) - 4 = 0$$

$$2x^2 - 3x + 1 = 0$$

$$(2x-1)(x-1) = 0$$

$$\mathbf{x = \frac{1}{2}, 1}$$

$$x^2 + 4x + 4 - 3x - 6 - 4 = 0$$

$$x^2 + x - 6 = 0$$

$$(x+3)(x-2) = 0$$

$$\mathbf{x = -3, 2}$$

$$\left[\begin{array}{l} \text{Alternative Solution} \\ [(x+2)-4][(x+2)+1] = 0 \\ (x-2)(x+3) = 0 \\ \mathbf{x = 2, -3} \end{array} \right]$$

J 105a KUMON

Quadratic Equations II

Solve the following equations either by factorisation or using the quadratic formula.

$$(1) \quad 4x^2 - 15x + 14 = 0$$

$$(4x - 7)(x - 2) = 0$$

$$x = \frac{7}{4}, 2$$

$$(4) \quad -4x^2 - 16x + 9 = 0$$

$$4x^2 + 16x - 9 = 0$$

$$(2x + 9)(2x - 1) = 0$$

$$x = -\frac{9}{2}, \frac{1}{2}$$

$$(2) \quad 15x^2 - 23x - 28 = 0$$

$$(5x + 4)(3x - 7) = 0$$

$$x = -\frac{4}{5}, \frac{7}{3}$$

$$(5) \quad x(x + 1) = 56$$

$$x^2 + x = 56$$

$$x^2 + x - 56 = 0$$

$$(x + 8)(x - 7) = 0$$

$$x = -8, 7$$

$$(3) \quad 3x^2 + 16x - 35 = 0$$

$$(3x - 5)(x + 7) = 0$$

$$x = \frac{5}{3}, -7$$

J 105b

Multiply both sides by 3.

$$(6) \quad x^2 = \frac{2}{3}(x+1)$$

$$3x^2 = 2(x+1)$$

$$3x^2 = 2x+2$$

$$3x^2 - 2x - 2 = 0$$

$$x = \frac{1 \pm \sqrt{(-1)^2 - 3 \times (-2)}}{3}$$

$$= \frac{1 \pm \sqrt{7}}{3}$$

Multiply both sides by 12.

$$(8) \quad \frac{x^2}{4} + \frac{x}{3} - \frac{1}{6} = 0$$

$$3x^2 + 4x - 2 = 0$$

$$x = \frac{-2 \pm \sqrt{2^2 - 3 \times (-2)}}{3}$$

$$= \frac{-2 \pm \sqrt{10}}{3}$$

Multiply both sides by 6.

$$(7) \quad \frac{x(x+1)}{2} = \frac{2x-3}{3} + 1$$

$$3x(x+1) = 2(2x-3) + 6$$

$$3x^2 + 3x = 4x$$

$$3x^2 - x = 0$$

$$x(3x-1) = 0$$

$$x = 0, \frac{1}{3}$$

Multiply both sides by 10.

$$(9) \quad 0.2x^2 + 0.7x + 0.3 = 0$$

$$2x^2 + 7x + 3 = 0$$

$$(2x+1)(x+3) = 0$$

$$x = -\frac{1}{2}, -3$$

J106a KUMON

Quadratic Equations II

Solve each equation both by using the quadratic formula and by factorisation.

$$(1) \quad 6x^2 + 13x + 6 = 0$$

Quadratic Formula

$$x = \frac{-13 \pm \sqrt{169 - 4 \times 6 \times 6}}{12}$$

$$= \frac{-13 \pm \sqrt{25}}{12}$$

$$= \frac{-13 \pm 5}{12}$$

$$x = -\frac{2}{3}, -\frac{3}{2}$$

Factorisation

$$(3x+2)(2x+3) = 0$$

$$x = -\frac{2}{3}, -\frac{3}{2}$$

$$(2) \quad 15x^2 + x - 2 = 0$$

Quadratic Formula

$$x = \frac{-1 \pm \sqrt{1 + 120}}{30}$$

$$= \frac{-1 \pm 11}{30}$$

$$x = \frac{1}{3}, -\frac{2}{5}$$

Factorisation

$$(5x+2)(3x-1) = 0$$

$$x = -\frac{2}{5}, \frac{1}{3}$$

J 106b

$$(3) \quad 4x^2 - 8x + 3 = 0$$

Quadratic Formula

$$x = \frac{4 \pm \sqrt{16 - 12}}{4}$$

$$= \frac{4 \pm 2}{4}$$

$$x = \frac{3}{2}, \frac{1}{2}$$

Factorisation

$$(2x - 3)(2x - 1) = 0$$

$$x = \frac{3}{2}, \frac{1}{2}$$

$$(4) \quad 12x^2 + 4x - 5 = 0$$

Quadratic Formula

$$x = \frac{-2 \pm \sqrt{4 + 60}}{12}$$

$$= \frac{-2 \pm 8}{12}$$

$$x = \frac{1}{2}, -\frac{5}{6}$$

Factorisation

$$(6x + 5)(2x - 1) = 0$$

$$x = -\frac{5}{6}, \frac{1}{2}$$

J107a KUMON

Quadratic Equations II

Factorise the following quadratic expressions using the quadratic formula.

Ex.

$$6x^2 - 13x + 6$$

[Sol] Solving $6x^2 - 13x + 6 = 0$

$$x = \frac{13 \pm \sqrt{169 - 144}}{12} = \frac{13 \pm 5}{12}$$

$$x = \frac{3}{2}, \frac{2}{3}$$

The quadratic equation with these solutions is:

$$6\left(x - \frac{3}{2}\right)\left(x - \frac{2}{3}\right) = 0 \quad \text{☞ Check that the coefficient of } x^2 \text{ is 6.}$$

$$2\left(x - \frac{3}{2}\right) \cdot 3\left(x - \frac{2}{3}\right) = 0 \quad \text{☞ Break up 6 into 2 and 3.}$$

$$(2x - 3)(3x - 2) = 0$$

Therefore,

$$6x^2 - 13x + 6 = (2x - 3)(3x - 2)$$

(1) $12x^2 + 25x + 12$

[Sol] Solving $12x^2 + 25x + 12 = 0$

$$x = \frac{-25 \pm \sqrt{625 - 576}}{24} = \frac{-25 \pm 7}{24}$$

$$x = -\frac{3}{4}, -\frac{4}{3}$$

The quadratic equation with these solutions is:

$$12\left(x + \frac{3}{4}\right)\left(x + \frac{4}{3}\right) = 0$$

$$4\left(x + \frac{3}{4}\right) \cdot 3\left(x + \frac{4}{3}\right) = 0$$

$$(4x + 3)(3x + 4) = 0$$

Therefore,

$$12x^2 + 25x + 12 = (4x + 3)(3x + 4)$$

J 107b

$$(2) \quad 15x^2 - 2x - 24$$

[Sol] Solving $15x^2 - 2x - 24 = 0$

$$x = \frac{1 \pm \sqrt{1 + 360}}{15} = \frac{1 \pm 19}{15}$$

$$x = \frac{4}{3}, -\frac{6}{5}$$

The quadratic equation with these solutions is:

$$15\left(x - \frac{4}{3}\right)\left(x + \frac{6}{5}\right) = 0$$

$$3\left(x - \frac{4}{3}\right) \cdot 5\left(x + \frac{6}{5}\right) = 0$$

$$(3x - 4)(5x + 6) = 0$$

Therefore,

$$15x^2 - 2x - 24 = \mathbf{(3x - 4)(5x + 6)}$$

$$(3) \quad 3x^2 + 7x - 48$$

[Sol] Solving $3x^2 + 7x - 48 = 0$

$$x = \frac{-7 \pm \sqrt{49 + 576}}{6} = \frac{-7 \pm 25}{6}$$

$$x = 3, -\frac{16}{3}$$

The quadratic equation with these solutions is:

$$3(x - 3)\left(x + \frac{16}{3}\right) = 0$$

$$(x - 3)(3x + 16) = 0$$

Therefore,

$$3x^2 + 7x - 48 = \mathbf{(x - 3)(3x + 16)}$$

J 108a KUMON

Quadratic Equations II

Factorise the following quadratic expressions using the quadratic formula.

$$(1) \quad x^2 - \sqrt{2}x - 12$$

[Sol] Solving $x^2 - \sqrt{2}x - 12 = 0$

$$x = \frac{\sqrt{2} \pm \sqrt{2 + 48}}{2} = \frac{\sqrt{2} \pm \sqrt{50}}{2} = \frac{\sqrt{2} \pm 5\sqrt{2}}{2}$$

$$x = 3\sqrt{2}, -2\sqrt{2}$$

Therefore,

$$x^2 - \sqrt{2}x - 12 = (x - 3\sqrt{2})(x + 2\sqrt{2})$$

$$(2) \quad x^2 - (2\sqrt{3} + 1)x + 2\sqrt{3}$$

[Sol] Solving $x^2 - (2\sqrt{3} + 1)x + 2\sqrt{3} = 0$

$$x = \frac{(2\sqrt{3} + 1) \pm \sqrt{(2\sqrt{3} + 1)^2 - 4 \times 2\sqrt{3}}}{2}$$

$$= \frac{(2\sqrt{3} + 1) \pm \sqrt{12 + 4\sqrt{3} + 1 - 8\sqrt{3}}}{2}$$

$$= \frac{(2\sqrt{3} + 1) \pm \sqrt{13 - 4\sqrt{3}}}{2}$$

$$= \frac{(2\sqrt{3} + 1) \pm \sqrt{13 - 2\sqrt{12}}}{2}$$

$$= \frac{(2\sqrt{3} + 1) \pm \sqrt{(\sqrt{12} - 1)^2}}{2}$$

$$= \frac{(2\sqrt{3} + 1) \pm (2\sqrt{3} - 1)}{2}$$

$$x = 2\sqrt{3}, 1$$

Therefore,

$$x^2 - (2\sqrt{3} + 1)x + 2\sqrt{3} = (x - 2\sqrt{3})(x - 1)$$

J 108b

$$(3) \quad x^2 - (a - 2b)x - 2ab$$

[Sol] Solving $x^2 - (a - 2b)x - 2ab = 0$

$$\begin{aligned} x &= \frac{(a - 2b) \pm \sqrt{(a - 2b)^2 + 4 \times 2ab}}{2} \\ &= \frac{(a - 2b) \pm \sqrt{a^2 - 4ab + 4b^2 + 8ab}}{2} \\ &= \frac{(a - 2b) \pm \sqrt{a^2 + 4ab + 4b^2}}{2} \\ &= \frac{(a - 2b) \pm \sqrt{(a + 2b)^2}}{2} \\ &= \frac{(a - 2b) \pm (a + 2b)}{2} \end{aligned}$$

$$x = a, -2b$$

Therefore,

$$x^2 - (a - 2b)x - 2ab = (\mathbf{x - a})(\mathbf{x + 2b})$$

$$(4) \quad x^2 - 2ax + a^2 - b^2$$

[Sol] Solving $x^2 - 2ax + a^2 - b^2 = 0$

$$\begin{aligned} x &= a \pm \sqrt{a^2 - (a^2 - b^2)} \\ &= a \pm \sqrt{b^2} \\ x &= a \pm b \end{aligned}$$

Therefore,


$$x^2 - 2ax + a^2 - b^2 = (\mathbf{x - a - b})(\mathbf{x - a + b})$$

Quadratic Equations II

Ex.


Given that one root of $x^2 + bx + c = 0$ is $1 + \sqrt{2}$, find the values of b and c , and the other root. (Assume b and c are integers.)

(A root is another word for the solution of an equation.)

[Sol] $(1 + \sqrt{2})^2 + b(1 + \sqrt{2}) + c = 0$  Substitute $x = 1 + \sqrt{2}$ into the quadratic equation.


$$3 + 2\sqrt{2} + b + \sqrt{2}b + c = 0$$

$$(3 + b + c) + (2 + b)\sqrt{2} = 0$$


 Group terms with a factor $\sqrt{2}$.

We can deduce that,

$$\begin{cases} 3 + b + c = 0 & \dots \textcircled{1} \\ 2 + b = 0 & \dots \textcircled{2} \end{cases}$$


 This equation holds when $3 + b + c = 0$ and $2 + b = 0$.

From $\textcircled{1}$ and $\textcircled{2}$, $b = -2$ and $c = -1$

Thus $x^2 - 2x - 1 = 0$  Put $b = -2$, $c = -1$ in the original equation.
and $x = 1 \pm \sqrt{2}$

Therefore, the other root is $x = 1 - \sqrt{2}$

1. Given that one root of $x^2 + bx + c = 0$ is $-1 + \sqrt{2}$, find the values of b and c , and the other root. (Assume b and c are integers.)

[Sol] $(-1 + \sqrt{2})^2 + b(-1 + \sqrt{2}) + c = 0$  Substitute $x = -1 + \sqrt{2}$ into the quadratic equation.

$$3 - 2\sqrt{2} - b + \sqrt{2}b + c = 0$$

$$(3 - b + c) + (b - 2)\sqrt{2} = 0$$

We can deduce that,

$$\begin{cases} 3 - b + c = 0 & \dots \textcircled{1} \\ b - 2 = 0 & \dots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}$, $b = 2$ and $c = -1$

Thus $x^2 + 2x - 1 = 0$

and $x = -1 \pm \sqrt{2}$

Therefore, the other root is $x = -1 - \sqrt{2}$

J 109b

2. Given that one root of $ax^2 + bx + 2 = 0$ is $2 - \sqrt{2}$, find the values of a and b , and the other root. (Assume a and b are integers.)

[Sol] $a(2 - \sqrt{2})^2 + b(2 - \sqrt{2}) + 2 = 0$



Substitute $x = 2 - \sqrt{2}$ into the quadratic equation.

$$a(6 - 4\sqrt{2}) + b(2 - \sqrt{2}) + 2 = 0$$

$$6a - 4\sqrt{2}a + 2b - \sqrt{2}b + 2 = 0$$

$$(6a + 2b + 2) + (-4a - b)\sqrt{2} = 0$$

We can deduce that,

$$\begin{cases} 6a + 2b + 2 = 0 \dots \textcircled{1} \\ -4a - b = 0 \dots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}$, $\mathbf{a = 1}$ and $\mathbf{b = -4}$

Thus $x^2 - 4x + 2 = 0$

and $x = 2 \pm \sqrt{2}$

Therefore, the other root is $\mathbf{x = 2 + \sqrt{2}}$

Note Summary

$\textcircled{1}$ • $x^2 - 6x + 8 = (x - 4)(x - 2)$

If the quadratic expression is set equal to zero, $x^2 - 6x + 8 = 0$, there are two solutions: $x = 4, 2$

• If we know that the roots of $x^2 - 6x + 8 = 0$ are $x = 4, 2$, we can use this to factorise $x^2 - 6x + 8 = (x - 4)(x - 2)$.

$\textcircled{2}$ If a, b and c are rational and the quadratic equation $ax^2 + bx + c = 0$ has an irrational solution $x = \alpha + \sqrt{\beta}$, then the other solution is $x = \alpha - \sqrt{\beta}$.

J110a KUMON

Quadratic Equations II

1. Solve the following quadratic equations.

$$(1) \quad 7x^2 + 32x - 15 = 0$$

$$(7x-3)(x+5) = 0$$

$$\mathbf{x = \frac{3}{7}, -5}$$

$$(4) \quad 0.2x^2 - 0.5x - 1.2 = 0$$

$$2x^2 - 5x - 12 = 0$$

$$(2x+3)(x-4) = 0$$

$$\mathbf{x = -\frac{3}{2}, 4}$$

$$(2) \quad x^2 - x - 3 = 0$$

$$\mathbf{x = \frac{1 \pm \sqrt{1+12}}{2}}$$

$$\mathbf{= \frac{1 \pm \sqrt{13}}{2}}$$

$$(5) \quad (x-4)(2x-3) = 12$$

$$2x^2 - 11x + 12 - 12 = 0$$

$$2x^2 - 11x = 0$$

$$x(2x-11) = 0$$

$$\mathbf{x = 0, \frac{11}{2}}$$

$$(3) \quad \frac{(x-1)^2}{3} - \frac{x^2-1}{2} - 1 = 0 \quad (6) \quad \frac{2}{3}x + \frac{1}{2} = \frac{1}{6}x^2$$

$$2(x-1)^2 - 3(x^2-1) - 6 = 0$$

$$-x^2 - 4x - 1 = 0$$

$$x^2 + 4x + 1 = 0$$

$$\mathbf{x = -2 \pm \sqrt{4-1}}$$

$$\mathbf{= -2 \pm \sqrt{3}}$$

$$4x + 3 = x^2$$


$$x^2 - 4x - 3 = 0$$

$$\mathbf{x = 2 \pm \sqrt{4+3}}$$

$$\mathbf{= 2 \pm \sqrt{7}}$$

J 110b

2. Given that one root of $x^2 - 3x + a = 0$ is $x = 6$, find the value of a and the other root.


[Sol] $36 - 18 + a = 0$  Substitute $x = 6$ into the quadratic equation.
 $18 + a = 0$
 $a = -18$

Therefore,

$$\begin{aligned}x^2 - 3x - 18 &= 0 \\(x - 6)(x + 3) &= 0 \\x &= 6, -3\end{aligned}$$

Thus, $a = -18$ and the other root is $x = -3$

3. Given that one root of $x^2 + 6x + a = 0$ is $x = -3 + \sqrt{17}$, find the value of a and the other root.

[Sol] $(-3 + \sqrt{17})^2 + 6(-3 + \sqrt{17}) + a = 0$  Substitute $x = -3 + \sqrt{17}$ into the quadratic equation.
 $26 - 6\sqrt{17} - 18 + 6\sqrt{17} + a = 0$
 $8 + a = 0$
 $a = -8$

Therefore,

$$\begin{aligned}x^2 + 6x - 8 &= 0 \\x &= -3 \pm \sqrt{9 + 8} \\&= -3 \pm \sqrt{17}\end{aligned}$$

Thus, $a = -8$ and the other root is $x = -3 - \sqrt{17}$

J111a KUMON

Quadratic Equations and Complex Numbers

1. Calculate the following expressions.

$$(1) \quad 2^2 = 4$$

$$(6) \quad (-2)^2 = 4$$

$$(2) \quad 5^2 = 25$$

$$(7) \quad (-5)^2 = 25$$

$$(3) \quad \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

$$(8) \quad \left(-\frac{1}{2}\right)^2 = \frac{1}{4}$$

$$(4) \quad (\sqrt{2})^2 = 2$$

$$(9) \quad (-\sqrt{2})^2 = 2$$

$$(5) \quad (\sqrt{3})^2 = 3$$

$$(10) \quad (-\sqrt{3})^2 = 3$$

Positive numbers, negative numbers and zero, form the set of *real numbers*.

All positive and negative numbers give positive answers when squared. Zero gives zero when squared.

To express numbers that give negative answers when squared, we define the “*non-real*” number i . (This is called an *imaginary number*.)

$$i^2 = -1 \quad (\sqrt{-1} = i)$$

J 111 b

2. Express the following square roots in terms of i .

Ex.

$$\sqrt{-5} = \sqrt{5}i, \quad \sqrt{-4} = \sqrt{4}i = 2i, \quad \sqrt{-8} = \sqrt{8}i = 2\sqrt{2}i$$

$$(1) \quad \sqrt{-3} = \sqrt{3}i$$

$$(2) \quad \sqrt{-6} = \sqrt{6}i$$

$$(3) \quad \sqrt{-9} = \sqrt{9}i = 3i$$

$$(4) \quad \sqrt{-12} = \sqrt{12}i = 2\sqrt{3}i$$

$$(5) \quad \sqrt{-18} = \sqrt{18}i = 3\sqrt{2}i$$

$$(6) \quad \sqrt{-\frac{1}{25}} = \sqrt{\frac{1}{25}}i = \frac{1}{5}i$$

$$(7) \quad \sqrt{-\frac{3}{16}} = \sqrt{\frac{3}{16}}i = \frac{\sqrt{3}}{4}i$$

When $a \geq 0$, $\sqrt{-a} = \sqrt{a}i$

Quadratic Equations and Complex Numbers

Using $i^2 = -1$, calculate the following expressions.

Ex.

$$(2i)^2 = 2i \times 2i = 2^2 \cdot i^2 = 4 \times (-1) = -4$$

$$(-2i)^2 = (-2)^2 \cdot i^2 = 4 \times (-1) = -4$$

$$(\sqrt{2}i)^2 = (\sqrt{2})^2 \cdot i^2 = 2 \times (-1) = -2$$

$$(1) \quad (3i)^2 = 3i \times 3i = 3^2 \cdot i^2 = 9 \times (-1) = -9$$

$$(2) \quad (-3i)^2 = (-3)^2 \cdot i^2 = 9 \times (-1) = -9$$

$$(3) \quad (4i)^2 = 16 \times (-1) = -16$$

$$(4) \quad (-7i)^2 = 49 \times (-1) = -49$$

$$(5) \quad (\sqrt{3}i)^2 = (\sqrt{3})^2 \cdot i^2 = 3 \times (-1) = -3$$

$$(6) \quad (-\sqrt{3}i)^2 = (-\sqrt{3})^2 \cdot i^2 = 3 \times (-1) = -3$$

$$(7) \quad (\sqrt{5}i)^2 = 5 \times (-1) = -5$$

$$(8) \quad (2\sqrt{2}i)^2 = 8 \times (-1) = -8$$

J 112b

Ex.

$$3i \times 2i = 6i^2 = -6 \quad \text{☞} \quad \text{Substitute } i^2 = -1.$$

$$(9) \quad 3i \times 4i = 12i^2 = -12$$

$$(10) \quad 3i \times (-4i) = -12i^2 = 12$$

$$(11) \quad \sqrt{3}i \times 2i = 2\sqrt{3}i^2 = -2\sqrt{3}$$

$$(12) \quad (-\sqrt{3}i) \times 4i = -4\sqrt{3}i^2 = 4\sqrt{3}$$

Ex.

$$\sqrt{-4} \times \sqrt{-25} = 2i \times 5i = 10i^2 = -10$$

$$(13) \quad \sqrt{-9} \times \sqrt{-16} = 3i \times 4i = 12i^2 = -12$$

$$(14) \quad 2\sqrt{-2} \times 3\sqrt{-3} = 2\sqrt{2}i \times 3\sqrt{3}i = 6\sqrt{6}i^2 = -6\sqrt{6}$$

$$(15) \quad \sqrt{-4} \times \sqrt{-9} = 2i \times 3i = 6i^2 = -6$$

$$(16) \quad \sqrt{(-4) \times (-9)} = \sqrt{36} = 6$$

Conclusion:

$$\sqrt{a} \sqrt{b} = \sqrt{ab} \quad \text{unless } a < 0 \text{ and } b < 0$$

Note the answers to (15) and (16) are not the same.

J113a KUMON

Quadratic Equations and Complex Numbers

Calculate the following expressions.

Ex.

$$\sqrt{-4} \div \sqrt{16} = \frac{2i}{4} = \frac{1}{2}i$$

$$\sqrt{4} \div \sqrt{-16} = \frac{2}{4i} = \frac{2i}{4i^2} = \frac{2i}{-4} = -\frac{1}{2}i$$

To eliminate i from the denominator, multiply the numerator and denominator of $\frac{2}{4i}$ by i .

$$(1) \quad \sqrt{-9} \div \sqrt{16} = \frac{3i}{4} = \frac{3}{4}i$$

$$(2) \quad \sqrt{9} \div \sqrt{-16} = \frac{3}{4i} = \frac{3i}{4i^2} = \frac{3i}{-4} = -\frac{3}{4}i$$

$$(3) \quad \sqrt{-9} \div \sqrt{25} = \frac{3i}{5} = \frac{3}{5}i$$

$$(4) \quad \sqrt{9} \div \sqrt{-25} = \frac{3}{5i} = \frac{3i}{5i^2} = \frac{3i}{-5} = -\frac{3}{5}i$$

$$(5) \quad \frac{\sqrt{9}}{\sqrt{-25}} = \frac{3}{5i} = \frac{3i}{5i^2} = \frac{3i}{-5} = -\frac{3}{5}i$$

$$(6) \quad \sqrt{\frac{9}{-25}} = \sqrt{-\frac{9}{25}} = \sqrt{\frac{9}{25}}i = \frac{3}{5}i$$

Conclusion:

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}} \text{ unless } a > 0 \text{ and } b < 0$$

Note the answers to (5) and (6) are not the same.

J 113b

Ex.

$$(2+i)(3+2i) = 6 + 7i + 2i^2 = 4 + 7i$$

$$(7) \quad (2-i)(3+2i) = 6 + i - 2i^2 = \mathbf{8+i}$$

$$(8) \quad (3-2i)(1+3i) = 3 + 7i - 6i^2 = \mathbf{9+7i}$$

$$(9) \quad (2+\sqrt{2}i)(3+\sqrt{2}i) = 6 + 5\sqrt{2}i + 2i^2 = \mathbf{4+5\sqrt{2}i}$$

$$(10) \quad (5+\sqrt{3}i)(4-\sqrt{3}i) = 20 - \sqrt{3}i - 3i^2 = \mathbf{23-\sqrt{3}i}$$

$$(11) \quad (3+2i)(3-2i) = 9 - 4i^2 = \mathbf{13}$$

$$(12) \quad (3+2i)^2 = 9 + 12i + 4i^2 = \mathbf{5+12i}$$

$3+2i$ and $4-3i$ are called **complex numbers**.

- $a+bi$ is the standard form of a **complex number**, (a and b are real numbers).
- When $b=0$, $a+bi$ is a **real number**, (since $a+0 \cdot i = a$).
- When $a=0$, $a+bi$ is an **imaginary number**.

J114a KUMON

Quadratic Equations and Complex Numbers

Calculate the following expressions.

$$(1) \quad (2 + \sqrt{-2})(3 + \sqrt{-2}) = 6 + 5\sqrt{2}i + 2i^2 = \mathbf{4 + 5\sqrt{2}i}$$

$$(2) \quad (2 - 5\sqrt{2}i)(3 + 4\sqrt{2}i) = 6 - 7\sqrt{2}i - 40i^2 = \mathbf{46 - 7\sqrt{2}i}$$

$$(3) \quad (4 - \sqrt{-2})(4 + \sqrt{-2}) = 16 - 2i^2 = \mathbf{18}$$

$$(4) \quad (\sqrt{3}i + 2)^2 = 3i^2 + 4\sqrt{3}i + 4 = \mathbf{1 + 4\sqrt{3}i}$$

$$(5) \quad (\sqrt{-5} + 1)^2 = (\sqrt{5}i + 1)^2 = 5i^2 + 2\sqrt{5}i + 1 = \mathbf{-4 + 2\sqrt{5}i}$$

$$(6) \quad (-3 - 2\sqrt{-2})^2 = (-3 - 2\sqrt{2}i)^2 = 9 + 12\sqrt{2}i + 8i^2 \\ = \mathbf{1 + 12\sqrt{2}i}$$

J 114b

Ex.

$$\frac{3-i}{3+2i} = \frac{(3-i)(3-2i)}{(3+2i)(3-2i)} = \frac{9-9i+2i^2}{9+4} = \frac{7-9i}{13}$$

Multiply the numerator and denominator of

$$\frac{3-i}{3+2i} \text{ by } (3-2i).$$

$$(7) \quad \frac{1+i}{3+2i} = \frac{(1+i)(3-2i)}{(3+2i)(3-2i)} = \frac{3+i-2i^2}{9+4} = \frac{5+i}{13}$$

$$(8) \quad \frac{5-i}{2-3i} = \frac{(5-i)(2+3i)}{(2-3i)(2+3i)} = \frac{10+13i-3i^2}{4+9} = \frac{13+13i}{13} = 1+i$$

$$(9) \quad \frac{1+i}{1-i} = \frac{(1+i)^2}{(1-i)(1+i)} = \frac{1+2i+i^2}{1+1} = \frac{2i}{2} = i$$

$$(10) \quad \frac{2-i}{2+i} = \frac{(2-i)^2}{(2+i)(2-i)} = \frac{4-4i+i^2}{4+1} = \frac{3-4i}{5}$$

$$(11) \quad \frac{1}{2-i} + \frac{1}{2+i} = \frac{2+i+2-i}{(2-i)(2+i)} = \frac{4}{4+1} = \frac{4}{5}$$

$$(12) \quad \frac{i}{2-i} - \frac{i}{2+i} = \frac{i(2+i)-i(2-i)}{(2-i)(2+i)} = \frac{2i+i^2-2i+i^2}{4+1} = -\frac{2}{5}$$

J115a KUMON

Quadratic Equations and Complex Numbers

1. Calculate the following expressions.

Ex.

$$i^3 = i^2 \cdot i = (-1) \cdot i = -i$$

$$i^4 = (i^2)^2 = (-1)^2 = 1$$

$$(1) \quad i^5 = (i^2)^2 \cdot i = (-1)^2 \cdot i = \mathbf{i}$$

$$(2) \quad i^6 = (i^2)^3 = (-1)^3 = \mathbf{-1}$$

$$(3) \quad i^7 = (i^2)^3 \cdot i = (-1)^3 \cdot i = \mathbf{-i}$$

$$(4) \quad i^8 = (i^2)^4 = (-1)^4 = \mathbf{1}$$

$$(5) \quad \frac{1}{i} = \frac{i}{i \cdot i} = \frac{i}{-1} = \mathbf{-i}$$

$$(6) \quad \frac{1}{i^2} = \frac{1}{-1} = \mathbf{-1}$$

$$(7) \quad \frac{1}{i^3} = \frac{1}{i^2 \cdot i} = -\frac{1}{i} = -\frac{i}{i^2} = -\frac{i}{-1} = \mathbf{i}$$

$$(8) \quad \frac{1}{i^4} = \frac{1}{(i^2)^2} = \frac{1}{(-1)^2} = \mathbf{1}$$

J 115b

2. Solve for the real numbers x and y .

Ex.

$$(x + y) + (x - y)i = 7 + 3i$$



This equation holds when

① $x + y = 7$ and ② $x - y = 3$.

[Sol] Comparing real and imaginary parts on each side:

$$\begin{cases} x + y = 7 & \dots \textcircled{1} \\ x - y = 3 & \dots \textcircled{2} \end{cases}$$

From ① and ②,
 $x = 5, y = 2$

(1) $(2x - y) + (y - x)i = 1 + 3i$

Comparing real and imaginary parts on each side:

$$\begin{cases} 2x - y = 1 & \dots \textcircled{1} \\ y - x = 3 & \dots \textcircled{2} \end{cases}$$

From ① and ②,

$$\mathbf{x = 4, y = 7}$$

First, group together real and imaginary parts.

(2) $(1 + i)x - (1 - 2i)y = 1 + 4i$

[Sol] $(\mathbf{x - y}) + (\mathbf{x + 2y})i = 1 + 4i$

Comparing real and imaginary parts on each side:

$$\begin{cases} x - y = 1 & \dots \textcircled{1} \\ x + 2y = 4 & \dots \textcircled{2} \end{cases}$$

From ① and ②,

$$\mathbf{x = 2, y = 1}$$

Two complex numbers, $a + bi$ and $c + di$, are equal when they have identical real parts and identical imaginary parts.

$$a + bi = c + di \Leftrightarrow a = c, b = d$$

Also $a + bi = 0 \Leftrightarrow a = 0, b = 0$

\Leftrightarrow means “if and only if”

J116a KUMON

Quadratic Equations and Complex Numbers

Quadratic Formula IWhen $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Quadratic Formula IIWhen $ax^2 + 2b'x + c = 0$,

$$x = \frac{-b' \pm \sqrt{b'^2 - ac}}{a}$$

Ex.

$$3x^2 + 5x - 7 = 0$$

$$x = \frac{-5 \pm \sqrt{25 - 4 \cdot 3 \cdot (-7)}}{6}$$

$$= \frac{-5 \pm \sqrt{109}}{6}$$

Ex.

$$5x^2 - 8x + 4 = 0$$

$$x = \frac{4 \pm \sqrt{16 - 20}}{5}$$

$$= \frac{4 \pm \sqrt{-4}}{5}$$

$$= \frac{4 \pm 2i}{5} \quad \leftarrow \begin{array}{l} \text{Substitute} \\ \sqrt{-4} = 2i. \end{array}$$

Solve the following equations by using the quadratic formula.

(1) $2x^2 + 5x + 4 = 0$

$$x = \frac{-5 \pm \sqrt{25 - 4 \cdot 2 \cdot 4}}{4}$$

$$= \frac{-5 \pm \sqrt{-7}}{4}$$

$$= \frac{-5 \pm \sqrt{7}i}{4}$$

(2) $4x^2 - 4x + 5 = 0$

$$x = \frac{2 \pm \sqrt{4 - 20}}{4}$$

$$= \frac{2 \pm \sqrt{-16}}{4}$$

$$= \frac{2 \pm 4i}{4}$$

$$= \frac{1 \pm 2i}{2}$$

When the solution of an equation is a complex number, we call it a **complex root** or **complex solution**.

J116b

$$(3) \quad 3x^2 - 8x + 7 = 0$$

$$\begin{aligned}x &= \frac{4 \pm \sqrt{16 - 21}}{3} \\&= \frac{4 \pm \sqrt{-5}}{3} \\&= \frac{4 \pm \sqrt{5}i}{3}\end{aligned}$$

$$(6) \quad 3x^2 - 12 = 0$$

$$\begin{aligned}x &= \frac{0 \pm \sqrt{0 - 3 \cdot (-12)}}{3} \\&= \pm \frac{6}{3} \\&= \pm 2\end{aligned}$$

$$(4) \quad 2x^2 + 8x + 9 = 0$$

$$\begin{aligned}x &= \frac{-4 \pm \sqrt{16 - 18}}{2} \\&= \frac{-4 \pm \sqrt{-2}}{2} \\&= \frac{-4 \pm \sqrt{2}i}{2}\end{aligned}$$

$$(7) \quad 2x^2 = 5x$$

$$\begin{aligned}2x^2 - 5x &= 0 \\x &= \frac{5 \pm \sqrt{25 - 4 \cdot 2 \cdot 0}}{4} \\&= \frac{5 \pm 5}{4} \\x &= \frac{5}{2}, 0\end{aligned}$$

$$(5) \quad 4x^2 - 12x + 9 = 0$$

$$\begin{aligned}x &= \frac{6 \pm \sqrt{36 - 36}}{4} \\&= \frac{6 \pm 0}{4} \\&= \frac{3}{2}\end{aligned}$$

$$(8) \quad -x^2 + 4x - 5 = 0$$

$$\begin{aligned}x^2 - 4x + 5 &= 0 \\x &= \frac{2 \pm \sqrt{4 - 5}}{1} \\&= 2 \pm \sqrt{-1} \\&= 2 \pm i\end{aligned}$$

J117a KUMON

Quadratic Equations and Complex Numbers

Solve the following quadratic equations.

$$(1) \quad 5x^2 - 6x + 4 = 0$$

$$\begin{aligned} x &= \frac{3 \pm \sqrt{9 - 20}}{5} \\ &= \frac{3 \pm \sqrt{11} i}{5} \end{aligned}$$

$$(4) \quad -x^2 + 3x - 5 = 0$$

$$\begin{aligned} x^2 - 3x + 5 &= 0 \\ x &= \frac{3 \pm \sqrt{9 - 20}}{2} \\ &= \frac{3 \pm \sqrt{11} i}{2} \end{aligned}$$

$$(2) \quad 9x^2 + 12x + 4 = 0$$

$$\begin{aligned} (3x + 2)^2 &= 0 \\ x &= -\frac{2}{3} \end{aligned}$$

$$(5) \quad -3x^2 + 8x - 7 = 0$$

$$\begin{aligned} 3x^2 - 8x + 7 &= 0 \\ x &= \frac{4 \pm \sqrt{16 - 21}}{3} \\ &= \frac{4 \pm \sqrt{5} i}{3} \end{aligned}$$

$$(3) \quad 2x^2 + 5x + 5 = 0$$

$$\begin{aligned} x &= \frac{-5 \pm \sqrt{25 - 40}}{4} \\ &= \frac{-5 \pm \sqrt{15} i}{4} \end{aligned}$$

J 117b

$$(6) \quad 2x(3-x) = 4 - 3x$$

$$2x^2 - 9x + 4 = 0$$

$$(2x-1)(x-4) = 0$$

$$\mathbf{x = \frac{1}{2}, 4}$$

$$(8) \quad x^2 - \sqrt{3}x - 2 = 0$$

$$\mathbf{x = \frac{\sqrt{3} \pm \sqrt{3+8}}{2}}$$

$$\mathbf{= \frac{\sqrt{3} \pm \sqrt{11}}{2}}$$

$$(7) \quad (x-5)(2x-1) = 5$$

$$2x^2 - 11x + 5 = 5$$

$$2x^2 - 11x = 0$$

$$x(2x-11) = 0$$

$$\mathbf{x = 0, \frac{11}{2}}$$

$$(9) \quad x^2 + \sqrt{5}x - 10 = 0$$

$$x = \frac{-\sqrt{5} \pm \sqrt{5+40}}{2}$$

$$= \frac{-\sqrt{5} \pm \sqrt{45}}{2} = \frac{-\sqrt{5} \pm 3\sqrt{5}}{2}$$

$$\mathbf{x = \sqrt{5}, -2\sqrt{5}}$$

$$\left[\begin{array}{l} \text{Alternative Solution} \\ (x + 2\sqrt{5})(x - \sqrt{5}) = 0 \\ \mathbf{x = -2\sqrt{5}, \sqrt{5}} \end{array} \right]$$

J118a KUMON

Quadratic Equations and Complex Numbers

Solve the following quadratic equations.

$$(1) \quad (x+1)^2 = (x+2)^2 + (x+3)^2 \quad (3) \quad x^2 = \frac{2}{3}(x+1)$$

$$x^2 + 2x + 1 = x^2 + 4x + 4 + x^2 + 6x + 9$$

$$3x^2 = 2(x+1)$$

$$x^2 + 8x + 12 = 0$$

$$3x^2 - 2x - 2 = 0$$

$$(x+6)(x+2) = 0$$

$$x = \frac{1 \pm \sqrt{1+6}}{3}$$

$$x = -6, -2$$

$$= \frac{1 \pm \sqrt{7}}{3}$$

$$(2) \quad \frac{1}{4} = (2+x)^2$$

$$(4) \quad \frac{x(x+3)}{3} = \frac{5x-1}{4}$$

$$1 = 4(2+x)^2$$

$$4x(x+3) = 3(5x-1)$$

$$4x^2 + 16x + 15 = 0$$

$$4x^2 + 12x = 15x - 3$$

$$(2x+5)(2x+3) = 0$$

$$4x^2 - 3x + 3 = 0$$

$$x = -\frac{5}{2}, -\frac{3}{2}$$

$$x = \frac{3 \pm \sqrt{9-48}}{8}$$

$$= \frac{3 \pm \sqrt{39}i}{8}$$

J 118b

$$(5) \quad 0.3x^2 - 2x + 3.6 = 0$$

$$3x^2 - 20x + 36 = 0$$

$$\begin{aligned} x &= \frac{10 \pm \sqrt{100 - 108}}{3} \\ &= \frac{10 \pm 2\sqrt{2}i}{3} \end{aligned}$$

$$(8) \quad x^2 - 4ax + 3a^2 = 0$$

$$(x - 3a)(x - a) = 0$$

$$\mathbf{x = 3a, a}$$

$$(6) \quad 0.2x^2 - 0.5x + 0.4 = 0$$

$$2x^2 - 5x + 4 = 0$$

$$\begin{aligned} x &= \frac{5 \pm \sqrt{25 - 32}}{4} \\ &= \frac{5 \pm \sqrt{7}i}{4} \end{aligned}$$

$$(9) \quad x^2 - 2ax + a^2 - b^2 = 0$$

$$[x - (a + b)][x - (a - b)] = 0$$

$$\mathbf{x = a + b, a - b}$$

$$[\mathbf{x = a \pm b}]$$

$$(7) \quad x^2 - 2mx + n^2 = 0$$

$$\mathbf{x = m \pm \sqrt{m^2 - n^2}}$$

Quadratic Equations and Complex Numbers

1. Factorise the following expressions using complex numbers.

Ex.

$$x^2 - 2x + 3$$

[Sol] Solving $x^2 - 2x + 3 = 0$

$$x = 1 \pm \sqrt{1-3}$$

$$= 1 \pm \sqrt{2}i$$

$$[x - (1 + \sqrt{2}i)][x - (1 - \sqrt{2}i)] = 0$$

Thus, $x^2 - 2x + 3 = (x - 1 - \sqrt{2}i)(x - 1 + \sqrt{2}i)$

(1) $x^2 - 2x + 5$

Solving $x^2 - 2x + 5 = 0$

$$x = 1 \pm \sqrt{1-5}$$

$$= 1 \pm \sqrt{-4}$$

$$= 1 \pm 2i$$

$$x = 1 + 2i, 1 - 2i$$

$$[x - (1 + 2i)][x - (1 - 2i)] = 0$$

$$(x - 1 - 2i)(x - 1 + 2i) = 0$$

Thus,

$$x^2 - 2x + 5$$

$$= (x - 1 - 2i)(x - 1 + 2i)$$

(2) $x^2 + 1$

Solving $x^2 + 1 = 0$

$$x^2 = -1$$

$$x = \pm i$$

$$(x - i)(x + i) = 0$$

Thus, $x^2 + 1 = (x - i)(x + i)$

Ex.

Given that one root of $ax^2 - 2x + b = 0$ is $1 + 2i$, find the values of a and b , and the other root. (Assume a and b are real numbers.)

[Sol] Substitute $x = 1 + 2i$ into the quadratic equation,

$$a(1 + 2i)^2 - 2(1 + 2i) + b = 0$$

$$-3a + 4ai - 2 - 4i + b = 0$$

$$-3a + b - 2 + (4a - 4)i = 0$$

↪ Group together real and imaginary terms.

Therefore,

$$\begin{cases} -3a + b - 2 = 0 & \dots \textcircled{1} \\ 4a - 4 = 0 & \dots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}$, $a = 1$, $b = 5$

$$x^2 - 2x + 5 = 0$$

$$x = 1 \pm 2i$$

↪ Substituting into $ax^2 - 2x + b = 0$.

Thus, the other root is $x = 1 - 2i$

2. Given that one root of $x^2 - ax + b = 0$ is $1 + i$, find the values of a and b , and the other root.

[Sol] Substitute $x = 1 + i$ in the quadratic equation,

$$(1 + i)^2 - a(1 + i) + b = 0$$

$$2i - a - ai + b = 0$$

$$(-a + b) + (2 - a)i = 0$$

Therefore,

$$\begin{cases} -a + b = 0 & \dots \textcircled{1} \\ 2 - a = 0 & \dots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}$, $a = 2$, $b = 2$

$$x^2 - 2x + 2 = 0$$

$$x = 1 \pm i$$

↪ Substituting into $x^2 - ax + b = 0$.

Thus, the other root is $x = 1 - i$

J 120a KUMON

Quadratic Equations and Complex Numbers

1. Solve the following quadratic equations.

$$(1) \quad -x^2 + 12x + 8 = 0$$

$$x^2 - 12x - 8 = 0$$

$$x = 6 \pm \sqrt{36 + 8}$$

$$= 6 \pm 2\sqrt{11}$$

$$(4) \quad x^2 - 2ax + 2a^2 = 0$$

$$x = a \pm \sqrt{a^2 - 2a^2}$$

$$= a \pm a\sqrt{-1}$$

$$= a \pm ai$$

$$(2) \quad \frac{1}{9} = (2+x)^2$$

$$1 = 9(2+x)^2$$

$$9x^2 + 36x + 35 = 0$$

$$(3x+7)(3x+5) = 0$$

$$x = -\frac{7}{3}, -\frac{5}{3}$$

$$(5) \quad x^2 + 2ax + (2a-1) = 0$$

$$(x+1)[x+(2a-1)] = 0$$

$$x = -1, -2a+1$$

$$(3) \quad 0.5x^2 + 0.2x + 0.1 = 0$$

$$5x^2 + 2x + 1 = 0$$

$$x = \frac{-1 \pm \sqrt{1-5}}{5}$$

$$= \frac{-1 \pm 2i}{5}$$

J | 20b

2. Calculate the following expressions.

$$(1) \quad \sqrt{-3} \sqrt{-5} = \sqrt{3}i \cdot \sqrt{5}i = \sqrt{15}i^2 = -\sqrt{15}$$

$$(2) \quad \sqrt{(-3) \times (-5)} = \sqrt{15}$$

$$(3) \quad \frac{\sqrt{3}}{\sqrt{-5}} = \frac{\sqrt{3}}{\sqrt{5}i} = \frac{\sqrt{3} \cdot \sqrt{5} \cdot i}{\sqrt{5} \cdot \sqrt{5} \cdot i^2} = \frac{\sqrt{15}i}{-5} = -\frac{\sqrt{15}}{5}i$$

$$(4) \quad \sqrt{\frac{3}{-5}} = \sqrt{\frac{3}{5}}i = \frac{\sqrt{15}}{5}i$$

$$(5) \quad (3+2i)(5-6i) = 15 - 8i - 12i^2 = \mathbf{27 - 8i}$$

$$(6) \quad (3 - \sqrt{-2})^2 = (3 - \sqrt{2}i)^2 = 9 - 6\sqrt{2}i + 2i^2 = \mathbf{7 - 6\sqrt{2}i}$$

$$(7) \quad \frac{5-i}{2-3i} = \frac{(5-i)(2+3i)}{(2-3i)(2+3i)} = \frac{10+13i-3i^2}{4+9} = \frac{13+13i}{13} = \mathbf{1+i}$$

Discriminant

Quadratic Formula I

$$\text{When } ax^2 + bx + c = 0,$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

1. Solve the following equations by using the quadratic formula.

$$(1) \quad x^2 + 8x + 15 = 0$$

$$x = \frac{-8 \pm \sqrt{64 - 60}}{2}$$

$$= -4 \pm 1$$

$$\mathbf{x = -3, -5}$$

$$(4) \quad x^2 - 4x + 4 = 0$$

$$\mathbf{x = \frac{4 \pm \sqrt{16 - 16}}{2}}$$

$$= 2$$

$$(2) \quad x^2 + 8x + 16 = 0$$

$$\mathbf{x = \frac{-8 \pm \sqrt{64 - 64}}{2}}$$

$$= -4$$

$$(5) \quad x^2 - 4x + 5 = 0$$

$$\mathbf{x = \frac{4 \pm \sqrt{16 - 20}}{2}}$$

$$= 2 \pm i$$

$$(3) \quad x^2 + 8x + 17 = 0$$

$$\mathbf{x = \frac{-8 \pm \sqrt{64 - 68}}{2}}$$

$$= -4 \pm i$$

$$(6) \quad x^2 - 4x + 2 = 0$$

$$\mathbf{x = \frac{4 \pm \sqrt{16 - 8}}{2}}$$

$$= 2 \pm \sqrt{2}$$

Note: It is also possible to solve these equations using Quadratic Formula II.

From the results above, there are three types of solutions:

> 2 different real solutions, (1) and **(6)**.

> a repeated real solution, (i.e. 1 distinct) (2) and **(4)**.

> 2 different complex solutions, (3) and **(5)**.

J121b

2. Solve the following equations by using the quadratic formula, then fill in the blank boxes .

(1) $x^2 - 5x + 5 = 0$

$$x = \frac{5 \pm \sqrt{25 - 20}}{2}$$

$$= \frac{5 \pm \sqrt{5}}{2}$$

The quadratic equation

$x^2 - 5x + 5 = 0$ has 2 different

solutions.

$$b^2 - 4ac = 5^2 - 4 \cdot 1 \cdot 5 = 5 > 0$$

(2) $x^2 - 10x + 25 = 0$

$$x = \frac{10 \pm \sqrt{100 - 100}}{2}$$

$$= 5$$

The quadratic equation

$x^2 - 10x + 25 = 0$ has a

real solution.

$$b^2 - 4ac = \input{type="text" value="0"} < 0$$

(3) $x^2 - 5x + 7 = 0$

$$x = \frac{5 \pm \sqrt{25 - 28}}{2}$$

$$= \frac{5 \pm \sqrt{3}i}{2}$$

The quadratic equation

$x^2 - 5x + 7 = 0$ has 2 different

solutions.

$$b^2 - 4ac = \input{type="text" value="-3"} < 0$$

Given a quadratic equation $ax^2 + bx + c = 0$, the solution is:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

We can calculate $D = b^2 - 4ac$ (the term inside the square root $\sqrt{\quad}$) to determine what types of solutions a quadratic equation may have.

- i) When $D > 0$, there are 2 different real solutions, e.g. (1).
- ii) When $D = 0$, there is a repeated real solution, e.g. (2).
- iii) When $D < 0$, there are 2 different complex solutions, e.g. (3).

* $D = b^2 - 4ac$ is called the **Discriminant**.

Discriminant

For the quadratic equation $ax^2 + bx + c = 0$, the discriminant is $D = b^2 - 4ac$.

When $D > 0 \Leftrightarrow$ There are 2 different real solutions.

When $D = 0 \Leftrightarrow$ There is a repeated real solution.

When $D < 0 \Leftrightarrow$ There are 2 different complex solutions.

In each question, calculate the discriminant, then state what type of solution(s) the given equation has.

Ex.

$$x^2 + 5x + 3 = 0$$

$$[\text{Sol}] D = 5^2 - 4 \cdot 1 \cdot 3 = 13 > 0$$

Therefore, there are 2 different real solutions.

$$(1) \quad x^2 + 5x - 1 = 0$$

$$\begin{aligned} D &= 5^2 - 4 \cdot 1 \cdot (-1) \\ &= 29 > 0 \end{aligned}$$

Therefore, **there are 2 different real solutions.**

$$(3) \quad 4x^2 - 12x + 9 = 0$$

$$\begin{aligned} D &= (-12)^2 - 4 \cdot 4 \cdot 9 \\ &= 0 \end{aligned}$$

Therefore, **there is a repeated real solution.**

$$(2) \quad 3x^2 + 3x + 1 = 0$$

$$\begin{aligned} D &= 3^2 - 4 \cdot 3 \cdot 1 \\ &= -3 < 0 \end{aligned}$$

Therefore, **there are 2 different complex solutions.**

$$(4) \quad 3x^2 + 7x + 4 = 0$$

$$\begin{aligned} D &= 7^2 - 4 \cdot 3 \cdot 4 \\ &= 1 > 0 \end{aligned}$$

Therefore, **there are 2 different real solutions.**

Note: To “discriminate the solutions” means to find whether the quadratic equation has 2 different real solutions, 1 repeated real solution, or 2 different complex solutions.

J | 22b

$$(5) \quad 2x^2 + 6x + 5 = 0$$

$$D = 6^2 - 4 \cdot 2 \cdot 5 \\ = -4 < 0$$

Therefore, **there are 2 different complex solutions.**

$$(8) \quad 2x^2 + \sqrt{3}x + 3 = 0$$

$$D = (\sqrt{3})^2 - 4 \cdot 2 \cdot 3 \\ = -21 < 0$$

Therefore, **there are 2 different complex solutions.**

$$(6) \quad 5x^2 - 10x + 5 = 0$$

$$D = (-10)^2 - 4 \cdot 5 \cdot 5 \\ = 0$$

Therefore, **there is a repeated real solution.**

$$(9) \quad x^2 + 2\sqrt{2}x + 2 = 0$$

$$D = (2\sqrt{2})^2 - 4 \cdot 1 \cdot 2 \\ = 0$$

Therefore, **there is a repeated real solution.**

$$(7) \quad (x+3)^2 + 7 = 0$$

$$x^2 + 6x + 16 = 0 \\ D = 6^2 - 4 \cdot 1 \cdot 16 \\ = -28 < 0$$

Therefore, **there are 2 different complex solutions.**

$$(10) \quad \sqrt{2}x^2 - 3x + 2\sqrt{2} = 0$$

$$D = (-3)^2 - 4 \cdot \sqrt{2} \cdot 2\sqrt{2} \\ = -7 < 0$$

Therefore, **there are 2 different complex solutions.**

Discriminant

For a quadratic equation $ax^2 + 2b'x + c = 0$ (where the coefficient of x is an even number), we can discriminate solutions using:

$$\frac{D}{4} = b'^2 - ac$$

Discriminate the solutions of the following equations.

Ex.

$$x^2 + 4x + 5 = 0 \quad \rightarrow \quad x^2 + 2 \cdot 2x + 5 = 0$$

$$[\text{Sol}] \quad \frac{D}{4} = 2^2 - 1 \cdot 5 = -1 < 0$$

Therefore, there are 2 different complex solutions.

$$(1) \quad 25x^2 + 10x + 1 = 0$$

$$[\text{Sol}] \quad \frac{D}{4} = 5^2 - 25 \cdot 1 = 0$$

Therefore, **there is a repeated real solution.**

$$(3) \quad 25x^2 - 30x + 9 = 0$$

$$\frac{D}{4} = (-15)^2 - 25 \cdot 9 = 0$$

Therefore, **there is a repeated real solution.**

$$(2) \quad 2x^2 + 6x + 3 = 0$$

$$\frac{D}{4} = 3^2 - 2 \cdot 3 = 3 > 0$$

Therefore, **there are 2 different real solutions.**

$$(4) \quad x^2 - 5x + 3 = 0$$

$$D = (-5)^2 - 4 \cdot 1 \cdot 3 = 13 > 0$$

Therefore, **there are 2 different real solutions.**

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$$(5) \quad 2x^2 - 4x + 3 = 0$$

$$\frac{D}{4} = (-2)^2 - 2 \cdot 3 = -2 < 0$$

Therefore, **there are 2 different complex solutions.**

$$(8) \quad 3x^2 - 12x + 13 = 0$$

$$\frac{D}{4} = (-6)^2 - 3 \cdot 13 = -3 < 0$$

Therefore, **there are 2 different complex solutions.**

$$(6) \quad x^2 - 3x - 2 = 0$$

$$D = (-3)^2 - 4 \cdot 1 \cdot (-2) = 17 > 0$$

Therefore, **there are 2 different real solutions.**

$$(9) \quad x^2 + 2\sqrt{3}x + 3 = 0$$

$$\frac{D}{4} = (\sqrt{3})^2 - 1 \cdot 3 = 0$$

Therefore, **there is a repeated real solution.**

$$(7) \quad 3x^2 - 7x + 6 = 0$$

$$D = (-7)^2 - 4 \cdot 3 \cdot 6 = -23 < 0$$

Therefore, **there are 2 different complex solutions.**

$$(10) \quad 2x^2 - 2\sqrt{3}x + 3 = 0$$

$$\frac{D}{4} = (-\sqrt{3})^2 - 2 \cdot 3 = -3 < 0$$

Therefore, **there are 2 different complex solutions.**

J 124a

Discriminant

Determine the value of k for which each equation has a *repeated real solution*.

For $ax^2 + bx + c = 0$,
the discriminant is

$$D = b^2 - 4ac$$

For $ax^2 + 2b'x + c = 0$,
use

$$\frac{D}{4} = b'^2 - ac$$

Ex.

$$2x^2 - 5x + k = 0$$

[Sol] $D = 25 - 8k = 0$



For a *repeated real solution*, the discriminant must equal zero, $D = 0$.

Therefore, $k = \frac{25}{8}$

(1) $3x^2 - 5x + k = 0$

$$D = 25 - 12k = 0$$

Therefore, $k = \frac{25}{12}$

(3) $2x^2 + kx + k - 2 = 0$

$$D = k^2 - 8(k - 2)$$

$$= k^2 - 8k + 16$$

$$= (k - 4)^2 = 0$$

Therefore, $k = 4$

(2) $3x^2 - 8x + k = 0$

$$\frac{D}{4} = 16 - 3k = 0$$

Therefore, $k = \frac{16}{3}$

(4) $3x^2 + 2kx + (2k - 3) = 0$

$$\frac{D}{4} = k^2 - 3(2k - 3)$$

$$= k^2 - 6k + 9$$

$$= (k - 3)^2 = 0$$

Therefore, $k = 3$

J 124b

$$(5) \quad x^2 + 2(k+1)x + 2(1+k^2) = 0 \quad (7) \quad kx^2 - x - (k-2) = 0$$

$$\begin{aligned} \frac{D}{4} &= (k+1)^2 - 2(1+k^2) \\ &= k^2 + 2k + 1 - 2 - 2k^2 \\ &= -k^2 + 2k - 1 = 0 \end{aligned}$$

$$k^2 - 2k + 1 = 0$$

$$(k-1)^2 = 0$$

Therefore, $k = 1$

$$\begin{aligned} D &= 1 + 4k(k-2) \\ &= 4k^2 - 8k + 1 = 0 \end{aligned}$$

Therefore,

$$k = \frac{4 \pm \sqrt{16-4}}{4}$$

$$= \frac{4 \pm 2\sqrt{3}}{4}$$

$$= \frac{2 \pm \sqrt{3}}{2}$$

$$(6) \quad 3x^2 + 4kx + 4k = 0$$

$$\begin{aligned} \frac{D}{4} &= 4k^2 - 12k \\ &= 4k(k-3) = 0 \end{aligned}$$

Therefore, $k = 0, 3$

$$(8) \quad kx^2 + (k+1)x - (k+2) = 0$$

$$\begin{aligned} D &= (k+1)^2 + 4k(k+2) \\ &= k^2 + 2k + 1 + 4k^2 + 8k \\ &= 5k^2 + 10k + 1 = 0 \end{aligned}$$

Therefore,

$$k = \frac{-5 \pm \sqrt{25-5}}{5}$$

$$= \frac{-5 \pm 2\sqrt{5}}{5}$$

Discriminant

Determine the value of a for which each equation has *real solutions*; ($D \geq 0$).

Ex.

$$x^2 - 3x + 2a = 0$$

$$[\text{Sol}] \quad D = 9 - 8a \geq 0 \quad \rightarrow$$

$$\text{Therefore, } a \leq \frac{9}{8}$$

A quadratic equation has 2 real solutions when $D > 0$. A quadratic equation has 1 repeated real solution when $D = 0$. For the equation to have *real solutions*, the discriminant must be greater or equal to zero, $D \geq 0$.

$$(1) \quad x^2 - 4x + 3a = 0$$

$$[\text{Sol}] \quad \frac{D}{4} = 4 - 3a \geq 0$$

$$\text{Therefore, } a \leq \frac{4}{3}$$

$$(3) \quad 2x^2 - 4x + a + 2 = 0$$

$$\frac{D}{4} = 4 - 2(a + 2)$$

$$= -2a \geq 0$$

$$\text{Therefore, } a \leq 0$$

$$(2) \quad 3x^2 - 7x + 2a - 1 = 0$$

$$[\text{Sol}] \quad D = 49 - 12(2a - 1) \\ = -24a + 61 \geq 0$$

$$\text{Therefore, } a \leq \frac{61}{24}$$

$$(4) \quad 3x^2 - 3x + 1 - a = 0$$

$$D = 9 - 12(1 - a) \\ = 12a - 3 \geq 0$$

$$\text{Therefore, } a \geq \frac{1}{4}$$

J 125b

$$(5) \quad 4x^2 - 4(a-3)x + a^2 = 0$$

$$\begin{aligned} \frac{D}{4} &= 4(a-3)^2 - 4a^2 \\ &= 4[(a-3)^2 - a^2] \\ &= 4(a^2 - 6a + 9 - a^2) \\ &= 4(-6a + 9) \geq 0 \end{aligned}$$

Therefore, $a \leq \frac{3}{2}$

$$(7) \quad x^2 - 2ax + (a+1)^2 = 0$$

$$\begin{aligned} \frac{D}{4} &= a^2 - (a+1)^2 \\ &= a^2 - a^2 - 2a - 1 \\ &= -2a - 1 \geq 0 \end{aligned}$$

Therefore, $a \leq -\frac{1}{2}$

$$(6) \quad x^2 - (2a-1)x + 1 + a^2 = 0$$

$$\begin{aligned} D &= (2a-1)^2 - 4(1+a^2) \\ &= 4a^2 - 4a + 1 - 4 - 4a^2 \\ &= -4a - 3 \geq 0 \end{aligned}$$

Therefore, $a \leq -\frac{3}{4}$

$$(8)^* \quad (a+1)x^2 - 2(a+1)x + (a-1) = 0 \quad (a \neq -1)$$

$$\begin{aligned} \frac{D}{4} &= (a+1)^2 - (a+1)(a-1) \\ &= a^2 + 2a + 1 - a^2 + 1 \\ &= 2a + 2 \geq 0 \end{aligned}$$

Therefore, $a \geq -1$

Since $a \neq -1$,
the answer is $a > -1$

Discriminant

Determine the value of k for which each equation has 2 different *complex solutions* ($D < 0$).

Ex.

$$x^2 - 3x + 3k = 0$$

$$[\text{Sol}] D = 9 - 12k < 0$$



To have 2 different *complex solutions* the discriminant must be less than zero, $D < 0$.

$$\text{Therefore, } k > \frac{3}{4}$$

$$(1) \quad x^2 - 2x + 3k = 0$$

$$[\text{Sol}] \quad \frac{D}{4} = 1 - 3k < 0$$

$$\text{Therefore, } k > \frac{1}{3}$$

$$(3) \quad 2x^2 - 4x + k + 1 = 0$$

$$\begin{aligned} \frac{D}{4} &= 4 - 2(k + 1) \\ &= 2 - 2k < 0 \end{aligned}$$

$$\text{Therefore, } k > 1$$

$$(2) \quad 2x^2 - 3x + k - 1 = 0$$

$$\begin{aligned} D &= 9 - 8(k - 1) \\ &= 17 - 8k < 0 \end{aligned}$$

$$\text{Therefore, } k > \frac{17}{8}$$

$$(4) \quad 3x^2 - 5x + 2 - k = 0$$

$$\begin{aligned} D &= 25 - 12(2 - k) \\ &= 1 + 12k < 0 \end{aligned}$$

$$\text{Therefore, } k < -\frac{1}{12}$$

J 126b

$$(5) \quad x^2 - 2(k-1)x + k^2 = 0$$

$$\begin{aligned} \frac{D}{4} &= (k-1)^2 - k^2 \\ &= k^2 - 2k + 1 - k^2 \\ &= -2k + 1 < 0 \end{aligned}$$

Therefore, $k > \frac{1}{2}$

$$(7) \quad x^2 - 4kx + (2k+1)^2 = 0$$

$$\begin{aligned} \frac{D}{4} &= 4k^2 - (2k+1)^2 \\ &= 4k^2 - 4k^2 - 4k - 1 \\ &= -4k - 1 < 0 \end{aligned}$$

Therefore, $k > -\frac{1}{4}$

$$(6) \quad x^2 - (2k+1)x - 1 + k^2 = 0$$

$$\begin{aligned} D &= (2k+1)^2 - 4(-1+k^2) \\ &= 4k^2 + 4k + 1 + 4 - 4k^2 \\ &= 4k + 5 < 0 \end{aligned}$$

Therefore, $k < -\frac{5}{4}$

$$(8) \quad kx^2 - 2(k+1)x + (k-1) = 0 \quad (k \neq 0)$$

$$\begin{aligned} \frac{D}{4} &= (k+1)^2 - k(k-1) \\ &= k^2 + 2k + 1 - k^2 + k \\ &= 3k + 1 < 0 \end{aligned}$$

Therefore, $k < -\frac{1}{3}$

Discriminant

Discriminate the solutions of the following quadratic equations.


Ex.

$$x^2 - 5x + k = 0$$


[Sol] $D = 25 - 4k$

Therefore,


When $k < \frac{25}{4}$,

there are 2 different real solutions.  From $D = 25 - 4k > 0$.

When $k = \frac{25}{4}$,

there is a repeated real solution.  From $D = 25 - 4k = 0$.

When $k > \frac{25}{4}$,

there are 2 different complex solutions.  From $D = 25 - 4k < 0$.

(1) $x^2 - 3x + k = 0$

$$D = 9 - 4k$$

Therefore,

When $k < \frac{9}{4}$, there are 2 different real solutions.

When $k = \frac{9}{4}$, there is a repeated real solution.

When $k > \frac{9}{4}$, there are 2 different complex solutions.

J | 27b

$$(2) \quad x^2 + 2x + 3a = 0$$

$$\frac{D}{4} = 1 - 3a$$

Therefore,

$$\left\{ \begin{array}{l} \text{When } a < \frac{1}{3}, \text{ there are 2 different real solutions.} \\ \text{When } a = \frac{1}{3}, \text{ there is a repeated real solution.} \\ \text{When } a > \frac{1}{3}, \text{ there are 2 different complex solutions.} \end{array} \right.$$

$$(3) \quad 2x^2 - 3x - k = 0$$

$$D = 9 + 8k$$

Therefore,

$$\left\{ \begin{array}{l} \text{When } k > -\frac{9}{8}, \text{ there are 2 different real solutions.} \\ \text{When } k = -\frac{9}{8}, \text{ there is a repeated real solution.} \\ \text{When } k < -\frac{9}{8}, \text{ there are 2 different complex solutions.} \end{array} \right.$$

$$(4) \quad 2x^2 - 4x - 3m = 0$$

$$\frac{D}{4} = 4 + 6m$$

Therefore,

$$\left\{ \begin{array}{l} \text{When } m > -\frac{2}{3}, \text{ there are 2 different real solutions.} \\ \text{When } m = -\frac{2}{3}, \text{ there is a repeated real solution.} \\ \text{When } m < -\frac{2}{3}, \text{ there are 2 different complex solutions.} \end{array} \right.$$

Discriminant

Discriminate the solutions of the following quadratic equations.

Ex.

$$2x^2 + x - 2a^2 = 0$$

[Sol] $D = 1 + 16a^2$

From $a^2 \geq 0$

$$D = 1 + 16a^2 > 0$$

A squared real number cannot be negative.

Thus, regardless of the value of a ,

$$1 + 16a^2 \geq 1 > 0.$$

Therefore, there are 2 different real solutions.

(1) $ax^2 - 3x - 2a = 0$

$$D = 9 + 8a^2$$

From $a^2 \geq 0$

$$D = 9 + 8a^2 > 0$$

Therefore, **there are 2 different real solutions.**

(2) $(a+5)x^2 - 3ax + (a-5) = 0$

$$D = 9a^2 - 4(a+5)(a-5) = 5a^2 + 100$$

From $a^2 \geq 0$

$$D = 5a^2 + 100 > 0$$

Therefore, **there are 2 different real solutions.**

(3)* $x^2 - 2ax + 2a^2 + 1 = 0$

$$\frac{D}{4} = a^2 - 2a^2 - 1 = -(a^2 + 1)$$

From $a^2 \geq 0$

$$D = -(a^2 + 1) < 0$$


Therefore, **there are 2 different complex solutions.**


Ex.

$$x^2 - 3ax + 2a^2 = 0$$

$$[\text{Sol}] D = 9a^2 - 8a^2 = a^2$$

Therefore,

When $a = 0$, there is a repeated real solution.  From $D = 0$.

When $a \neq 0$, there are 2 different real solutions.  From $D > 0$.

$$(4) \quad x^2 - 4ax + 3a^2 = 0$$

$$\frac{D}{4} = 4a^2 - 3a^2 = a^2$$

Therefore,

{ When $a = 0$, there is a repeated real solution.
 When $a \neq 0$, there are 2 different real solutions.

$$(5) \quad x^2 - ax + a^2 = 0$$

$$D = a^2 - 4a^2 = -3a^2$$

Therefore,

{ When $a = 0$, there is a repeated real solution.
 When $a \neq 0$, there are 2 different complex solutions.

$$(6) \quad x^2 - 2ax + a^2 = 0$$

$$\frac{D}{4} = a^2 - a^2 = 0$$

Therefore, there is a repeated real solution for all real values of a .

Discriminant

Ex.

The equation $x^2 + (m+1)x + (2m-1) = 0$ has a repeated real solution.

(1) Find the value(s) of m .

$$\begin{aligned} \text{[Sol]} \quad D &= (m+1)^2 - 4(2m-1) \\ &= m^2 - 6m + 5 \\ &= (m-5)(m-1) = 0 \end{aligned}$$

Thus, $m = 5, 1$

(2) Find the repeated real solution(s).

[Sol] When $m = 5$,

$$x = -\frac{m+1}{2} = -3 \quad \rightarrow$$

When $m = 1$,

$$x = -\frac{m+1}{2} = -1$$

When $ax^2 + bx + c = 0$,
 $D = b^2 - 4ac = 0$, therefore,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{2a}$$

1. The equation $3x^2 + 4mx + m = 0$ has a repeated real solution. Find the value(s) of m , and the solution(s).

$$\begin{aligned} \text{[Sol]} \quad \frac{D}{4} &= 4m^2 - 3m \\ &= m(4m-3) = 0 \end{aligned}$$

Thus, $m = 0, \frac{3}{4}$

When $m = 0$,

$$x = -\frac{4m}{2 \cdot 3} = -\frac{0}{6} = 0$$

When $m = \frac{3}{4}$,

$$x = -\frac{4m}{2 \cdot 3} = -\frac{3}{6} = -\frac{1}{2}$$

J 129b

2.* Given the equation $x^2 + (2k - 3)x + 2k = 0$ has a repeated negative real solution, find the solution.

$$\begin{aligned} \text{[Sol]} \quad D &= (2k - 3)^2 - 8k \\ &= 4k^2 - 12k + 9 - 8k \\ &= 4k^2 - 20k + 9 = 0 \end{aligned}$$

$$\begin{aligned} \text{Thus, } k &= \frac{10 \pm \sqrt{100 - 36}}{4} \\ &= \frac{10 \pm 8}{4} \end{aligned}$$

$$k = \frac{9}{2}, \frac{1}{2}$$

$$\text{When } k = \frac{9}{2},$$

$$x = -\frac{2k - 3}{2} = -\frac{6}{2} = -3$$

$$\text{When } k = \frac{1}{2},$$

$$x = -\frac{2k - 3}{2} = -\frac{1 - 3}{2} = 1$$

$x = 1$ is positive and, thus, it is not the answer to this problem.

Thus, $x = -3$

Discriminant

1. Discriminate the solutions of the following equations.

$$(1) \quad x^2 + 5x + 3 + a = 0$$

$$D = 25 - 4(3 + a)$$

$$= 13 - 4a$$

Therefore,

$$\left\{ \begin{array}{l} \text{When } a < \frac{13}{4}, \text{ there are 2 different real solutions.} \\ \text{When } a = \frac{13}{4}, \text{ there is a repeated real solution.} \\ \text{When } a > \frac{13}{4}, \text{ there are 2 different complex solutions.} \end{array} \right.$$

$$(2) \quad x^2 + (a+b)x + ab = 0$$

$$D = (a+b)^2 - 4ab$$

$$= a^2 - 2ab + b^2$$

$$= (a-b)^2$$

Therefore,

$$\left\{ \begin{array}{l} \text{When } a = b, \text{ there is a repeated real solution.} \\ \text{When } a \neq b, \text{ there are 2 different real solutions.} \end{array} \right.$$

J | 30b

2. The quadratic equation $(b-c)x^2 + (c-a)x + (a-b) = 0$ has a repeated real solution. Prove that $b = \frac{a+c}{2}$.

(Hint: We can rearrange $b = \frac{a+c}{2}$ as $a+c-2b=0$.)

$$\begin{aligned} \text{[Sol]} \quad D &= (c-a)^2 - 4(b-c)(a-b) \\ &= (c^2 - 2ac + a^2) - 4(ab - b^2 - ac + bc) \\ &= c^2 - 2ac + a^2 - 4ab + 4b^2 + 4ac - 4bc \\ &= a^2 + 2(c-2b)a + (c-2b)^2 \\ &= (a+c-2b)^2 \end{aligned}$$

For a repeated real solution, $D = 0$

$$D = (a+c-2b)^2 = 0$$

$$a+c-2b = 0$$

Thus,

$$\mathbf{b = \frac{a+c}{2}}$$

J131a KUMON

Root-Coefficient Relationships

1. By first finding the roots of each quadratic equation, obtain the sum and product of these roots.

Ex.

$$x^2 - 4x + 3 = 0$$

$$[\text{Sol}] (x-3)(x-1) = 0$$

$$x = 3, 1$$

$$\alpha + \beta = 3 + 1 = 4 \quad \text{Use } \alpha \text{ and } \beta \text{ for the two roots.}$$

$$\alpha\beta = 3 \times 1 = 3$$

$$(1) \quad x^2 - 2x - 8 = 0$$

$$(x-4)(x+2) = 0$$

$$x = 4, -2$$

$$\alpha + \beta = 4 - 2 = 2$$

$$\alpha\beta = 4 \times (-2) = -8$$

$$(3) \quad 3x^2 + 2x + 5 = 0$$

$$x = \frac{-1 \pm \sqrt{1-15}}{3}$$

$$= \frac{-1 \pm \sqrt{14}i}{3}$$

$$\alpha + \beta = \frac{-1 + \sqrt{14}i}{3} + \frac{-1 - \sqrt{14}i}{3} = -\frac{2}{3}$$

$$\alpha\beta = \left(\frac{-1 + \sqrt{14}i}{3}\right)\left(\frac{-1 - \sqrt{14}i}{3}\right) = \frac{5}{3}$$

$$(2) \quad x^2 - 7x + 5 = 0$$

$$x = \frac{7 \pm \sqrt{49-20}}{2}$$

$$= \frac{7 \pm \sqrt{29}}{2}$$

$$\alpha + \beta = \frac{7 + \sqrt{29}}{2} + \frac{7 - \sqrt{29}}{2} = 7$$

$$\alpha\beta = \left(\frac{7 + \sqrt{29}}{2}\right)\left(\frac{7 - \sqrt{29}}{2}\right) = 5$$

$$(4) \quad ax^2 + bx + c = 0$$

$$\alpha + \beta = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$= -\frac{b}{a}$$

$$\alpha\beta = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)$$

$$= \frac{b^2 - b^2 + 4ac}{4a^2} = \frac{c}{a}$$

J 13 | b

From result of J 13 | a (4), the formulas for the relationships between the roots and coefficients of quadratic equations are:

Root-Coefficient Relationships

Given $ax^2 + bx + c = 0$ ($a \neq 0$), if the roots are α and β ,

$$\alpha + \beta = -\frac{b}{a} \quad \alpha\beta = \frac{c}{a}$$

2. If the roots of the quadratic equation are α and β , obtain their sum and product using the formulas above.

Ex.

$$x^2 - 3x + 9 = 0$$

$$\alpha + \beta = -\frac{-3}{1} = 3$$

$$\alpha\beta = \frac{9}{1} = 9$$

$$(2) \quad 3x^2 + 6x + 7 = 0$$

$$\alpha + \beta = -2$$

$$\alpha\beta = \frac{7}{3}$$

$$(1) \quad 3x^2 - 7x + 6 = 0$$

$$\alpha + \beta = \frac{7}{3}$$

$$\alpha\beta = 2$$

$$(3) \quad 7x^2 - 6x + 1 = 0$$

$$\alpha + \beta = \frac{6}{7}$$

$$\alpha\beta = \frac{1}{7}$$

Root-Coefficient Relationships

Root-Coefficient Relationships

Given $ax^2 + bx + c = 0$ ($a \neq 0$), if the roots are α and β ,

$$\alpha + \beta = -\frac{b}{a} \quad \alpha\beta = \frac{c}{a}$$

1. If the roots of the quadratic equation are α and β , obtain their sum and product using the formulas above.

$$(1) \quad 3x^2 + 7x - 6 = 0$$

$$\alpha + \beta = -\frac{7}{3}$$

$$\alpha\beta = -2$$

$$(4) \quad x^2 + ax + b = 0$$

$$\alpha + \beta = -a$$

$$\alpha\beta = b$$

$$(2) \quad x^2 + 3 = 0$$

$$\alpha + \beta = 0$$

$$\alpha\beta = 3$$

$$(5) \quad x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$\alpha + \beta = -\frac{b}{a}$$

$$\alpha\beta = \frac{c}{a}$$

$$(3) \quad 5x^2 - 3x = 0$$

$$\alpha + \beta = \frac{3}{5}$$

$$\alpha\beta = 0$$

$$(6) \quad x^2 - \frac{5}{6}x + \frac{1}{6} = 0$$

$$\alpha + \beta = \frac{5}{6}$$

$$\alpha\beta = \frac{1}{6}$$

2. Obtain each quadratic equation using the 2 given roots.

Ex.

$$\frac{1}{2}, \frac{1}{3}$$

[Sol] If the roots are α and β ,

$$\alpha + \beta = \frac{5}{6} \quad \alpha\beta = \frac{1}{6}$$

Therefore,

$$x^2 - \frac{5}{6}x + \frac{1}{6} = 0$$

$$6x^2 - 5x + 1 = 0$$

See question (6) on side a
— the same values are used.

Multiply both sides by 6.

(1) $-\frac{1}{4}, \frac{2}{3}$

If the roots are α and β ,

$$\alpha + \beta = \frac{5}{12}$$

$$\alpha\beta = -\frac{1}{6}$$

Therefore,

$$x^2 - \frac{5}{12}x - \frac{1}{6} = 0$$

$$12x^2 - 5x - 2 = 0$$

(2) $2, -\frac{2}{3}$

If the roots are α and β ,

$$\alpha + \beta = \frac{4}{3}$$

$$\alpha\beta = -\frac{4}{3}$$

Therefore,

$$x^2 - \frac{4}{3}x - \frac{4}{3} = 0$$

$$3x^2 - 4x - 4 = 0$$

(3) $3, -5$

If the roots are α and β ,

$$\alpha + \beta = -2$$

$$\alpha\beta = -15$$

Therefore,

$$x^2 + 2x - 15 = 0$$

(4) $-3, -8$

If the roots are α and β ,

$$\alpha + \beta = -11$$

$$\alpha\beta = 24$$

Therefore,

$$x^2 + 11x + 24 = 0$$

Root-Coefficient Relationships

Obtain each quadratic equation using the 2 given roots.

Ex.

$$1 + \sqrt{2}, 1 - \sqrt{2}$$

[Sol] If the roots are α and β ,

$$\alpha + \beta = 2$$

$$\alpha\beta = -1$$

Therefore,

$$x^2 - 2x - 1 = 0$$

$$(3) \quad 2 + \sqrt{3}i, 2 - \sqrt{3}i$$

$$\alpha + \beta = 4$$

$$\alpha\beta = 7$$

$$x^2 - 4x + 7 = 0$$

$$(1) \quad 3 + \sqrt{5}, 3 - \sqrt{5}$$

$$\alpha + \beta = 6$$

$$\alpha\beta = 4$$

$$x^2 - 6x + 4 = 0$$

$$(4) \quad 2 + 3i, 2 - 3i$$

$$\alpha + \beta = 4$$

$$\alpha\beta = 13$$

$$x^2 - 4x + 13 = 0$$

$$(2) \quad -\sqrt{3}, \sqrt{3}$$

$$\alpha + \beta = 0$$

$$\alpha\beta = -3$$

$$x^2 - 3 = 0$$

$$(5) \quad 1 + i, 1 - i$$

$$\alpha + \beta = 2$$

$$\alpha\beta = 2$$

$$x^2 - 2x + 2 = 0$$

J 133b

$$(6) \quad \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}$$

$$\alpha + \beta = 1$$

$$\alpha\beta = -1$$

$$\mathbf{x^2 - x - 1 = 0}$$

$$(8) \quad a + bi, a - bi$$

$$\alpha + \beta = 2a$$

$$\alpha\beta = a^2 + b^2$$

$$\mathbf{x^2 - 2ax + a^2 + b^2 = 0}$$

$$(7) \quad a, b$$

$$\alpha + \beta = a + b$$

$$\alpha\beta = ab$$

$$\mathbf{x^2 - (a+b)x + ab = 0}$$

$$(9) \quad 3a + 2\sqrt{5}b, 3a - 2\sqrt{5}b$$

$$\alpha + \beta = 6a$$

$$\alpha\beta = 9a^2 - 20b^2$$

$$\mathbf{x^2 - 6ax + 9a^2 - 20b^2 = 0}$$

Note: A quadratic equation with solutions $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$ can be written

$$\left(x - \frac{1+\sqrt{5}}{2}\right)\left(x - \frac{1-\sqrt{5}}{2}\right) = 0.$$

This can be expanded and simplified; however it is simpler to find the quadratic equation using the method shown.

Root-Coefficient Relationships

1. If the roots of $x^2 - 4x + 5 = 0$ are α and β , evaluate the following expressions.

Ex.

$$\alpha^2 + \beta^2$$

$$[\text{Sol}] \alpha + \beta = 4, \alpha\beta = 5$$

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = 4^2 - 2 \times 5 = 6$$

$$(1) (\alpha - \beta)^2$$

$$\begin{aligned} [\text{Sol}] (\alpha - \beta)^2 &= (\alpha + \beta)^2 - 4\alpha\beta \\ &= 4^2 - 4 \times 5 \\ &= -4 \end{aligned}$$

$$(2) \alpha^2\beta + \alpha\beta^2$$

$$\begin{aligned} [\text{Sol}] \alpha^2\beta + \alpha\beta^2 &= \alpha\beta(\alpha + \beta) \\ &= 5 \times 4 \\ &= \mathbf{20} \end{aligned}$$

$$\begin{aligned} (3) \alpha^2 + \alpha\beta + \beta^2 &= (\alpha + \beta)^2 - \alpha\beta \\ &= 16 - 5 \\ &= \mathbf{11} \end{aligned}$$

J 134b

2. If the roots of $3x^2 + 3x - 1 = 0$ are α and β , evaluate the following expressions.

$$(1) \quad \frac{1}{\alpha} + \frac{1}{\beta}$$

$$\alpha + \beta = -1, \quad \alpha\beta = -\frac{1}{3}$$

$$\begin{aligned} \frac{1}{\alpha} + \frac{1}{\beta} &= \frac{1}{\alpha\beta}(\alpha + \beta) \\ &= (-3) \times (-1) \\ &= \mathbf{3} \end{aligned}$$

$$(2) \quad \frac{\beta}{\alpha} + \frac{\alpha}{\beta} = \frac{1}{\alpha\beta}(\alpha^2 + \beta^2)$$

$$\begin{aligned} &= \frac{1}{\alpha\beta}[(\alpha + \beta)^2 - 2\alpha\beta] \\ &= -3 \times \left[(-1)^2 - 2 \times \left(-\frac{1}{3}\right) \right] \\ &= \mathbf{-5} \end{aligned}$$

$$\begin{aligned} (3) \quad \frac{1}{\alpha+1} + \frac{1}{\beta+1} &= \frac{(\beta+1) + (\alpha+1)}{(\alpha+1)(\beta+1)} \\ &= \frac{(\alpha+\beta) + 2}{\alpha\beta + (\alpha+\beta) + 1} \\ &= \frac{-1+2}{-\frac{1}{3} + (-1) + 1} \\ &= \mathbf{-3} \end{aligned}$$

J | 35a KUMON

Root-Coefficient Relationships

1. If the roots of $3x^2 + 3x - 1 = 0$ are α and β , evaluate the following expressions.

$$(1) \quad \alpha^2 + \beta^2$$

$$\alpha + \beta = -1, \quad \alpha\beta = -\frac{1}{3}$$

$$\begin{aligned} \alpha^2 + \beta^2 &= (\alpha + \beta)^2 - 2\alpha\beta \\ &= (-1)^2 - 2 \times \left(-\frac{1}{3}\right) \\ &= \frac{5}{3} \end{aligned}$$

$$\begin{aligned} (2) \quad \alpha^3 + \beta^3 &= (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta) \\ &= (-1)^3 - 3 \times \left(-\frac{1}{3}\right) \times (-1) \\ &= -2 \end{aligned}$$

Hint

$$\begin{aligned} (3) \quad \alpha^4 + \beta^4 &= (\alpha^2 + \beta^2)^2 - 2\alpha^2\beta^2 \\ &= (\alpha^2 + \beta^2)^2 - 2(\alpha\beta)^2 \\ &= \left(\frac{5}{3}\right)^2 - 2 \times \left(-\frac{1}{3}\right)^2 \\ &= \frac{23}{9} \end{aligned}$$

Hint**Hints**

$$(2) \quad a^3 + b^3 = (a + b)^3 - 3ab(a + b)$$

$$(3) \quad a^4 + b^4 = (a^2 + b^2)^2 - 2a^2b^2$$

J 135b

2. If the roots of $x^2 + 5x - 1 = 0$ are α and β , evaluate the following expressions.

$$(1) \quad \frac{1}{\alpha^2} + \frac{1}{\beta^2}$$

$$\alpha + \beta = -5, \quad \alpha\beta = -1$$

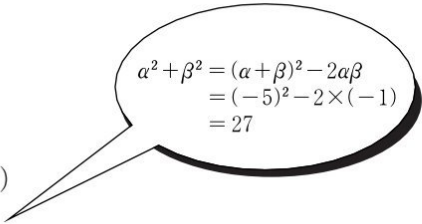
$$\begin{aligned} \frac{1}{\alpha^2} + \frac{1}{\beta^2} &= \frac{1}{\alpha^2\beta^2}(\alpha^2 + \beta^2) \\ &= \frac{1}{(\alpha\beta)^2}[(\alpha + \beta)^2 - 2\alpha\beta] \\ &= 1 \times [(-5)^2 - 2 \times (-1)] \\ &= \mathbf{27} \end{aligned}$$

$$(2) \quad \frac{1}{\alpha^3} + \frac{1}{\beta^3} = \frac{1}{\alpha^3\beta^3}(\alpha^3 + \beta^3)$$

$$\begin{aligned} &= \frac{1}{(\alpha\beta)^3}[(\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)] \\ &= (-1) \times [(-5)^3 - 3 \times (-1) \times (-5)] \\ &= \mathbf{140} \end{aligned}$$

$$(3) \quad \frac{1}{\alpha^4} + \frac{1}{\beta^4} = \frac{1}{\alpha^4\beta^4}(\alpha^4 + \beta^4)$$

$$\begin{aligned} &= \frac{1}{(\alpha\beta)^4}[(\alpha^2 + \beta^2)^2 - 2\alpha^2\beta^2] \\ &= 1 \times [27^2 - 2 \times (-1)^2] \\ &= \mathbf{727} \end{aligned}$$


$$\begin{aligned} \alpha^2 + \beta^2 &= (\alpha + \beta)^2 - 2\alpha\beta \\ &= (-5)^2 - 2 \times (-1) \\ &= 27 \end{aligned}$$

Root-Coefficient Relationships

Ex.

If the roots of $x^2 - 4x + 7 = 0$ are α and β , create a quadratic equation that has the roots α^2 and β^2 . (Note that the coefficient of x^2 is 1.)

[Sol] $\alpha + \beta = 4$

$\alpha\beta = 7$



From the root-coefficient relationships of the given quadratic equation.

$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$= 16 - 2 \cdot 7 = 2 \dots \textcircled{1}$

$\alpha^2\beta^2 = (\alpha\beta)^2 = 49 \dots \textcircled{2}$



From the relationships between the sum and product of α^2 and β^2 .

From $\textcircled{1}$ and $\textcircled{2}$,

$x^2 - 2x + 49 = 0$

1. If the roots of $x^2 - 4x + 7 = 0$ are α and β , create a quadratic equation that has the roots $\frac{1}{\alpha}$ and $\frac{1}{\beta}$. (Note that the coefficient of x^2 is 1.)

[Sol] $\alpha + \beta = 4$

$\alpha\beta = 7$

$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{1}{\alpha\beta}(\alpha + \beta)$

$= \frac{1}{7} \cdot 4 = \frac{4}{7} \dots \textcircled{1}$

$\frac{1}{\alpha} \cdot \frac{1}{\beta} = \frac{1}{\alpha\beta} = \frac{1}{7} \dots \textcircled{2}$

From $\textcircled{1}$ and $\textcircled{2}$,

$x^2 - \frac{4}{7}x + \frac{1}{7} = 0$

J | 36b

2. If the roots of $x^2 - x + 1 = 0$ are α and β , create a quadratic equation that has the roots $\frac{1}{\alpha+1}$ and $\frac{1}{\beta+1}$. (Note that the coefficient of x^2 is 1.)

[Sol] $\alpha + \beta = 1$

$$\alpha\beta = 1$$

$$\begin{aligned}\frac{1}{\alpha+1} + \frac{1}{\beta+1} &= \frac{(\beta+1) + (\alpha+1)}{(\alpha+1)(\beta+1)} \\ &= \frac{(\alpha+\beta) + 2}{\alpha\beta + (\alpha+\beta) + 1} \\ &= 1 \quad \dots \textcircled{1}\end{aligned}$$

$$\begin{aligned}\left(\frac{1}{\alpha+1}\right)\left(\frac{1}{\beta+1}\right) &= \frac{1}{\alpha\beta + (\alpha+\beta) + 1} \\ &= \frac{1}{3} \quad \dots \textcircled{2}\end{aligned}$$

From $\textcircled{1}$ and $\textcircled{2}$,

$$x^2 - x + \frac{1}{3} = 0$$

Root-Coefficient Relationships

1. Given that $x^2 - 3x + 1 = 0$ has 2 different roots, create a quadratic equation with roots 1 greater than the corresponding roots of the original equation.

[Sol] If the roots of $x^2 - 3x + 1 = 0$ are α and β ,

Hint

$$\alpha + \beta = 3$$

$$\alpha\beta = 1$$

Find the sum and product of the roots of the new quadratic equation.

Therefore,

$$(\alpha + 1) + (\beta + 1) = (\alpha + \beta) + 2 = 5 \quad \dots \textcircled{1}$$

$$(\alpha + 1)(\beta + 1) = \alpha\beta + (\alpha + \beta) + 1 = 5 \quad \dots \textcircled{2}$$

From $\textcircled{1}$ and $\textcircled{2}$,

$$x^2 - 5x + 5 = 0$$

Hint

If the roots of $x^2 - 3x + 1 = 0$ are α and β , the roots of the new equation will be $\alpha + 1$ and $\beta + 1$.

J | 37b

2. Given that $x^2 - 2x - 5 = 0$ has 2 different roots, and $x^2 + ax + b = 0$ has 2 roots, each 2 less than the corresponding root of the equation, find the values of a and b .

[Sol] If the roots of $x^2 - 2x - 5 = 0$ are α and β ,

Hint

$$\alpha + \beta = 2$$

$$\alpha\beta = -5$$

Find the sum and product of the roots of the second quadratic equation.

Therefore,

$$(\alpha - 2) + (\beta - 2) = (\alpha + \beta) - 4 = -2 \quad \dots \textcircled{1}$$

$$(\alpha - 2)(\beta - 2) = \alpha\beta - 2(\alpha + \beta) + 4 = -5 \quad \dots \textcircled{2}$$

From $\textcircled{1}$ and $\textcircled{2}$,

$$x^2 + 2x - 5 = 0$$

Since the second equation is $x^2 + ax + b = 0$,

$$\mathbf{a = 2 \text{ and } b = -5}$$

Hint

If the roots of $x^2 - 2x - 5 = 0$ are α and β , the roots of the second quadratic equation will be $\alpha - 2$ and $\beta - 2$.

Root-Coefficient Relationships

Ex.

Given that $x^2 - 6x + k = 0$ has roots α and α^2 , find the value of α , then find the value of k .

$$[\text{Sol}] \begin{cases} \alpha + \alpha^2 = 6 & \dots \textcircled{1} \\ \alpha \cdot \alpha^2 = \alpha^3 = k & \dots \textcircled{2} \end{cases}$$



From the Root-Coefficient Relationships of the given quadratic equation.

From $\textcircled{1}$,

$$\alpha^2 + \alpha - 6 = 0$$

$$(\alpha + 3)(\alpha - 2) = 0$$

Therefore, $\alpha = -3, 2$

Substituting into $\textcircled{2}$,

$$\text{When } \alpha = -3, k = -27$$

$$\text{When } \alpha = 2, k = 8$$

1. Given that $x^2 - 12x + k = 0$ has roots α and α^2 , find the value of α , then find the value of k .

$$[\text{Sol}] \begin{cases} \alpha + \alpha^2 = 12 & \dots \textcircled{1} \\ \alpha \cdot \alpha^2 = \alpha^3 = k & \dots \textcircled{2} \end{cases}$$

From $\textcircled{1}$,

$$\alpha^2 + \alpha - 12 = 0$$

$$(\alpha + 4)(\alpha - 3) = 0$$

$$\alpha = -4, 3$$

Substituting into $\textcircled{2}$,

$$\text{When } \alpha = -4, k = -64$$

$$\text{When } \alpha = 3, k = 27$$

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2. Given that $x^2 - 12x + k = 0$ has roots α and 2α , find the value of α , then find the value of k .

$$[\text{Sol}] \begin{cases} \alpha + 2\alpha = 12 & \dots \textcircled{1} \\ \alpha \cdot 2\alpha = 2\alpha^2 = k & \dots \textcircled{2} \end{cases}$$

From $\textcircled{1}$,

$$3\alpha = 12$$

$$\alpha = 4$$

Substituting into $\textcircled{2}$,

$$k = 32$$

3. Given that $x^2 + kx + 12 = 0$ has roots α and 2α , find the value of α , then find the value of k .

$$[\text{Sol}] \begin{cases} \alpha + 2\alpha = -k & \dots \textcircled{1} \\ \alpha \cdot 2\alpha = 12 & \dots \textcircled{2} \end{cases}$$

From $\textcircled{2}$,

$$\alpha^2 = 6$$

$$\alpha = \pm\sqrt{6}$$

Substituting into $\textcircled{1}$,

$$\text{When } \alpha = \sqrt{6}, \quad k = -3\sqrt{6}$$

$$\text{When } \alpha = -\sqrt{6}, \quad k = 3\sqrt{6}$$

Root-Coefficient Relationships

Ex.

Determine the range of k for which $x^2 - 6x + 3k = 0$ has 2 different positive real solutions.

[Sol] If $x^2 - 6x + 3k = 0$ have roots α and β .

$$\alpha + \beta = 6, \alpha\beta = 3k \quad \text{From the Root-Coefficient Relationships.}$$

From $\alpha\beta = 3k > 0$, $\text{If } \alpha \text{ and } \beta \text{ are both positive, then } \alpha\beta \text{ must be positive.}$

$$k > 0 \quad \dots \textcircled{1}$$

And, from $\frac{D}{4} = 9 - 3k > 0$, $\text{In order to have 2 different real number solutions, } D > 0.$

$$k < 3 \quad \dots \textcircled{2}$$

From $\textcircled{1}$ and $\textcircled{2}$,

$$0 < k < 3$$

1. Determine the range of k for which $3x^2 + 8x + 2k = 0$ has 2 different negative real solutions.

[Sol] If $3x^2 + 8x + 2k = 0$ have roots α and β .

$$\alpha + \beta = -\frac{8}{3}, \alpha\beta = \frac{2}{3}k$$

From $\alpha\beta = \frac{2}{3}k > 0$, $\text{If } \alpha \text{ and } \beta \text{ are both negative, then } \alpha\beta \text{ must be positive.}$

$$k > 0 \quad \dots \textcircled{1}$$

And, from $\frac{D}{4} = 16 - 6k > 0$,

$$k < \frac{8}{3} \dots \textcircled{2}$$

From $\textcircled{1}$ and $\textcircled{2}$,

$$0 < k < \frac{8}{3}$$

2. Given that $x^2 - 2x - m + 1 = 0$ has 2 positive real number roots, α and β , determine the range of values of m .

[Sol] $\alpha + \beta = 2$, $\alpha\beta = -m + 1$

From $\alpha\beta = -m + 1 > 0$,

$m < 1 \quad \dots \textcircled{1}$

And, from $\frac{D}{4} = 1 - (-m + 1) > 0$,

$m > 0 \quad \dots \textcircled{2}$


From $\textcircled{1}$ and $\textcircled{2}$,

$0 < m < 1$

Consider this!

- i) Given that $x^2 - 6x + 3k = 0$ has 2 real number roots, α and β , with opposite signs (one root is positive and the other root is negative), determine the range of values of k .

[Sol] $\alpha + \beta = 6$, $\alpha\beta = 3k$

From $\alpha\beta = 3k < \boxed{0}$, 

A positive number times a negative number is negative.

$k < \boxed{0} \quad \dots \textcircled{1}$

And, from $\frac{D}{4} = 9 - 3k > 0$,

$k < 3 \quad \dots \textcircled{2}$

From $\textcircled{1}$ and $\textcircled{2}$,

$k < \boxed{0}$

- ii) In order for $ax^2 + bx + c = 0$ to have 2 different real number roots, α and β , with opposite signs,

$$\begin{cases} \alpha\beta = \frac{c}{a} < 0 \\ D = b^2 - 4ac > 0 \end{cases}$$

- If $\frac{c}{a} < 0$, then $ac < 0$, so the discriminant is always positive.
- Therefore we only need the condition $\alpha\beta < 0$.

J140a KUMON

Root-Coefficient Relationships

1. Find the quadratic equation with the two given roots.

$$(1) \quad \frac{2+\sqrt{5}}{3}, \quad \frac{2-\sqrt{5}}{3}$$

$$\alpha + \beta = \frac{4}{3}$$

$$\alpha\beta = -\frac{1}{9}$$

Therefore,

$$x^2 - \frac{4}{3}x - \frac{1}{9} = 0$$

$$\mathbf{9x^2 - 12x - 1 = 0} \quad \curvearrowright \text{Multiply by 9 to eliminate the denominator.}$$

$$(2) \quad \frac{-1+\sqrt{3}i}{2}, \quad \frac{-1-\sqrt{3}i}{2}$$

$$\alpha + \beta = -1$$

$$\alpha\beta = 1$$

Therefore,

$$\mathbf{x^2 + x + 1 = 0}$$

2. Given that $x^2 + 5x - 1 = 0$ has 2 roots α and β , evaluate the following expressions.

$$(1) \quad \alpha^2 - \alpha\beta + \beta^2$$

$$\alpha + \beta = -5$$

$$\alpha\beta = -1$$

$$\begin{aligned} \alpha^2 - \alpha\beta + \beta^2 &= \alpha^2 + \beta^2 - \alpha\beta \\ &= (\alpha + \beta)^2 - 3\alpha\beta \\ &= 25 - 3 \times (-1) = \mathbf{28} \end{aligned}$$

$$\begin{aligned} (2) \quad (\alpha - 2\beta)(2\alpha - \beta) &= 2\alpha^2 - 5\alpha\beta + 2\beta^2 \\ &= 2(\alpha + \beta)^2 - 9\alpha\beta \\ &= 2 \times 25 - 9 \times (-1) = \mathbf{59} \end{aligned}$$

J 140b

3. Given that $x^2 + 2mx + m^2 - 2m + 3 = 0$ has 2 roots α and β (with $\alpha < \beta$),

(1) Find $\alpha + \beta$ and $\alpha\beta$ in terms of m .

$$\alpha + \beta = -2m$$

$$\alpha\beta = m^2 - 2m + 3$$

(2) Find $(\alpha - \beta)^2$ in terms of m .

$$\begin{aligned}(\alpha - \beta)^2 &= (\alpha + \beta)^2 - 4\alpha\beta \\ &= 4m^2 - 4(m^2 - 2m + 3) \\ &= \mathbf{8m - 12}\end{aligned}$$

(3) If $\beta - \alpha = 2$, find m .

From question (2),

$$(\alpha - \beta)^2 = 8m - 12 \dots \textcircled{1}$$

We can also write the equation:

$$(\alpha - \beta)^2 = (\beta - \alpha)^2 \dots \textcircled{2}$$

From $\textcircled{1}$ and $\textcircled{2}$,

$$8m - 12 = (\beta - \alpha)^2$$

$$8m - 12 = 2^2$$

$$\mathbf{m = 2}$$

Simultaneous Equations

Solve the following simultaneous equations.

Ex.

$$\begin{cases} y = 2x - 1 & \dots\dots ① \\ x^2 + y^2 = 29 & \dots\dots ② \end{cases}$$

[Sol] Substituting ① into ②,

$$x^2 + (2x - 1)^2 = 29$$

$$5x^2 - 4x - 28 = 0$$

$$(5x - 14)(x + 2) = 0$$

Therefore,

$$x = \frac{14}{5}, -2$$

Substituting $x = \frac{14}{5}$ into ①,

$$y = \frac{23}{5}$$

Substituting $x = -2$ into ①,

$$y = -5$$

$$\text{Ans. } \begin{cases} x = \frac{14}{5} \\ y = \frac{23}{5} \end{cases} \quad \begin{cases} x = -2 \\ y = -5 \end{cases}$$

$$(1) \quad \begin{cases} y = 2x - 1 & \dots\dots ① \\ x^2 + y^2 = 13 & \dots\dots ② \end{cases}$$

Substituting ① into ②,

$$x^2 + (2x - 1)^2 = 13$$

$$5x^2 - 4x - 12 = 0$$

$$(5x + 6)(x - 2) = 0$$

Therefore,

$$x = -\frac{6}{5}, 2$$

Note: Solving a group of simultaneous equations is often called solving a “system of equations”.

Substituting $x = -\frac{6}{5}$ into ①,

$$y = -\frac{17}{5}$$

Substituting $x = 2$ into ①,

$$y = 3$$

$$\text{Ans. } \begin{cases} x = -\frac{6}{5} \\ y = -\frac{17}{5} \end{cases} \quad \begin{cases} x = 2 \\ y = 3 \end{cases}$$

$$(2) \begin{cases} y = x + 3 & \dots\dots ① \\ x^2 + y^2 = 5 & \dots\dots ② \end{cases}$$

Substituting ① into ②,

$$x^2 + (x + 3)^2 = 5$$

$$2x^2 + 6x + 4 = 0$$

$$x^2 + 3x + 2 = 0$$

$$(x + 2)(x + 1) = 0$$

Therefore,

$$x = -2, -1$$

Substituting $x = -2$ into ①,

$$y = 1$$

Substituting $x = -1$ into ①,

$$y = 2$$

$$\text{Ans. } \begin{cases} x = -2 \\ y = 1 \end{cases} \quad \begin{cases} x = -1 \\ y = 2 \end{cases}$$

$$(3) \begin{cases} x = y + 1 & \dots\dots ① \\ x^2 + y^2 = 5 & \dots\dots ② \end{cases}$$

Substituting ① into ②,

$$(y + 1)^2 + y^2 = 5$$

$$2y^2 + 2y - 4 = 0$$

$$y^2 + y - 2 = 0$$

$$(y + 2)(y - 1) = 0$$

Therefore,

$$y = -2, 1$$

Substituting $y = -2$ into ①,

$$x = -1$$

Substituting $y = 1$ into ①,

$$x = 2$$

$$\text{Ans. } \begin{cases} x = -1 \\ y = -2 \end{cases} \quad \begin{cases} x = 2 \\ y = 1 \end{cases}$$

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Simultaneous Equations

Solve the following simultaneous equations.

$$(1) \begin{cases} y = 2x + 1 & \dots\dots(1) \\ x^2 - y^2 = -21 & \dots\dots(2) \end{cases}$$

Substituting (1) into (2),

$$x^2 - (2x + 1)^2 = -21$$

$$3x^2 + 4x - 20 = 0$$

$$(3x + 10)(x - 2) = 0$$

Therefore,

$$x = -\frac{10}{3}, 2$$

Substituting $x = -\frac{10}{3}$ into (1),

$$y = -\frac{17}{3}$$

Substituting $x = 2$ into (1),

$$y = 5$$

$$\text{Ans.} \begin{cases} x = -\frac{10}{3} \\ y = -\frac{17}{3} \end{cases} \quad \begin{cases} x = 2 \\ y = 5 \end{cases}$$

$$(2) \begin{cases} y = 2x - 1 & \dots\dots(1) \\ x^2 - 2y^2 = -1 & \dots\dots(2) \end{cases}$$

Substituting (1) into (2),

$$x^2 - 2(2x - 1)^2 = -1$$

$$7x^2 - 8x + 1 = 0$$

$$(7x - 1)(x - 1) = 0$$

Therefore,

$$x = \frac{1}{7}, 1$$

Substituting $x = \frac{1}{7}$ into (1),

$$y = -\frac{5}{7}$$

Substituting $x = 1$ into (1),

$$y = 1$$

$$\text{Ans.} \begin{cases} x = \frac{1}{7} \\ y = -\frac{5}{7} \end{cases} \quad \begin{cases} x = 1 \\ y = 1 \end{cases}$$

J 142b

$$(3) \begin{cases} y = x - 3 & \dots\dots ① \\ x^2 - y^2 = 15 & \dots\dots ② \end{cases}$$

Substituting ① into ②,

$$x^2 - (x - 3)^2 = 15$$

$$6x = 24$$

$$x = 4$$

Substituting $x = 4$ into ①,

$$y = 1$$

There is only one solution.

$$\text{Ans. } \begin{cases} x = 4 \\ y = 1 \end{cases}$$

$$(4) \begin{cases} 2x - y = 3 & \dots\dots ① \\ x^2 - y^2 = 3 & \dots\dots ② \end{cases}$$

After rearranging ①,

$$y = 2x - 3 \dots\dots ③$$

Substituting ③ into ②,

$$x^2 - (2x - 3)^2 = 3$$

$$x^2 - 4x + 4 = 0$$

$$(x - 2)^2 = 0$$

$$x = 2$$

Substituting $x = 2$ into ③,

$$y = 1$$

$$\text{Ans. } \begin{cases} x = 2 \\ y = 1 \end{cases}$$

J143a KUMON

Simultaneous Equations

Solve the following simultaneous equations.

$$(1) \begin{cases} 3x = 2y & \dots\dots(1) \\ \frac{x^2}{4} + \frac{y^2}{9} = 8 & \dots\dots(2) \end{cases}$$

From (1),

$$x = \frac{2}{3}y \quad \dots\dots(3)$$

Substituting (3) into (2),

$$\frac{\left(\frac{2}{3}y\right)^2}{4} + \frac{y^2}{9} = 8$$

$$\frac{1}{4} \times \frac{4}{9}y^2 + \frac{y^2}{9} = 8$$

$$y^2 = 36$$

$$y = \pm 6$$

Substituting $y = 6$ into (3),

$$x = 4$$

Substituting $y = -6$ into (3),

$$x = -4$$

$$\text{Ans. } \begin{cases} x = \pm \boxed{4} \\ y = \pm \boxed{6} \end{cases}$$

Solutions with the same numbers but different signs can be written as follows:

$$\begin{cases} x = 1 \\ y = 2 \end{cases} \quad \text{and} \quad \begin{cases} x = -1 \\ y = -2 \end{cases} \quad \text{can be written as} \quad \begin{cases} x = \pm 1 \\ y = \pm 2 \end{cases}$$

$$\begin{cases} x = 1 \\ y = -2 \end{cases} \quad \text{and} \quad \begin{cases} x = -1 \\ y = 2 \end{cases} \quad \text{can be written as} \quad \begin{cases} x = \pm 1 \\ y = \mp 2 \end{cases}$$

Keep the positive and negative signs in the correct order.

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$$(2) \begin{cases} y = x + 1 & \dots\dots ① \\ x^2 + y^2 = 3 & \dots\dots ② \end{cases}$$

The solutions are irrational numbers.

Substituting ① into ②,

$$x^2 + (x + 1)^2 = 3$$

$$2x^2 + 2x - 2 = 0$$

$$x^2 + x - 1 = 0$$

Therefore,

$$x = \frac{-1 \pm \sqrt{5}}{2}$$

Substituting $x = \frac{-1 + \sqrt{5}}{2}$ into ①,

$$y = \frac{1 + \sqrt{5}}{2}$$

Substituting $x = \frac{-1 - \sqrt{5}}{2}$ into ①,

$$y = \frac{1 - \sqrt{5}}{2}$$

$$\text{Ans.} \begin{cases} x = \frac{-1 \pm \sqrt{5}}{2} \\ y = \frac{1 \pm \sqrt{5}}{2} \end{cases}$$

$$(3) \begin{cases} 4x - 2y = 2 & \dots\dots ① \\ x^2 - y^2 + 3 = 0 & \dots\dots ② \end{cases}$$

The solutions are irrational numbers.

From ①,

$$y = 2x - 1 \quad \dots\dots ③$$

Substituting ③ into ②,

$$x^2 - (2x - 1)^2 + 3 = 0$$

$$3x^2 - 4x - 2 = 0$$

Therefore,

$$x = \frac{2 \pm \sqrt{10}}{3}$$

Substituting $x = \frac{2 + \sqrt{10}}{3}$ into ③,

$$y = \frac{1 + 2\sqrt{10}}{3}$$

Substituting $x = \frac{2 - \sqrt{10}}{3}$ into ③,

$$y = \frac{1 - 2\sqrt{10}}{3}$$

$$\text{Ans.} \begin{cases} x = \frac{2 \pm \sqrt{10}}{3} \\ y = \frac{1 \pm 2\sqrt{10}}{3} \end{cases}$$

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Simultaneous Equations

Solve the following simultaneous equations.

Ex.

$$\begin{cases} xy + x - y = 1 & \dots\dots ① \\ xy - 3x + y = 5 & \dots\dots ② \end{cases}$$

[Sol] From ① - ②,

$$4x - 2y = -4$$

Therefore, $y = 2x + 2$ ③

Substituting ③ into ①,

$$x(2x + 2) + x - (2x + 2) = 1$$

$$2x^2 + x - 3 = 0$$

$$(2x + 3)(x - 1) = 0$$

Therefore,

$$x = -\frac{3}{2}, 1$$

Substituting $x = -\frac{3}{2}$ into ③,

$$y = -1$$

Substituting $x = 1$ into ③,

$$y = 4$$

$$\text{Ans. } \begin{cases} x = -\frac{3}{2} \\ y = -1 \end{cases} \quad \begin{cases} x = 1 \\ y = 4 \end{cases}$$

$$(1) \begin{cases} xy + 3x - 3y + 3 = 0 & \dots\dots ① \\ xy - x - y + 1 = 0 & \dots\dots ② \end{cases}$$

From ① - ②,

$$4x - 2y + 2 = 0$$

Therefore, $y = 2x + 1$ ③

Substituting ③ into ①,

$$x(2x + 1) + 3x - 3(2x + 1) + 3 = 0$$

$$2x^2 + x + 3x - 6x - 3 + 3 = 0$$

$$2x^2 - 2x = 0$$

$$x(x - 1) = 0$$

Therefore,

$$x = 0, 1$$

Substituting $x = 0$ into ③,

$$y = 1$$

Substituting $x = 1$ into ③,

$$y = 3$$

$$\text{Ans. } \begin{cases} x = 0 \\ y = 1 \end{cases} \quad \begin{cases} x = 1 \\ y = 3 \end{cases}$$

J 144b

$$(2) \begin{cases} xy + x + y = 5 & \dots\dots(1) \\ 2xy + x + 3y = 9 & \dots\dots(2) \end{cases}$$

From (1) $\times 2$,

$$2xy + 2x + 2y = 10 \dots\dots(3)$$

$$(3) - (2) : x - y = 1$$

Therefore, $y = x - 1 \dots\dots(4)$

Substituting (4) into (1),

$$x(x-1) + x + (x-1) = 5$$

$$x^2 + x - 6 = 0$$

$$(x+3)(x-2) = 0$$

Therefore,

$$x = -3, 2$$

Substituting $x = -3$ into (4),

$$y = -4$$

Substituting $x = 2$ into (4),

$$y = 1$$

$$\text{Ans. } \begin{cases} x = -3 \\ y = -4 \end{cases} \quad \begin{cases} x = 2 \\ y = 1 \end{cases}$$

$$(3) \begin{cases} xy + 2x + 2y + 1 = 0 & \dots\dots(1) \\ 3xy + 3x + 5y + 3 = 0 & \dots\dots(2) \end{cases}$$

From (1) $\times 3$,

$$3xy + 6x + 6y + 3 = 0 \dots\dots(3)$$

$$(3) - (2) : 3x + y = 0$$

Therefore, $y = -3x \dots\dots(4)$

Substituting (4) into (1),

$$x(-3x) + 2x + 2(-3x) + 1 = 0$$

$$-3x^2 - 4x + 1 = 0$$

$$3x^2 + 4x - 1 = 0$$

Therefore,

$$x = \frac{-2 \pm \sqrt{7}}{3}$$

Substituting $x = \frac{-2 + \sqrt{7}}{3}$ into (4),

$$y = 2 - \sqrt{7}$$

Substituting $x = \frac{-2 - \sqrt{7}}{3}$ into (4),

$$y = 2 + \sqrt{7}$$

$$\text{Ans. } \begin{cases} x = \frac{-2 \pm \sqrt{7}}{3} \\ y = 2 \mp \sqrt{7} \end{cases}$$

J145a KUMON

Simultaneous Equations

Solve the following equations.

Ex.

$$\begin{cases} x^2 + y^2 - 2x + 2y = 7 & \dots\dots ① \\ x^2 + y^2 + 4x - 4y = 1 & \dots\dots ② \end{cases}$$

[Sol] From ① - ②,

$$-6x + 6y = 6$$

Therefore, $x = y - 1$ ③

Substituting ③ into ①,

$$(y-1)^2 + y^2 - 2(y-1) + 2y = 7$$

$$2y^2 - 2y - 4 = 0$$

$$(y-2)(y+1) = 0$$

Therefore, $y = 2, -1$

Substituting $y = 2$ into ③,

$$x = 1$$

Substituting $y = -1$ into ③,

$$x = -2$$

$$\text{Ans. } \begin{cases} x = 1 \\ y = 2 \end{cases} \quad \begin{cases} x = -2 \\ y = -1 \end{cases}$$

$$(1) \begin{cases} x^2 + y^2 - 4x - 4y - 10 = 0 & \dots\dots ① \\ x^2 + y^2 + 4x + 4y - 10 = 0 & \dots\dots ② \end{cases}$$

From ① - ②,

$$-8x - 8y = 0$$

Therefore, $x = -y$ ③

Substituting ③ into ①,

$$y^2 + y^2 + 4y - 4y - 10 = 0$$

$$y^2 = 5$$

Therefore, $y = \pm\sqrt{5}$

Substituting $y = \sqrt{5}$ into ③,

$$x = -\sqrt{5}$$

Substituting $y = -\sqrt{5}$ into ③,

$$x = \sqrt{5}$$

$$\text{Ans. } \begin{cases} x = \pm\sqrt{5} \\ y = \mp\sqrt{5} \end{cases}$$

J 145b

$$(2) \begin{cases} x^2 + y^2 + x - y - 6 = 0 & \dots\dots(1) \\ 2x^2 + 2y^2 + 3x + y - 12 = 0 & \dots\dots(2) \end{cases}$$

From (1) $\times 2$,

$$2x^2 + 2y^2 + 2x - 2y - 12 = 0 \dots\dots(3)$$

$$(3) - (2): -x - 3y = 0$$

$$x = -3y \dots\dots(4)$$

Substituting (4) into (1),

$$9y^2 + y^2 - 3y - y - 6 = 0$$

$$10y^2 - 4y - 6 = 0$$

$$(5y + 3)(y - 1) = 0$$

Therefore, $y = -\frac{3}{5}, 1$

Substituting $y = -\frac{3}{5}$ into (4),

$$x = \frac{9}{5}$$

Substituting $y = 1$ into (4),

$$x = -3$$

$$\text{Ans. } \begin{cases} x = \frac{9}{5} \\ y = -\frac{3}{5} \end{cases} \quad \begin{cases} x = -3 \\ y = 1 \end{cases}$$

$$(3) \begin{cases} 3x^2 + 2y^2 - x + 4y - 8 = 0 & \dots\dots(1) \\ 6x^2 + 4y^2 + 4x + 11y - 25 = 0 & \dots\dots(2) \end{cases}$$

From (1) $\times 2$,

$$6x^2 + 4y^2 - 2x + 8y - 16 = 0 \dots\dots(3)$$

$$(3) - (2): -6x - 3y + 9 = 0$$

$$y = -2x + 3 \dots\dots(4)$$

Substituting (4) into (1),

$$3x^2 + 2(-2x + 3)^2 - x + 4(-2x + 3) - 8 = 0$$

$$11x^2 - 33x + 22 = 0$$

$$(x - 2)(x - 1) = 0$$

Therefore, $x = 2, 1$

Substituting $x = 2$ into (4),

$$y = -1$$

Substituting $x = 1$ into (4),

$$y = 1$$

$$\text{Ans. } \begin{cases} x = 2 \\ y = -1 \end{cases} \quad \begin{cases} x = 1 \\ y = 1 \end{cases}$$

J146a KUMON

Simultaneous Equations

Solve the following equations.

Ex.

$$\begin{cases} (x-y)(x+2y) = 0 & \dots\dots ① \\ x^2 - xy + 2y^2 = 16 & \dots\dots ② \end{cases}$$

[Sol] From ①, $x = y$ or $x = -2y$

$$\begin{cases} x = y & \dots\dots ③ \\ x^2 - xy + 2y^2 = 16 & \dots\dots ② \end{cases}$$

Substituting ③ into ②,

$$y^2 - y^2 + 2y^2 = 16$$

$$y^2 = 8$$

$$y = \pm 2\sqrt{2}$$

From ③, $x = \pm 2\sqrt{2}$

$$\begin{cases} x = -2y & \dots\dots ③' \\ x^2 - xy + 2y^2 = 16 & \dots\dots ② \end{cases}$$

Substituting ③' into ②,

$$4y^2 + 2y^2 + 2y^2 = 16$$

$$y^2 = 2$$

$$y = \pm\sqrt{2}$$

From ③', $x = \mp 2\sqrt{2}$

$$\text{Ans. } \begin{cases} x = \pm 2\sqrt{2} \\ y = \pm 2\sqrt{2} \end{cases} \quad \begin{cases} x = \pm 2\sqrt{2} \\ y = \mp\sqrt{2} \end{cases}$$

$$(1) \begin{cases} (x-3y)(x-2y) = 0 & \dots\dots ① \\ x^2 - 2xy + 6y^2 = 6 & \dots\dots ② \end{cases}$$

From ①, $x = 3y$ or $x = 2y$

$$\begin{cases} x = 3y & \dots\dots ③ \\ x^2 - 2xy + 6y^2 = 6 & \dots\dots ② \end{cases}$$

Substituting ③ into ②,

$$9y^2 - 6y^2 + 6y^2 = 6$$

$$y = \pm \frac{\sqrt{6}}{3}$$

From ③, $x = \pm\sqrt{6}$

$$\begin{cases} x = 2y & \dots\dots ③' \\ x^2 - 2xy + 6y^2 = 6 & \dots\dots ② \end{cases}$$

Substituting ③' into ②,

$$4y^2 - 4y^2 + 6y^2 = 6$$

$$y = \pm 1$$

From ③', $x = \pm 2$

$$\text{Ans. } \begin{cases} x = \pm\sqrt{6} \\ y = \pm \frac{\sqrt{6}}{3} \end{cases} \quad \begin{cases} x = \pm 2 \\ y = \pm 1 \end{cases}$$

Ex.

$$\begin{cases} x^2 - 4xy + 3y^2 = 0 & \dots\dots ① \\ x^2 - 5xy + 6y^2 = 2 & \dots\dots ② \end{cases}$$

[Sol] From ①, $(x - y)(x - 3y) = 0$ therefore $x = y$ or $x = 3y$

$$\begin{cases} x = y & \dots ③ \\ x^2 - 5xy + 6y^2 = 2 & \dots ② \end{cases}$$

Substituting ③ into ②,

$$\begin{aligned} y^2 - 5y^2 + 6y^2 &= 2 \\ y &= \pm 1 \end{aligned}$$

From ③, $x = \pm 1$

$$\begin{cases} x = 3y & \dots ③' \\ x^2 - 5xy + 6y^2 = 2 & \dots ② \end{cases}$$

Substituting ③' into ②,

$$\begin{aligned} 9y^2 - 15y^2 + 6y^2 &= 2 \\ 0 \cdot y^2 &= 2 \end{aligned}$$

There is no value of y that would satisfy this. Therefore, there is no solution.

$$\text{Ans. } \begin{cases} x = \pm 1 \\ y = \pm 1 \end{cases}$$

$$(2) \begin{cases} x^2 + 3xy + 2y^2 = 0 & \dots\dots ① \\ x^2 - 2xy - 3y^2 = 5 & \dots\dots ② \end{cases}$$

From ①, $(x + 2y)(x + y) = 0$ therefore $x = -2y$ or $x = -y$

$$\begin{cases} x = -2y & \dots ③ \\ x^2 - 2xy - 3y^2 = 5 & \dots ② \end{cases}$$

Substituting ③ into ②,

$$\begin{aligned} 4y^2 + 4y^2 - 3y^2 &= 5 \\ y &= \pm 1 \end{aligned}$$

From ③, $x = \mp 2$

$$\begin{cases} x = -y & \dots ③' \\ x^2 - 2xy - 3y^2 = 5 & \dots ② \end{cases}$$

Substituting ③' into ②,

$$\begin{aligned} y^2 + 2y^2 - 3y^2 &= 5 \\ 0 \cdot y^2 &= 5 \end{aligned}$$

There is no value of y that would satisfy this. Therefore, there is no solution.

$$\text{Ans. } \begin{cases} x = \pm 2 \\ y = \mp 1 \end{cases}$$

J147a KUMON

Simultaneous Equations


Solve the following equations.

Ex.

$$\begin{cases} x + y = 9 & \dots\dots ① \\ xy = 7 & \dots\dots ② \end{cases}$$

[Sol] x and y are the roots of

$$t^2 - 9t + 7 = 0$$

Therefore, $t = \frac{9 \pm \sqrt{53}}{2}$ 

When a quadratic equation has roots α and β , we can write the equation as

$$t^2 - (\alpha + \beta)t + \alpha\beta = 0$$

In this case, $\alpha = x$ and $\beta = y$.

$$\text{Ans.} \begin{cases} x = \frac{9 + \sqrt{53}}{2} \\ y = \frac{9 - \sqrt{53}}{2} \end{cases} \quad \begin{cases} x = \frac{9 - \sqrt{53}}{2} \\ y = \frac{9 + \sqrt{53}}{2} \end{cases}$$

$$(1) \begin{cases} x + y = 4 \\ xy = 2 \end{cases}$$

x and y are the roots of

$$t^2 - 4t + 2 = 0$$

Therefore, $t = 2 \pm \sqrt{2}$

$$\text{Ans.} \begin{cases} x = 2 + \sqrt{2} \\ y = 2 - \sqrt{2} \end{cases} \quad \begin{cases} x = 2 - \sqrt{2} \\ y = 2 + \sqrt{2} \end{cases}$$

Alternative method for example: From ①, $y = 9 - x$. We can substitute this into ②, and solve as usual. However, with terms such as $x + y$ and xy , it is easier to use the **Root-Coefficient Relationships**.

J 147b

$$(2) \begin{cases} x + y = 5 \\ xy = -6 \end{cases}$$

x and y are the roots of

$$t^2 - 5t - 6 = 0$$

$$(t - 6)(t + 1) = 0$$

Therefore,

$$t = 6, -1$$

$$\text{Ans. } \begin{cases} \mathbf{x = 6} \\ \mathbf{y = -1} \end{cases} \quad \begin{cases} \mathbf{x = -1} \\ \mathbf{y = 6} \end{cases}$$

$$(3) \begin{cases} x + y = \sqrt{3} + 1 \\ xy = \sqrt{3} \end{cases}$$

x and y are the roots of

$$t^2 - (\sqrt{3} + 1)t + \sqrt{3} = 0$$

$$(t - 1)(t - \sqrt{3}) = 0$$

Therefore,

$$t = 1, \sqrt{3}$$

$$\text{Ans. } \begin{cases} \mathbf{x = 1} \\ \mathbf{y = \sqrt{3}} \end{cases} \quad \begin{cases} \mathbf{x = \sqrt{3}} \\ \mathbf{y = 1} \end{cases}$$

J148a KUMON

Simultaneous Equations

Solve the following equations.

Ex.

$$\begin{cases} x + y = 4 & \dots\dots(1) \\ x^2 + xy + y^2 = 13 & \dots\dots(2) \end{cases}$$

[Sol] From (2),

$$(x + y)^2 - xy = 13 \dots\dots(3)$$

Substituting (1) into (3),

$$16 - xy = 13$$

$$xy = 3 \dots\dots(4)$$

From (1) and (4), x and y are the roots of

$$t^2 - 4t + 3 = 0$$

$$(t - 3)(t - 1) = 0$$

Therefore, $t = 3, 1$

$$\text{Ans. } \begin{cases} x = 3 \\ y = 1 \end{cases} \quad \begin{cases} x = 1 \\ y = 3 \end{cases}$$

$$(1) \begin{cases} x + y = 4 & \dots\dots(1) \\ x^2 - xy + y^2 = 22 & \dots\dots(2) \end{cases}$$

From (2),

$$(x + y)^2 - 3xy = 22 \dots\dots(3)$$

Substituting (1) into (3),

$$16 - 3xy = 22$$

$$xy = -2 \dots\dots(4)$$

From (1) and (4), x and y are the roots of

$$t^2 - 4t - 2 = 0$$

Therefore, $t = 2 \pm \sqrt{6}$

$$\text{Ans. } \begin{cases} x = 2 + \sqrt{6} \\ y = 2 - \sqrt{6} \end{cases} \quad \begin{cases} x = 2 - \sqrt{6} \\ y = 2 + \sqrt{6} \end{cases}$$

$$(2) \begin{cases} xy = 3 & \dots\dots ① \\ x^2 + y^2 = 10 & \dots\dots ② \end{cases}$$

There are 4 solutions.

[Sol] From ②,

$$(x+y)^2 - 2xy = 10 \dots\dots ③$$

Substituting ① into ③,

$$(x+y)^2 - 6 = 10$$

$$(x+y)^2 = 16$$

$$x+y = \pm 4 \dots\dots ④$$

From ① and ④, x and y are the roots of

$$t^2 - 4t + 3 = 0$$

$$t^2 + 4t + 3 = 0$$

$$(t-3)(t-1) = 0$$

$$(t+3)(t+1) = 0$$

Therefore, $t = 3, 1$

Therefore, $t = -3, -1$

$$\text{Ans. } \begin{cases} \mathbf{x = 3} \\ \mathbf{y = 1} \end{cases} \quad \begin{cases} \mathbf{x = 1} \\ \mathbf{y = 3} \end{cases} \quad \begin{cases} \mathbf{x = -3} \\ \mathbf{y = -1} \end{cases} \quad \begin{cases} \mathbf{x = -1} \\ \mathbf{y = -3} \end{cases}$$

$$(3) \begin{cases} xy = 2 & \dots\dots ① \\ x^2 + y^2 = 4 & \dots\dots ② \end{cases}$$

From ②,

$$(x+y)^2 - 2xy = 4 \dots\dots ③$$

Substituting ① into ③,

$$(x+y)^2 - 4 = 4$$

$$(x+y)^2 = 8$$

$$x+y = \pm 2\sqrt{2} \dots\dots ④$$

From ① and ④, x and y are the roots of

$$t^2 - 2\sqrt{2}t + 2 = 0$$

$$t^2 + 2\sqrt{2}t + 2 = 0$$

$$(t-\sqrt{2})^2 = 0$$

$$(t+\sqrt{2})^2 = 0$$

Therefore, $t = \sqrt{2}$

Therefore, $t = -\sqrt{2}$

$$\text{Ans. } \begin{cases} \mathbf{x = \sqrt{2}} \\ \mathbf{y = \sqrt{2}} \end{cases} \quad \begin{cases} \mathbf{x = -\sqrt{2}} \\ \mathbf{y = -\sqrt{2}} \end{cases}$$

Simultaneous Equations

Solve the following equations.

$$(1) \begin{cases} x^2 + 9xy + y^2 = 23 & \dots\dots(1) \\ xy + x + y = 5 & \dots\dots(2) \end{cases}$$

[Sol] Assuming that $x + y = A$, $xy = B$.

From (1) and (2),

$$\begin{cases} A^2 + 7B = 23 & \dots\dots(3) \\ A + B = 5 & \dots\dots(4) \end{cases}$$

$$\begin{aligned} x^2 + 9xy + y^2 & \\ = x^2 + 2xy + y^2 + 7xy & \\ = A^2 + 7B & \end{aligned}$$

From (4), $B = 5 - A$

Substituting this into (3),

$$A^2 + 7(5 - A) = 23$$

$$A^2 - 7A + 12 = 0$$

$$(A - 4)(A - 3) = 0$$

When $A = 4$, $B = 1$

When $A = 3$, $B = 2$

(i) When $\begin{cases} x + y = 4 \\ xy = 1 \end{cases}$

x and y are the roots of

$$t^2 - 4t + 1 = 0$$

Therefore, $t = 2 \pm \sqrt{3}$

(ii) When $\begin{cases} x + y = 3 \\ xy = 2 \end{cases}$

x and y are the roots of

$$t^2 - 3t + 2 = 0$$

$$(t - 2)(t - 1) = 0$$

Therefore, $t = 2, 1$

Ans. $\begin{cases} x = 2 + \sqrt{3} \\ y = 2 - \sqrt{3} \end{cases}$ $\begin{cases} x = 2 - \sqrt{3} \\ y = 2 + \sqrt{3} \end{cases}$ $\begin{cases} x = 2 \\ y = 1 \end{cases}$ $\begin{cases} x = 1 \\ y = 2 \end{cases}$

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$$(2) \begin{cases} xy = 12 & \dots\dots ① \\ x^2 + y^2 = 25 & \dots\dots ② \end{cases}$$

Assuming that $x + y = A$, $xy = B$.

From ① and ②,

$$\begin{cases} B = 12 & \dots\dots ③ \\ A^2 - 2B = 25 & \dots\dots ④ \end{cases}$$

$$x^2 + y^2 = (x + y)^2 - 2xy = A^2 - 2B$$

Substituting ③ into ④,

$$A^2 - 24 = 25$$

$$A = \pm 7$$

$$\begin{cases} A = \pm 7 \\ B = 12 \end{cases}$$

(i) When $\begin{cases} x + y = 7 \\ xy = 12 \end{cases}$

x and y are the roots of

$$t^2 - 7t + 12 = 0$$

$$(t - 4)(t - 3) = 0$$

Therefore, $t = 4, 3$

(ii) When $\begin{cases} x + y = -7 \\ xy = 12 \end{cases}$

x and y are the roots of

$$t^2 + 7t + 12 = 0$$

$$(t + 4)(t + 3) = 0$$

Therefore, $t = -4, -3$

Ans. $\begin{cases} x = 4 \\ y = 3 \end{cases} \quad \begin{cases} x = 3 \\ y = 4 \end{cases} \quad \begin{cases} x = -4 \\ y = -3 \end{cases} \quad \begin{cases} x = -3 \\ y = -4 \end{cases}$

J 150a KUMON

Simultaneous Equations

Solve the following equations.

$$(1) \begin{cases} x - y = 3 & \dots\dots(1) \\ x^2 + y^2 = 9 & \dots\dots(2) \end{cases}$$

From (1),

$$y = x - 3 \quad \dots\dots(3)$$

Substituting (3) into (2),

$$x^2 + (x - 3)^2 = 9$$

$$2x^2 - 6x = 0$$

$$x(x - 3) = 0$$

$$x = 0, 3$$

Substituting $x = 0$ into (3),

$$y = -3$$

Substituting $x = 3$ into (3),

$$y = 0$$

$$\text{Ans. } \begin{cases} x = 0 \\ y = -3 \end{cases} \quad \begin{cases} x = 3 \\ y = 0 \end{cases}$$

$$(2) \begin{cases} x^2 + y^2 - 2x + 2y = 3 & \dots\dots(1) \\ x^2 + y^2 - 4x + 4y = 1 & \dots\dots(2) \end{cases}$$

From (1) - (2),

$$2x - 2y = 2$$

$$x = y + 1 \quad \dots\dots(3)$$

Substituting (3) into (1),

$$(y + 1)^2 + y^2 - 2(y + 1) + 2y = 3$$

$$2y^2 + 2y - 1 = 3$$

$$y^2 + y - 2 = 0$$

$$(y + 2)(y - 1) = 0$$

$$y = -2, 1$$

Substituting $y = -2$ into (3),

$$x = -1$$

Substituting $y = 1$ into (3),

$$x = 2$$

$$\text{Ans. } \begin{cases} x = -1 \\ y = -2 \end{cases} \quad \begin{cases} x = 2 \\ y = 1 \end{cases}$$

J 150b

$$(3) \begin{cases} (x-y)(x+2y) = 0 & \dots\dots ① \\ x^2 - 2xy + 4y^2 = 12 & \dots\dots ② \end{cases}$$

From ①, $x = y$ or $x = -2y$

Substituting $x = y$ into ②,

$$y^2 - 2y^2 + 4y^2 = 12$$

$$y^2 = 4$$

$$y = \pm 2$$

Since $x = y$,

$$x = \pm 2$$

Substituting $x = -2y$ into ②,

$$4y^2 + 4y^2 + 4y^2 = 12$$

$$y^2 = 1$$

$$y = \pm 1$$

Since $x = -2y$,

$$x = \mp 2$$

$$\text{Ans. } \begin{cases} x = \pm 2 \\ y = \pm 2 \end{cases} \quad \begin{cases} x = \pm 2 \\ y = \mp 1 \end{cases}$$

$$(4) \begin{cases} x + y = 5 & \dots\dots ① \\ x^3 + y^3 = 35 & \dots\dots ② \end{cases}$$

From ②,

$$(x+y)^3 - 3xy(x+y) = 35 \dots\dots ③$$

Substituting ① into ③,

$$125 - 15xy = 35$$

$$xy = 6 \quad \dots\dots ④$$

From ① and ④, x and y are the roots of

$$t^2 - 5t + 6 = 0$$

$$(t-3)(t-2) = 0$$

Therefore,

$$t = 3, 2$$

$$\text{Ans. } \begin{cases} x = 3 \\ y = 2 \end{cases} \quad \begin{cases} x = 2 \\ y = 3 \end{cases}$$

J151a KUMON

Dividing Polynomials

Divide the following polynomials.

Ex.

$$\begin{array}{r}
 x + 2 \\
 x + 3 \overline{) x^2 + 5x + 6} \\
 \underline{x^2 + 3x} \quad \text{☞ } (x+3) \times x \\
 2x + 6 \\
 \underline{2x + 6} \quad \text{☞ } (x+3) \times 2 \\
 0
 \end{array}$$

$$(1) \quad \begin{array}{r}
 x + 3 \\
 x + 5 \overline{) x^2 + 8x + 15} \\
 \underline{x^2 + 5x} \\
 3x + 15 \\
 \underline{3x + 15} \\
 0
 \end{array}$$

$$(3) \quad \begin{array}{r}
 2x + 5y \\
 3x - 2y \overline{) 6x^2 + 11xy - 10y^2} \\
 \underline{6x^2 - 4xy} \\
 15xy - 10y^2 \\
 \underline{15xy - 10y^2} \\
 0
 \end{array}$$

$$(2) \quad \begin{array}{r}
 x + 3 \\
 2x - 3 \overline{) 2x^2 + 3x - 9} \\
 \underline{2x^2 - 3x} \\
 6x - 9 \\
 \underline{6x - 9} \\
 0
 \end{array}$$

$$(4) \quad \begin{array}{r}
 2x - 3y \\
 3x + 4y \overline{) 6x^2 - xy - 12y^2} \\
 \underline{6x^2 + 8xy} \\
 -9xy - 12y^2 \\
 \underline{-9xy - 12y^2} \\
 0
 \end{array}$$

Ex.

$$\begin{array}{r}
 x^2 + 2x + 3 \\
 x^2 - 2x + 2 \overline{) x^4 + 0x^3 + x^2 - 2x + 6} \\
 \underline{x^4 - 2x^3 + 2x^2} \\
 2x^3 - x^2 - 2x \\
 \underline{2x^3 - 4x^2 + 4x} \\
 3x^2 - 6x + 6 \\
 \underline{3x^2 - 6x + 6} \\
 0
 \end{array}$$

☞ Leave a space as there is no x^3 term.

You can either leave a space or write $+0x^3$.

$$\begin{array}{r}
 (5) \quad \quad \quad \mathbf{4x^2 + 2x - 3} \\
 x^2 + 2x + 3 \overline{) 4x^4 + 10x^3 + 13x^2 + 0x - 9} \\
 \underline{4x^4 + 8x^3 + 12x^2} \\
 2x^3 + x^2 + 0x \\
 \underline{2x^3 + 4x^2 + 6x} \\
 - 3x^2 - 6x - 9 \\
 \underline{- 3x^2 - 6x - 9} \\
 0
 \end{array}$$

$$\begin{array}{r}
 (6) \quad \quad \quad \mathbf{x^2 + 2x + 3} \\
 x^2 - 2x + 2 \overline{) x^4 + 0x^3 + x^2 - 2x + 6} \\
 \underline{x^4 - 2x^3 + 2x^2} \\
 2x^3 - x^2 - 2x \\
 \underline{2x^3 - 4x^2 + 4x} \\
 3x^2 - 6x + 6 \\
 \underline{3x^2 - 6x + 6} \\
 0
 \end{array}$$

J 152a KUMON

Dividing Polynomials

Divide the following polynomials.

(1) $(x^4 + 4x^2 + 16) \div (x^2 - 2x + 4)$

$$\begin{array}{r}
 x^2 - 2x + 4 \overline{) x^4 + 0x^3 + 4x^2 + 0x + 16} \\
 \underline{x^4 - 2x^3 + 4x^2} \\
 2x^3 + 0x^2 + 0x \\
 \underline{2x^3 - 4x^2 + 8x} \\
 4x^2 - 8x + 16 \\
 \underline{4x^2 - 8x + 16} \\
 0
 \end{array}$$



Leave spaces as there are no x^3 or x terms.

Alternatively, write in $+0x^3$ and $+0x$.

Ans. $x^2 + 2x + 4$

(2) $(x^3 + 8) \div (x + 2)$

$$\begin{array}{r}
 x + 2 \overline{) x^3 + 0x^2 + 0x + 8} \\
 \underline{x^3 + 2x^2} \\
 -2x^2 + 0x \\
 \underline{-2x^2 - 4x} \\
 4x + 8 \\
 \underline{4x + 8} \\
 0
 \end{array}$$

Ans. $x^2 - 2x + 4$

J 152b

(3) $(a^4 + a^2b^2 + b^4) \div (a^2 + ab + b^2)$

$$\begin{array}{r}
 a^2 - ab + b^2 \\
 a^2 + ab + b^2 \overline{) a^4 + 0a^3b + a^2b^2 + 0ab^3 + b^4} \\
 \underline{a^4 + a^3b + a^2b^2} \\
 - a^3b + 0a^2b^2 + 0ab^3 \\
 \underline{- a^3b - a^2b^2 - ab^3} \\
 a^2b^2 + ab^3 + b^4 \\
 \underline{a^2b^2 + ab^3 + b^4} \\
 0
 \end{array}$$

Ans. $a^2 - ab + b^2$

(4) $(x^4 - y^4) \div (x - y)$

$$\begin{array}{r}
 x^3 + x^2y + xy^2 + y^3 \\
 x - y \overline{) x^4 + 0x^3y + 0x^2y^2 + 0xy^3 - y^4} \\
 \underline{x^4 - x^3y} \\
 x^3y + 0x^2y^2 \\
 \underline{x^3y - x^2y^2} \\
 x^2y^2 + 0xy^3 \\
 \underline{x^2y^2 - xy^3} \\
 xy^3 - y^4 \\
 \underline{xy^3 - y^4} \\
 0
 \end{array}$$

Ans. $x^3 + x^2y + xy^2 + y^3$

J 153a KUMON

Dividing Polynomials

Divide the following polynomials.

Ex.

$$(2x^3 + 5x^2 + 9) \div (2x^2 + x + 3)$$

[Sol]

$$\begin{array}{r}
 x + 2 \\
 2x^2 + x + 3 \overline{) 2x^3 + 5x^2 + 0x + 9} \\
 \underline{2x^3 + x^2 + 3x} \\
 4x^2 - 3x + 9 \\
 \underline{4x^2 + 2x + 6} \\
 -5x + 3
 \end{array}$$

Ans. Quotient $x + 2$ Remainder $-5x + 3$

$$(1) \quad (2x^4 - 18x^2 + 3x - 4) \div (x^3 - 3x^2 - 2)$$

$$\begin{array}{r}
 2x + 6 \\
 x^3 - 3x^2 - 2 \overline{) 2x^4 + 0x^3 - 18x^2 + 3x - 4} \\
 \underline{2x^4 - 6x^3} \\
 6x^3 - 18x^2 + 7x - 4 \\
 \underline{6x^3 - 18x^2} \\
 7x + 8
 \end{array}$$

Ans. Quotient $2x + 6$ Remainder $7x + 8$

J 153b

$$(2) \quad (2a^3 + a^2 - 5a + 2) \div (a + 1)$$

$$\begin{array}{r}
 2a^2 - a - 4 \\
 a + 1 \overline{) 2a^3 + a^2 - 5a + 2} \\
 \underline{2a^3 + 2a^2} \\
 -a^2 - 5a \\
 \underline{-a^2 - a} \\
 -4a + 2 \\
 \underline{-4a - 4} \\
 6
 \end{array}$$

Ans. Quotient $2a^2 - a - 4$ Remainder 6

$$(3) \quad (x^2 - 3xy + 2y^2 + 3x - 6y - 8) \div (x - y + 4)$$

$$\begin{array}{r}
 x - 2y - 1 \\
 x - y + 4 \overline{) x^2 - 3xy + 2y^2 + 3x - 6y - 8} \\
 \underline{x^2 - xy + 4x} \\
 -2xy + 2y^2 - x - 6y - 8 \\
 \underline{-2xy + 2y^2} \\
 -x + 2y - 8 \\
 \underline{-x + y - 4} \\
 y - 4
 \end{array}$$

Ans. Quotient $x - 2y - 1$ Remainder $y - 4$

J154a KUMON

Dividing Polynomials

$$17 \div 5 = 3 \text{ Remainder } 2$$

This relationship may be written as $17 = 5 \times 3 + 2$.

Similarly,

$$(x^3 - x^2 - 1) \div (x - 2)$$

$$= x^2 + x + 2 \text{ Remainder } 3$$

This relationship may be written as

$$x^3 - x^2 - 1 = (x - 2)(x^2 + x + 2) + 3.$$

1. Complete the following exercises.

(1) Divide $x^3 + 4x^2 - 5x + 3$ by $x - 2$.

$$\begin{array}{r} x^2 + 6x + 7 \\ x - 2 \overline{) x^3 + 4x^2 - 5x + 3} \\ \underline{x^3 - 2x^2} \\ 6x^2 - 5x \\ \underline{6x^2 - 12x} \\ 7x + 3 \\ \underline{7x - 14} \\ 17 \end{array}$$

Note:

For $26 \div 7 = 3 \text{ R}5$

7 is the divisor,

3 is the quotient,

5 is the remainder.

Ans. Quotient $x^2 + 6x + 7$ Remainder 17

(2) Using the result of (1), write the relationship between $x^3 + 4x^2 - 5x + 3$ and $x - 2$.

Ans. $x^3 + 4x^2 - 5x + 3 = (x - 2)(x^2 + 6x + 7) + 17$

J 154b

2. In each question, divide A by B , and write the relationship between the two.

(1) $A = 2x^3 + x^2 - 5x + 2$ $B = x + 1$

$$\begin{array}{r}
 \frac{2x^2 - x - 4}{x+1} \overline{) 2x^3 + x^2 - 5x + 2} \\
 \underline{2x^3 + 2x^2} \\
 - x^2 - 5x \\
 \underline{- x^2 - x} \\
 - 4x + 2 \\
 \underline{- 4x - 4} \\
 6
 \end{array}$$

Ans. $A = B(2x^2 - x - 4) + 6$

(2) $A = x^3 - 3x^2 + 4x + 30$ $B = x - 2$

$$\begin{array}{r}
 \frac{x^2 - x + 2}{x-2} \overline{) x^3 - 3x^2 + 4x + 30} \\
 \underline{x^3 - 2x^2} \\
 - x^2 + 4x \\
 \underline{- x^2 + 2x} \\
 2x + 30 \\
 \underline{2x - 4} \\
 34
 \end{array}$$

Ans. $A = B(x^2 - x + 2) + 34$

J 155a KUMON

Dividing Polynomials

In each question, divide A by B , and write the relationship between the two.

$$(1) \quad A = x^4 + x^3 - 8x^2 - 9x + 9 \quad B = x - 3$$

$$\begin{array}{r}
 x^3 + 4x^2 + 4x + 3 \\
 x - 3 \overline{) x^4 + x^3 - 8x^2 - 9x + 9} \\
 \underline{x^4 - 3x^3} \\
 4x^3 - 8x^2 \\
 \underline{4x^3 - 12x^2} \\
 4x^2 - 9x \\
 \underline{4x^2 - 12x} \\
 3x + 9 \\
 \underline{3x - 9} \\
 18
 \end{array}$$

$$\text{Ans. } A = B(x^3 + 4x^2 + 4x + 3) + 18$$

$$(2) \quad A = 2x^4 + 25x^3 + 4x + 10 \quad B = 2x + 1$$

$$\begin{array}{r}
 x^3 + 12x^2 - 6x + 5 \\
 2x + 1 \overline{) 2x^4 + 25x^3 + 0x^2 + 4x + 10} \\
 \underline{2x^4 + x^3} \\
 24x^3 + 0x^2 \\
 \underline{24x^3 + 12x^2} \\
 -12x^2 + 4x \\
 \underline{-12x^2 - 6x} \\
 10x + 10 \\
 \underline{10x + 5} \\
 5
 \end{array}$$

$$\text{Ans. } A = B(x^3 + 12x^2 - 6x + 5) + 5$$

J 155b

(3) $A = x^4 + 6x^3 - 7x^2 + 3x + 9$ $B = x^2 + 3x - 2$

$$\begin{array}{r}
 x^2 + 3x - 14 \\
 x^2 + 3x - 2 \overline{) x^4 + 6x^3 - 7x^2 + 3x + 9} \\
 \underline{x^4 + 3x^3 - 2x^2} \\
 3x^3 - 5x^2 + 3x \\
 \underline{3x^3 + 9x^2 - 6x} \\
 -14x^2 + 9x + 9 \\
 \underline{-14x^2 - 42x + 28} \\
 51x - 19
 \end{array}$$

Ans. $A = B(x^2 + 3x - 14) + 51x - 19$

(4) $A = x^3 + 6x^2 + 6x - 6$ $B = x^2 + 3x - 3$


$$\begin{array}{r}
 x + 3 \\
 x^2 + 3x - 3 \overline{) x^3 + 6x^2 + 6x - 6} \\
 \underline{x^3 + 3x^2 - 3x} \\
 3x^2 + 9x - 6 \\
 \underline{3x^2 + 9x - 9} \\
 3
 \end{array}$$

Ans. $A = B(x + 3) + 3$

Dividing Polynomials

Ex.

When A is divided by $2x^2 - 1$, the quotient is $2x - 1$ with remainder $x - 2$. Find A .

[Sol] $A = (2x^2 - 1)(2x - 1) + x - 2$  Refer to J 154a.

Therefore,

$$A = 4x^3 - 2x^2 - x - 1$$

1. When A is divided by $x + 1$, the quotient is $x^2 + 2$ with remainder 3. Find A .

[Sol] $A = (x + 1)(x^2 + 2) + 3$

Therefore,

$$\begin{aligned} A &= x^3 + 2x + x^2 + 2 + 3 \\ &= x^3 + x^2 + 2x + 5 \end{aligned}$$

2. When A is divided by $2x + 3$, the quotient is $x^2 - 2x + 1$ with remainder -2 . Find A .

[Sol] $A = (2x + 3)(x^2 - 2x + 1) - 2$

Therefore,

$$\begin{aligned} A &= 2x^3 - 4x^2 + 2x + 3x^2 - 6x + 3 - 2 \\ &= 2x^3 - x^2 - 4x + 1 \end{aligned}$$

J 156b

3. When A is divided by $x^2 + 1$, the quotient is $x^2 + 1$ with remainder $x + 1$. Find A .

$$[\text{Sol}] A = (x^2 + 1)(x^2 + 1) + x + 1$$

Therefore,

$$\begin{aligned} \mathbf{A} &= x^4 + 2x^2 + 1 + x + 1 \\ &= \mathbf{x^4 + 2x^2 + x + 2} \end{aligned}$$

4. When A is divided by $x^2 + x + 1$, the quotient is $x^2 + 1$ with remainder $2x$. Find A .

$$[\text{Sol}] A = (x^2 + x + 1)(x^2 + 1) + 2x$$

Therefore,

$$\begin{aligned} \mathbf{A} &= x^4 + x^2 + x^3 + x + x^2 + 1 + 2x \\ &= \mathbf{x^4 + x^3 + 2x^2 + 3x + 1} \end{aligned}$$

Dividing Polynomials

Ex.

When $x^3 + x^2 + 4x - 1$ is divided by B , the quotient is $x + 1$ with remainder $3x - 2$. Find B .

$$[\text{Sol}] \quad x^3 + x^2 + 4x - 1 = B(x + 1) + 3x - 2$$

$$x^3 + x^2 + x + 1 = B(x + 1)$$

Move $3x - 2$ to the left side.

Therefore,

$$B = (x^3 + x^2 + x + 1) \div (x + 1)$$

$$\begin{array}{r} x^2 \quad + 1 \\ x + 1 \overline{) x^3 + x^2 + x + 1} \\ \underline{x^3 + x^2} \\ x + 1 \\ \underline{x + 1} \\ 0 \end{array}$$

$$\text{Ans. } B = x^2 + 1$$

1. When $2x^3 + x^2 - x + 1$ is divided by B , the quotient is $2x + 3$ with remainder -2 . Find B .

$$[\text{Sol}] \quad 2x^3 + x^2 - x + 1 = B(2x + 3) - 2$$

$$2x^3 + x^2 - x + 3 = B(2x + 3)$$

Therefore,

$$B = (2x^3 + x^2 - x + 3) \div (2x + 3)$$

$$\begin{array}{r} x^2 - x \quad + 1 \\ 2x + 3 \overline{) 2x^3 + x^2 - x + 3} \\ \underline{2x^3 + 3x^2} \\ -2x^2 - x \\ \underline{-2x^2 - 3x} \\ 2x + 3 \\ \underline{2x + 3} \\ 0 \end{array}$$

$$\text{Ans. } B = x^2 - x + 1$$

Ex.

When $2x^3 - x^2 - ax + b$ is divided by $x^2 - 2x + 1$, the remainder is $3x + 2$. Find the values of a and b .

[Sol]

$$\begin{array}{r}
 2x + 3 \\
 x^2 - 2x + 1 \overline{) 2x^3 - x^2 - ax + b} \\
 \underline{2x^3 - 4x^2} \\
 3x^2 - (a+2)x + b \\
 \underline{3x^2} \\
 (-a+4)x + b - 3
 \end{array}$$

Since the remainder is $3x + 2$,

$$(-a+4)x + b - 3 = 3x + 2$$

Comparing left and right hand sides,

$$\begin{cases} -a+4 = 3 & \dots \textcircled{1} \\ b-3 = 2 & \dots \textcircled{2} \end{cases}$$



Match the coefficients of x on both sides, and the constant terms on both sides.

From $\textcircled{1}$ and $\textcircled{2}$,

$$a = 1, b = 5$$

2. When $x^4 + ax^2 + b$ is divided by $x^2 + x + 1$, the remainder is $-2x + 2$. Find the values of a and b .

[Sol]

$$\begin{array}{r}
 x^2 - x + a \\
 x^2 + x + 1 \overline{) x^4 + 0x^3 + ax^2 + 0x + b} \\
 \underline{x^4 + x^3 + x^2} \\
 -x^3 + (a-1)x^2 + 0x \\
 \underline{-x^3} \\
 ax^2 + x + b \\
 \underline{ax^2 + ax + a} \\
 (1-a)x + b - a
 \end{array}$$

Since the remainder is $-2x + 2$,

$$(1-a)x + b - a = -2x + 2$$

Comparing left and right hand sides,

$$\begin{cases} 1-a = -2 & \dots \textcircled{1} \\ b-a = 2 & \dots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}$,

$$a = 3, b = 5$$

J 158a KUMON

Dividing Polynomials

1. Simplify the expressions.

Ex.

$$\frac{4x-5}{x-2} = 4 + \frac{3}{x-2} \quad \Rightarrow \quad \begin{array}{r} 4 \\ x-2 \overline{)4x-5} \\ \underline{4x-8} \\ 3 \end{array}$$

$$(1) \quad \frac{2x-1}{x-2} = 2 + \frac{3}{x-2}$$

$$\begin{array}{r} 2 \\ x-2 \overline{)2x-1} \\ \underline{2x-4} \\ 3 \end{array}$$

$$(2) \quad \frac{-2x+1}{x+3} = -2 + \frac{7}{x+3}$$

$$\begin{array}{r} -2 \\ x+3 \overline{)-2x+1} \\ \underline{-2x-6} \\ 7 \end{array}$$

$$(3) \quad \frac{4x+5}{2x-1} = 2 + \frac{7}{2x-1}$$

$$\begin{array}{r} 2 \\ 2x-1 \overline{)4x+5} \\ \underline{4x-2} \\ 7 \end{array}$$

Note: You can also simplify in this way:

$$\frac{4x-5}{x-2} = \frac{4(x-2)+3}{x-2} = \frac{4(x-2)}{x-2} + \frac{3}{x-2} = 4 + \frac{3}{x-2}$$

J 158b

2. Simplify the expressions.

Ex.

$$\frac{x^2 - 3x + 3}{x - 1} = x - 2 + \frac{1}{x - 1}$$



$$\begin{array}{r} x - 2 \\ x - 1 \overline{) x^2 - 3x + 3} \\ \underline{x^2 - x} \\ -2x + 3 \\ \underline{-2x + 2} \\ 1 \end{array}$$

$$(1) \quad \frac{x^2 - x + 4}{x - 2} = x + 1 + \frac{6}{x - 2}$$

$$\begin{array}{r} x + 1 \\ x - 2 \overline{) x^2 - x + 4} \\ \underline{x^2 - 2x} \\ x + 4 \\ \underline{x - 2} \\ 6 \end{array}$$

$$(2) \quad \frac{2x^2 - x + 1}{x + 1} = 2x - 3 + \frac{4}{x + 1}$$

$$\begin{array}{r} 2x - 3 \\ x + 1 \overline{) 2x^2 - x + 1} \\ \underline{2x^2 + 2x} \\ -3x + 1 \\ \underline{-3x - 3} \\ 4 \end{array}$$

$$(3) \quad \frac{2x^2 - 3x + 2}{2x + 1} = x - 2 + \frac{4}{2x + 1}$$

$$\begin{array}{r} x - 2 \\ 2x + 1 \overline{) 2x^2 - 3x + 2} \\ \underline{2x^2 + x} \\ -4x + 2 \\ \underline{-4x - 2} \\ 4 \end{array}$$

J 159a KUMON

Dividing Polynomials

Simplify the following expressions.

Ex.

$$\begin{aligned}\frac{a-4}{a-5} - \frac{a-5}{a-6} &= \left(1 + \frac{1}{a-5}\right) - \left(1 + \frac{1}{a-6}\right) \\ &= \frac{1}{a-5} - \frac{1}{a-6} = -\frac{1}{(a-5)(a-6)}\end{aligned}$$

Note: Simplifying the fractions first makes the calculations easier.

$$\begin{aligned}(1) \quad \frac{x-8}{x-4} + \frac{x-5}{x-7} - 2 &= \left(1 + \frac{-4}{x-4}\right) + \left(1 + \frac{2}{x-7}\right) - 2 \\ &= \frac{-4}{x-4} + \frac{2}{x-7} \\ &= \frac{-4(x-7) + 2(x-4)}{(x-4)(x-7)} \\ &= \frac{-2x + 20}{(x-4)(x-7)} \\ &= -\frac{2(x-10)}{(x-4)(x-7)}\end{aligned}$$

J 159b

$$\begin{aligned}
 (2) \quad \frac{2a}{a-1} - \frac{a-1}{a+1} - \frac{a+1}{a} &= \left(2 + \frac{2}{a-1}\right) - \left(1 + \frac{-2}{a+1}\right) - \left(1 + \frac{1}{a}\right) \\
 &= \frac{2}{a-1} - \frac{-2}{a+1} - \frac{1}{a} \\
 &= \frac{2(a+1) + 2(a-1)}{(a-1)(a+1)} - \frac{1}{a} \\
 &= \frac{4a}{(a-1)(a+1)} - \frac{1}{a} \\
 &= \frac{4a^2 - (a^2 - 1)}{a(a-1)(a+1)} \\
 &= \frac{\mathbf{3a^2 + 1}}{\mathbf{a(a-1)(a+1)}}
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad \frac{x+2}{x+1} - \frac{x+3}{x+2} - \frac{x+5}{x+4} + \frac{x+6}{x+5} \\
 &= \left(1 + \frac{1}{x+1}\right) - \left(1 + \frac{1}{x+2}\right) - \left(1 + \frac{1}{x+4}\right) + \left(1 + \frac{1}{x+5}\right) \\
 &= \frac{(x+2) - (x+1)}{(x+1)(x+2)} - \frac{(x+5) - (x+4)}{(x+4)(x+5)} \\
 &= \frac{1}{(x+1)(x+2)} - \frac{1}{(x+4)(x+5)} \\
 &= \frac{(x^2 + 9x + 20) - (x^2 + 3x + 2)}{(x+1)(x+2)(x+4)(x+5)} \\
 &= \frac{\mathbf{6x + 18}}{\mathbf{(x+1)(x+2)(x+4)(x+5)}} \\
 &\left[= \frac{\mathbf{6(x+3)}}{\mathbf{(x+1)(x+2)(x+4)(x+5)}} \right]
 \end{aligned}$$

J160a KUMON

Dividing Polynomials

1. Divide the following polynomial.

$$(x^4 + 2x^3 - 12x^2 - 4) \div (x^2 - 2x - 6)$$

[Sol]

$$\begin{array}{r} x^2 + 4x + 2 \\ x^2 - 2x - 6 \overline{) x^4 + 2x^3 - 12x^2 + 0x - 4} \\ \underline{x^4 - 2x^3 - 6x^2} \\ 4x^3 - 6x^2 + 0x \\ \underline{4x^3 - 8x^2 - 24x} \\ 2x^2 + 24x - 4 \\ \underline{2x^2 - 4x - 12} \\ 28x + 8 \end{array}$$

Ans. Quotient $x^2 + 4x + 2$ Remainder $28x + 8$

2. When $x^3 + ax^2 - 4x - a^2$ is divided by $x^2 - x - a$, the remainder is $x + b$. Find the values of a and b .

[Sol]

$$\begin{array}{r} x + (a+1) \\ x^2 - x - a \overline{) x^3 - 4x - a^2} \\ \underline{x^3 - x^2 - ax} \\ (a+1)x^2 + (a-4)x - a^2 \\ \underline{(a+1)x^2 - (a+1)x - a(a+1)} \\ (2a-3)x + a \end{array}$$

Since the remainder is $x + b$,

$$(2a-3)x + a = x + b$$

Comparing left and right hand sides,

$$\begin{cases} 2a - 3 = 1 & \dots \textcircled{1} \\ a = b & \dots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}$,

$$\mathbf{a = 2, \quad b = 2}$$

J 160b

3. When A is divided by $x^2 + x + 1$, the quotient is $x^2 - 2$ with remainder $2x + 7$. Find the quotient and the remainder when A is divided by $x^2 - 2x + 4$.

$$\begin{aligned} \text{[Sol]} \quad A &= (x^2 + x + 1)(x^2 - 2) + 2x + 7 \\ &= x^4 - 2x^2 + x^3 - 2x + x^2 - 2 + 2x + 7 \\ &= x^4 + x^3 - x^2 + 5 \end{aligned}$$

Dividing A by $x^2 - 2x + 4$:

$$\begin{array}{r} x^2 - 2x + 4 \overline{) x^4 + x^3 - x^2 + 0x + 5} \\ \underline{x^4 - 2x^3 + 4x^2} \\ 3x^3 - 5x^2 + 0x \\ \underline{3x^3 - 6x^2 + 12x} \\ x^2 - 12x + 5 \\ \underline{x^2 - 2x + 4} \\ -10x + 1 \end{array}$$

Ans. Quotient $x^2 + 3x + 1$ Remainder $-10x + 1$

Remainder Theorem

Ex.

Determine the value of a for which $x^2 + ax + 3$ is exactly divisible by $x - 1$.

(i.e. the remainder is equal to zero)

$$\begin{array}{r}
 \text{[Sol]} \quad \frac{x + (a+1)}{x-1} \overline{) x^2 + ax + 3} \\
 \underline{x^2 - x} \\
 (a+1)x + 3 \\
 \underline{(a+1)x - a - 1} \\
 a + 4
 \end{array}$$

$$\begin{aligned}
 \text{Therefore, } a + 4 &= 0 \\
 a &= -4
 \end{aligned}$$

“To be divisible by” means the remainder is zero.

1. Determine the value of a for which $x^2 + 2x - a$ is exactly divisible by $x + 4$.

$$\begin{array}{r}
 \text{[Sol]} \quad \frac{x - 2}{x+4} \overline{) x^2 + 2x - a} \\
 \underline{x^2 + 4x} \\
 -2x - a \\
 \underline{-2x - 8} \\
 -a + 8
 \end{array}$$

$$\begin{aligned}
 \text{Therefore, } -a + 8 &= 0 \\
 \mathbf{a} &= \mathbf{8}
 \end{aligned}$$

J161b

2. Determine the value of a for which $2x^2 + 7x + a$ is exactly divisible by $x + 1$.

$$\begin{array}{r}
 \text{[Sol]} \quad \frac{2x + 5}{x + 1} \overline{) 2x^2 + 7x + a} \\
 \underline{2x^2 + 2x} \\
 5x + a \\
 \underline{5x + 5} \\
 a - 5
 \end{array}$$

Therefore, $a - 5 = 0$

$$\mathbf{a = 5}$$

3. Determine the value of a for which $x^2 - ax + 6$ is exactly divisible by $x - 2$.

$$\begin{array}{r}
 \text{[Sol]} \quad \frac{x + (-a + 2)}{x - 2} \overline{) x^2 - ax + 6} \\
 \underline{x^2 - 2x} \\
 (-a + 2)x + 6 \\
 \underline{(-a + 2)x + 2a - 4} \\
 -2a + 10
 \end{array}$$

Therefore, $-2a + 10 = 0$

$$\mathbf{a = 5}$$

4. Determine the value of a for which $x^2 + 2x - a$ is divided by $x + 4$ with remainder 3.

$$\begin{array}{r}
 \text{[Sol]} \quad \frac{x - 2}{x + 4} \overline{) x^2 + 2x - a} \\
 \underline{x^2 + 4x} \\
 -2x - a \\
 \underline{-2x - 8} \\
 -a + 8
 \end{array}$$

Therefore, $-a + 8 = 3$

$$\mathbf{a = 5}$$

J162a KUMON

Remainder Theorem

Notation

- $P(x)$ and $Q(x)$ are used to represent polynomial expressions in x .
- When 2 is substituted for x , this is written as $P(2)$.

Complete the following exercises on $P(x) = x^3 + x^2 - 5x + 6$.

Ex.

First find $P(2)$, then find the quotient and remainder when $P(x)$ is divided by $x - 2$.

[Sol] $P(2) = 2^3 + 2^2 - 5 \times 2 + 6 = 8$

☞ Substitute $x = 2$ into
 $P(x) = x^3 + x^2 - 5x + 6$.

$$\begin{array}{r}
 x^2 + 3x + 1 \\
 x - 2 \overline{) x^3 + x^2 - 5x + 6} \\
 \underline{x^3 - 2x^2} \\
 3x^2 - 5x \\
 \underline{3x^2 - 6x} \\
 x + 6 \\
 \underline{x - 2} \\
 8
 \end{array}$$

Ans. $\frac{\text{Quotient } x^2 + 3x + 1}{\text{Remainder } 8}$

► When $P(x)$ is divided by $x - 2$, the remainder is $P(2)$.

- (1) First find $P(3)$, then find the quotient and remainder when $P(x)$ is divided by $x - 3$.

[Sol] $P(3) = \boxed{3}^3 + \boxed{3}^2 - 5 \times \boxed{3} + 6 = 27$

$$\begin{array}{r}
 x^2 + 4x + 7 \\
 x - 3 \overline{) x^3 + x^2 - 5x + 6} \\
 \underline{x^3 - 3x^2} \\
 4x^2 - 5x \\
 \underline{4x^2 - 12x} \\
 7x + 6 \\
 \underline{7x - 21} \\
 27
 \end{array}$$

Ans. $\frac{\text{Quotient } x^2 + 4x + 7}{\text{Remainder } 27}$

► When $P(x)$ is divided by $x - 3$, the remainder is $\boxed{P(3)}$.

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(2) First find $P(-2)$, then find the quotient and remainder when $P(x)$ is divided by $x+2$.

[Sol] $P(-2) = (-2)^3 + (-2)^2 - 5 \times (-2) + 6 = 12$

$$\begin{array}{r}
 x^2 - x - 3 \\
 x+2 \overline{) x^3 + x^2 - 5x + 6} \\
 \underline{x^3 + 2x^2} \\
 -x^2 - 5x \\
 \underline{-x^2 - 2x} \\
 -3x + 6 \\
 \underline{-3x - 6} \\
 12
 \end{array}$$

Ans. $\frac{\text{Quotient } x^2 - x - 3}{\text{Remainder } 12}$

► When $P(x)$ is divided by $x+2$, the remainder is $P(-2)$.

(3) First find $P(-1)$, then find the quotient and remainder when $P(x)$ is divided by $x+1$.

[Sol] $P(-1) = (-1)^3 + (-1)^2 - 5 \times (-1) + 6 = 11$

$$\begin{array}{r}
 x^2 - 5 \\
 x+1 \overline{) x^3 + x^2 - 5x + 6} \\
 \underline{x^3 + x^2} \\
 -5x + 6 \\
 \underline{-5x - 5} \\
 11
 \end{array}$$

Ans. $\frac{\text{Quotient } x^2 - 5}{\text{Remainder } 11}$

► When $P(x)$ is divided by $x+1$, the remainder is $P(-1)$.

- Looking at the example on side a, $P(x) = (x-2)(x^2 + 3x + 1) + 8$, and substitution gives $P(2) = 8$.
- If a polynomial $P(x)$ is divided by $x-a$, we can express the relationship with the quotient $Q(x)$ and remainder R as:

$$P(x) = \underline{(x-a)Q(x)} + R$$

- When $x = a$, the underlined portion of the above equation becomes $(a-a)Q(x) = 0$. Thus, $P(a) = (a-a)Q(x) + R = R$

Conclusion: When $P(x)$ is divided by $x-a$, the remainder is $P(a)$.

Remainder Theorem

The Remainder Theorem

When $P(x)$ is divided by $x - a$, the remainder is $P(a)$.

1. Use the *Remainder Theorem* to find the remainder when the given $P(x)$ is divided by the term on the right.

$$(1) \quad P(x) = x^3 - 7x^2 + 4x - 3 \qquad x - 2$$

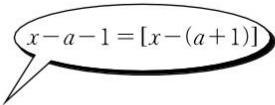
$$\begin{aligned} P(2) &= 2^3 - 7 \times 2^2 + 4 \times 2 - 3 \\ &= -15 \end{aligned}$$

$$(2) \quad P(x) = 2x^3 - x + 1 \qquad x + 3$$

$$\begin{aligned} P(-3) &= 2 \times (-3)^3 - (-3) + 1 \\ &= -50 \end{aligned}$$

$$(3) \quad P(x) = x^3 - 2ax^2 + 1 \qquad x - a - 1$$

$$\begin{aligned} P(a+1) &= (a+1)^3 - 2a(a+1)^2 + 1 \\ &= a^3 + 3a^2 + 3a + 1 - 2a(a^2 + 2a + 1) + 1 \\ &= -a^3 - a^2 + a + 2 \end{aligned}$$



$$x - a - 1 = [x - (a + 1)]$$

$$(4) \quad P(x) = x^4 - 2x^2 - 1 \qquad 2x - 1$$

$$\begin{aligned} P\left(\frac{1}{2}\right) &= \left(\frac{1}{2}\right)^4 - 2 \times \left(\frac{1}{2}\right)^2 - 1 \\ &= \frac{1}{16} - \frac{1}{2} - 1 = -\frac{23}{16} \end{aligned}$$

$$(5) \quad P(x) = x^5 - 6x^2 + x + 6 \qquad x + 2$$

$$\begin{aligned} P(-2) &= (-2)^5 - 6 \times (-2)^2 + (-2) + 6 \\ &= -52 \end{aligned}$$

J 163b

2. Find the remainders when $P(x) = 2x^2 - 3x + 4$ is divided by each of the following terms.

(1) $x - 1$

$$\begin{aligned} P(1) &= 2 \times 1^2 - 3 \times 1 + 4 \\ &= \mathbf{3} \end{aligned}$$

(2) $x + 2$

$$\begin{aligned} P(-2) &= 2 \times (-2)^2 - 3 \times (-2) + 4 \\ &= \mathbf{18} \end{aligned}$$

(3) $3x - 4$

$$\begin{aligned} P\left(\frac{4}{3}\right) &= 2 \times \left(\frac{4}{3}\right)^2 - 3 \times \left(\frac{4}{3}\right) + 4 \\ &= \frac{\mathbf{32}}{\mathbf{9}} \end{aligned}$$

(4) $x + a$

$$\begin{aligned} P(-a) &= 2(-a)^2 - 3(-a) + 4 \\ &= \mathbf{2a^2 + 3a + 4} \end{aligned}$$

(5) $x - a - 1$ 

$$\begin{aligned} P(a+1) &= 2(a+1)^2 - 3(a+1) + 4 \\ &= 2(a^2 + 2a + 1) - 3a - 3 + 4 \\ &= \mathbf{2a^2 + a + 3} \end{aligned}$$

J164a KUMON

Remainder Theorem

1. By using the *Remainder Theorem*, find the value of a so that $P(x) = 2x^2 + 7x + a$ can be divided by $x + 1$ with no remainder.

$$\begin{aligned} \text{[Sol]} \quad P(-1) &= 2 \times (-1)^2 + 7 \times (-1) + a \\ &= a - 5 \end{aligned}$$

For there to be no remainder, $P(-1) = 0$

$$a - 5 = 0$$

$$\mathbf{a = 5}$$

2. Find the value of a so that $P(x) = x^2 - ax + 6$ can be divided by $x - 2$ with no remainder.

$$\begin{aligned} \text{[Sol]} \quad P(2) &= 4 - 2a + 6 \\ &= -2a + 10 \end{aligned}$$

For there to be no remainder, $P(2) = 0$

$$-2a + 10 = 0$$

$$\mathbf{a = 5}$$

3. Find the value of a so that $P(x) = a^2x^3 - 3ax + 1$ can be divided by $x - 2$ with no remainder.

$$\text{[Sol]} \quad P(2) = 8a^2 - 6a + 1$$

For there to be no remainder, $P(2) = 0$

$$8a^2 - 6a + 1 = 0$$

$$(4a - 1)(2a - 1) = 0$$

$$\mathbf{a = \frac{1}{4}, \frac{1}{2}}$$

The **Remainder Theorem** can only be used when dividing by a linear expression. When dividing by a quadratic expression, use long division.

Ex.

When $x^3 + 6x^2 + ax + b$ is divided by $x^2 + 3x - 2$, the remainder is 3. Find the values of a and b .

[Sol]

$$\begin{array}{r}
 x + 3 \\
 x^2 + 3x - 2 \overline{) x^3 + 6x^2 + ax + b} \\
 \underline{x^3 + 3x^2 } \\
 3x^2 + (a+2)x + b \\
 \underline{3x^2 + 9x - 6} \\
 (a-7)x + b + 6
 \end{array}$$

Since the remainder is 3,

$$(a-7)x + b + 6 = 3$$

$$\begin{cases}
 a-7 = 0 & \dots \textcircled{1} \\
 b+6 = 3 & \dots \textcircled{2}
 \end{cases}$$

In order for the remainder to be 3,
 $a-7 = 0$ and $b+6 = 3$ must be true.

From $\textcircled{1}$ and $\textcircled{2}$,

$$a = 7, b = -3$$

4. Find the values of a and b for which $x^3 + 2x^2 + ax + b$ is divisible by $x^2 + 3x - 1$. (Remember that " $x^3 + 2x^2 + ax + b$ is divisible by $x^2 + 3x - 1$ " means that the remainder equals zero.)

[Sol]

$$\begin{array}{r}
 x - 1 \\
 x^2 + 3x - 1 \overline{) x^3 + 2x^2 + ax + b} \\
 \underline{x^3 + 3x^2 } \\
 -x^2 + (a+1)x + b \\
 \underline{-x^2 + 1} \\
 (a+4)x + b - 1
 \end{array}$$

Since the remainder is 0,

$$(a+4)x + b - 1 = 0$$

$$\begin{cases}
 a+4 = 0 & \dots \textcircled{1} \\
 b-1 = 0 & \dots \textcircled{2}
 \end{cases}$$

In order for the remainder to be 0,
 $a+4 = 0$ and $b-1 = 0$ must be true.


From $\textcircled{1}$ and $\textcircled{2}$,

$$a = -4, b = 1$$

Remainder Theorem

Ex.

Given that $P(x)$ is divided by $(x-2)(x-3)$, express the relationship with the quotient and the remainder.

[Sol] Let the quotient be $Q(x)$,
and let the remainder be $ax+b$. 

Since we are dividing by a quadratic, we write the remainder as $ax+b$.

$$P(x) = (x-2)(x-3)Q(x) + ax + b$$

1. Given that $P(x)$ is divided by $(x-2)(x+3)$, express the relationship with the quotient and the remainder.

[Sol] Let the quotient be $Q(x)$, and let the remainder be $ax+b$.

$$P(x) = (x-2)(x+3)Q(x) + ax + b$$

2. Given that $P(x)$ is divided by $(x+1)(x+4)$, express the relationship with the quotient and the remainder.

[Sol] Let the quotient be $Q(x)$, and let the remainder be $ax+b$.

$$P(x) = (x+1)(x+4)Q(x) + ax + b$$

3. Given that $P(x)$ is divided by x^2-x-2 , express the relationship with the quotient and the remainder.

[Sol] Let the quotient be $Q(x)$, and let the remainder be $ax+b$.

$$P(x) = (x^2-x-2)Q(x) + ax + b$$

$$= (x-2)(x+1)Q(x) + ax + b$$

Ex.

Given that $P(x)$ is divided by $(x-1)(x-2)(x-3)$, express the relationship with the quotient and the remainder.

[Sol] Let the quotient be $Q(x)$, and
let the remainder be $ax^2 + bx + c$.

Since we are dividing by a cubic, we write the remainder as $ax^2 + bx + c$.

$$P(x) = (x-1)(x-2)(x-3)Q(x) + ax^2 + bx + c$$

4. Given that $P(x)$ is divided by $(x-1)(x-2)(x+3)$, express the relationship with the quotient and the remainder.

[Sol] Let the quotient be $Q(x)$, and let the remainder be $ax^2 + bx + c$.

$$P(x) = (x-1)(x-2)(x+3)Q(x) + ax^2 + bx + c$$

5. Given that $P(x)$ is divided by $(x+1)(x+2)(x+3)$, express the relationship with the quotient and the remainder.

[Sol] Let the quotient be $Q(x)$, and let the remainder be $ax^2 + bx + c$.

$$P(x) = (x+1)(x+2)(x+3)Q(x) + ax^2 + bx + c$$

6. Given that $P(x)$ is divided by $x^3 - x - 1$, express the relationship with the quotient and the remainder.

[Sol] Let the quotient be $Q(x)$, and let the remainder be $ax^2 + bx + c$.

$$P(x) = (x^3 - x - 1)Q(x) + ax^2 + bx + c$$

Note Summary

The general form of a remainder can be written:

- ① Dividing by a second-degree expression (a quadratic), the remainder must be of degree one or less. We write this as $ax + b$.
- ② Dividing by a third-degree expression (a cubic), the remainder must be of degree two or less. We write this as $ax^2 + bx + c$.

Remainder Theorem

Ex.

When $P(x)$ is divided by $x-2$, the remainder is 5. When divided by $x-3$, the remainder is 9. Find the remainder when $P(x)$ is divided by $(x-2)(x-3)$.

[Sol] Let the quotient be $Q(x)$, and let the remainder be $ax+b$.

$$P(x) = (x-2)(x-3)Q(x) + ax + b$$

First express the relationship between quotient and remainder.

$$\begin{cases} P(2) = 2a + b = 5 & \dots \textcircled{1} \\ P(3) = 3a + b = 9 & \dots \textcircled{2} \end{cases}$$

From the **Remainder Theorem**.

From $\textcircled{1}$ and $\textcircled{2}$,

$$a = 4, b = -3$$

The remainder is $4x-3$

Substituting into $ax+b$.

1. When $P(x)$ is divided by $x-2$, the remainder is 3. When divided by $x+3$, the remainder is -7 . Find the remainder when $P(x)$ is divided by $(x-2)(x+3)$.

[Sol] Let the quotient be $Q(x)$, and let the remainder be $ax+b$.

$$P(x) = (x-2)(x+3)Q(x) + ax + b$$

$$\begin{cases} P(2) = 2a + b = 3 & \dots \textcircled{1} \\ P(-3) = -3a + b = -7 & \dots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}$,

$$a = 2, b = -1$$

The remainder is $2x-1$

J 166b

2. When $P(x)$ is divided by $x-3$, the remainder is 3. When divided by $x+4$, the remainder is -4 . Find the remainder when $P(x)$ is divided by $(x-3)(x+4)$.

[Sol] Let the quotient be $Q(x)$, and let the remainder be $ax+b$.

$$P(x) = (x-3)(x+4)Q(x) + ax + b$$

$$\begin{cases} P(3) = 3a + b = 3 & \dots \textcircled{1} \\ P(-4) = -4a + b = -4 & \dots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}$,

$$a = 1, b = 0$$

The remainder is x

3. $P(x)$ is divisible by $x-4$. When divided by $x+3$, the remainder is 14. Find the remainder when $P(x)$ is divided by x^2-x-12 . Hint

[Sol] Let the quotient be $Q(x)$, and let the remainder be $ax+b$.

$$\begin{aligned} P(x) &= (x^2-x-12)Q(x) + ax + b \\ &= (x-4)(x+3)Q(x) + ax + b \end{aligned}$$

$$\begin{cases} P(4) = 4a + b = 0 & \dots \textcircled{1} \\ P(-3) = -3a + b = 14 & \dots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}$,

$$a = -2, b = 8$$

The remainder is $-2x+8$

Hint

Use the factorised form of x^2-x-12 .

J167a KUMON

Remainder Theorem

1. When $P(x)$ is divided by $x+1$, $x+2$ and $x+3$, the remainders are 2, 3 and 6 respectively. Find the remainder when $P(x)$ is divided by $(x+1)(x+2)(x+3)$.

(Remember that, when the divisor is a third-degree expression, the remainder is written as $ax^2 + bx + c$.)

[Sol] Let the quotient be $Q(x)$, and let the remainder be $ax^2 + bx + c$.

$$P(x) = (x+1)(x+2)(x+3)Q(x) + ax^2 + bx + c$$

$$\begin{cases} P(-1) = a - b + c = 2 & \dots \textcircled{1} \\ P(-2) = 4a - 2b + c = 3 & \dots \textcircled{2} \\ P(-3) = 9a - 3b + c = 6 & \dots \textcircled{3} \end{cases}$$

From $\textcircled{3} - \textcircled{1}$,

$$8a - 2b = 4 \quad \dots \textcircled{4}$$

From $\textcircled{2} - \textcircled{1}$,

$$3a - b = 1$$

Therefore, $b = 3a - 1 \quad \dots \textcircled{5}$

Substituting $\textcircled{5}$ into $\textcircled{4}$,

$$8a - 2(3a - 1) = 4$$

$$a = 1$$

Substituting into $\textcircled{5}$,

$$b = 2$$

Substituting into $\textcircled{1}$,

$$c = 3$$

The remainder is $x^2 + 2x + 3$

2. When $P(x)$ is divided by $x-1$, the remainder is 1. When divided by $(x-2)(x-3)$, the remainder is 5. Find the remainder when $P(x)$ is divided by $(x-1)(x-2)(x-3)$.

[Sol] Let the quotient be $Q(x)$, and let the remainder be $ax^2 + bx + c$ \textcircled{a}

$$P(x) = (x-1)(x-2)(x-3)Q(x) + ax^2 + bx + c \quad \dots\textcircled{a}$$

$$\begin{cases} P(1) = a + b + c = 1 & \dots\textcircled{1} \\ P(2) = 4a + 2b + c = 5 & \dots\textcircled{2} \\ P(3) = 9a + 3b + c = 5 & \dots\textcircled{3} \end{cases}$$

From $\textcircled{3} - \textcircled{1}$,

$$8a + 2b = 4 \quad \dots\textcircled{4}$$

From $\textcircled{2} - \textcircled{1}$,

$$3a + b = 4$$

Therefore, $b = -3a + 4$... $\textcircled{5}$

Substituting $\textcircled{5}$ into $\textcircled{4}$,

$$8a + 2(-3a + 4) = 4$$

$$a = -2$$

Substituting into $\textcircled{5}$,

$$b = 10$$

Substituting into $\textcircled{1}$,

$$c = -7$$

The remainder is $-2x^2 + 10x - 7$

Note: When $P(x)$ is divided by $(x-2)(x-3)$, the remainder is 5, so $P(x)$ can be written in the form

$$P(x) = (x-2)(x-3)Q_1(x) + 5$$

Substituting $x = 2$ into this equation gives $P(2) = 5$. Also, substituting $x = 2$ into equation \textcircled{a} above gives $P(2) = 4a + 2b + c$. Therefore we can conclude that $P(2) = 4a + 2b + c = 5$.

Remainder Theorem

1. Find the remainder when $P(x) = x^{10} - 3$ is divided by $x^2 - 1$.

[Sol] Let the quotient be $Q(x)$, and let the remainder be $ax + b$.

$$\begin{aligned} P(x) = x^{10} - 3 &= (x^2 - 1)Q(x) + ax + b \\ &= (x+1)(x-1)Q(x) + ax + b \end{aligned}$$

$$\begin{cases} P(1) = 1^{10} - 3 = -2 = a + b & \dots \textcircled{1} \\ P(-1) = (-1)^{10} - 3 = -2 = -a + b & \dots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}$,

$$a = 0, b = -2$$

The remainder is -2

J 168b

2. Find the remainder when $P(x) = 32x^5 - 1$ is divided by $4x^2 - 1$.

[Sol] Let the quotient be $Q(x)$, and let the remainder be $ax + b$.

$$\begin{aligned}P(x) = 32x^5 - 1 &= (4x^2 - 1)Q(x) + ax + b \\ &= (2x + 1)(2x - 1)Q(x) + ax + b\end{aligned}$$

$$\left\{ \begin{aligned}P\left(\frac{1}{2}\right) &= 32 \times \left(\frac{1}{2}\right)^5 - 1 = 0 = \frac{1}{2}a + b \quad \dots \textcircled{1}\end{aligned} \right.$$

$$\left\{ \begin{aligned}P\left(-\frac{1}{2}\right) &= 32 \times \left(-\frac{1}{2}\right)^5 - 1 = -2 = -\frac{1}{2}a + b \quad \dots \textcircled{2}\end{aligned} \right.$$

From $\textcircled{1}$ and $\textcircled{2}$,

$$a = 2, \quad b = -1$$

The remainder is $2x - 1$

J 169b

1. Find the value of $x^4 - x^2 + 6x - 4$ when $x = \frac{1 + \sqrt{3}i}{2}$.

[Sol] From $x = \frac{1 + \sqrt{3}i}{2}$,

$$2x - 1 = \sqrt{3}i \quad \dots \textcircled{1}$$

Squaring both sides of $\textcircled{1}$,

$$4x^2 - 4x + 1 = -3$$

$$4x^2 - 4x + 4 = 0$$

$$x^2 - x + 1 = 0 \quad \dots \textcircled{2}$$

Let $P(x) = x^4 - x^2 + 6x - 4$

When $P(x)$ is divided by $x^2 - x + 1$,

the quotient is: $x^2 + x - 1$

and the remainder is: $4x - 3$

Thus, $P(x) = (x^2 - x + 1)(x^2 + x - 1) + 4x - 3 \quad \dots \textcircled{3}$

From $\textcircled{2}$ and $\textcircled{3}$,

$$P\left(\frac{1 + \sqrt{3}i}{2}\right) = 4 \times \left(\frac{1 + \sqrt{3}i}{2}\right) - 3 = -1 + 2\sqrt{3}i$$

$$\begin{array}{r} x^2 + x - 1 \\ x^2 - x + 1 \overline{) x^4 \quad - x^2 + 6x - 4} \\ \underline{x^4 - x^3 + x^2} \\ x^3 - 2x^2 + 6x \\ \underline{x^3 - x^2 + x} \\ -x^2 + 5x - 4 \\ \underline{-x^2 + x - 1} \\ 4x - 3 \end{array}$$

J 170a KUMON

Remainder Theorem

1. Find the value of a for each of the following cases.

(1) When $x^2 - 2x + a$ is divisible by $x + 2$.

$$\text{Let } P(x) = x^2 - 2x + a$$

$$P(-2) = 4 + 4 + a = 0$$

$$\mathbf{a = -8}$$

(2) When $x^3 - 3x^2 + 4x + a$ is divisible by $x - 2$.

$$\text{Let } P(x) = x^3 - 3x^2 + 4x + a$$

$$P(2) = 8 - 12 + 8 + a = 0$$

$$\mathbf{a = -4}$$

(3) When $x^3 + ax^2 + 3x + 1$ is divided by $x + 3$ with remainder 1.

$$\text{Let } P(x) = x^3 + ax^2 + 3x + 1$$

$$P(-3) = -27 + 9a - 9 + 1 = 1$$

$$9a = 36$$

$$\mathbf{a = 4}$$

(4) When $3x^3 - 8x^2 + ax - 4$ is divided by $3x - 2$ with remainder -2 .

$$\text{Let } P(x) = 3x^3 - 8x^2 + ax - 4$$

$$P\left(\frac{2}{3}\right) = \frac{8}{9} - \frac{32}{9} + \frac{2}{3}a - 4 = -2$$

$$\frac{2}{3}a = \frac{14}{3}$$

$$\mathbf{a = 7}$$

J 170b

2. When $P(x)$ is divided by $x-1$, the remainder is 3. When divided by $x+2$, the remainder is 6. When divided by $x-3$, the remainder is 11. Find the remainder when $P(x)$ is divided by $(x-1)(x+2)(x-3)$.

[Sol] Let the quotient be $Q(x)$, and let the remainder be ax^2+bx+c .

$$P(x) = (x-1)(x+2)(x-3)Q(x) + ax^2 + bx + c$$

$$\begin{cases} P(1) = a + b + c = 3 & \dots \textcircled{1} \\ P(-2) = 4a - 2b + c = 6 & \dots \textcircled{2} \\ P(3) = 9a + 3b + c = 11 & \dots \textcircled{3} \end{cases}$$

From $\textcircled{3} - \textcircled{1}$,

$$8a + 2b = 8 \quad \dots \textcircled{4}$$

From $\textcircled{2} - \textcircled{1}$,

$$3a - 3b = 3$$

$$a - b = 1$$

Therefore, $b = a - 1 \quad \dots \textcircled{5}$

Substituting $\textcircled{5}$ into $\textcircled{4}$,

$$8a + 2(a - 1) = 8$$

$$a = 1$$

Substituting into $\textcircled{5}$,

$$b = 0$$

Substituting into $\textcircled{1}$,

$$c = 2$$

The remainder is $x^2 + 2$

Factor Theorem

Solve the following equations.

Ex.

$$x^3 - x^2 + x - 1 = 0$$

$$[\text{Sol}] x^2(x-1) + (x-1) = 0$$

$$(x-1)(x^2+1) = 0$$

$$x = 1 \text{ or } x^2 = -1$$

$$\text{Thus, } x = 1, \pm i$$

$$x-1 = 0 \text{ or } x^2+1 = 0$$

$$\text{If } x^2 = -1, \text{ then } x = \pm i.$$

$$(1) \quad x^3 - 2x^2 + x - 2 = 0$$

$$x^2(x-2) + (x-2) = 0$$

$$(x-2)(x^2+1) = 0$$

$$x = 2, \pm i$$

$$(2) \quad 2x^3 - 5x^2 + 5x - 2 = 0$$

$$2(x^3 - 1) - 5x(x-1) = 0$$

$$2(x-1)(x^2+x+1) - 5x(x-1) = 0$$

$$(x-1)(2x^2 - 3x + 2) = 0$$

$$x = 1, \frac{3 \pm \sqrt{7}i}{4}$$

Use the formulas

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

[J47a]

J171b

$$(3) \quad x^3 + 1 = 0$$

$$(x+1)(x^2 - x + 1) = 0$$

$$\mathbf{x = -1, \frac{1 \pm \sqrt{3}i}{2}}$$

Use the formulas

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

[J47a]

$$(4) \quad x^3 - 1 = 0$$

$$(x-1)(x^2 + x + 1) = 0$$

$$\mathbf{x = 1, \frac{-1 \pm \sqrt{3}i}{2}}$$

$$(5) \quad x^3 - 8 = 0$$

$$(x-2)(x^2 + 2x + 4) = 0$$

$$\mathbf{x = 2, -1 \pm \sqrt{3}i}$$

$$(6) \quad x^4 - 1 = 0$$

$$(x^2 + 1)(x^2 - 1) = 0$$

$$\mathbf{x = \pm i, \pm 1}$$

Factor Theorem

When $P(a) = 0$, then $P(x)$ can be divided exactly by $x - a$. From this, we can conclude:

The Factor Theorem

$P(x)$ has a factor $(x - a) \Leftrightarrow P(a) = 0$
 (\Leftrightarrow means “if and only if”)

Factorise $P(x)$, using the *Factor Theorem*.

Ex.

$$P(x) = x^3 - 7x + 6$$

[Sol] From $P(1) = 1^3 - 7 \times 1 + 6 = 0$



Try the factors of 6:

$(\pm 1, \pm 2, \pm 3, \pm 6)$ until you find one that makes $P(x) = 0$.

So $P(x)$ is divisible by $x - 1$.

Therefore,

$$\begin{aligned} P(x) &= (x-1)(x^2 + x - 6) \\ &= (x-1)(x+3)(x-2) \end{aligned}$$

$$\begin{array}{r} x^2 + x - 6 \\ x-1 \overline{) x^3 + 0x^2 - 7x + 6} \\ \underline{x^3 - x^2} \\ x^2 - 7x \\ \underline{x^2 - x} \\ -6x + 6 \\ \underline{-6x + 6} \\ 0 \end{array}$$

(1) $P(x) = x^3 - 3x - 2$

From $P(-1) = (-1)^3 - 3 \times (-1) - 2 = 0$

$P(x)$ is divisible by $(x + 1)$.

Therefore,

$$\begin{aligned} P(x) &= (x+1)(x^2 - x - 2) \\ &= (x+1)^2(x-2) \end{aligned}$$

$$\begin{array}{r} x^2 - x - 2 \\ x+1 \overline{) x^3 + 0x^2 - 3x - 2} \\ \underline{x^3 + x^2} \\ -x^2 - 3x \\ \underline{-x^2 - x} \\ -2x - 2 \\ \underline{-2x - 2} \\ 0 \end{array}$$

Alternative Solution

$$P(2) = 0$$

$$P(x) = (x-2)(x+1)^2$$

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(2) $P(x) = x^3 + 2x^2 - 3$

From $P(1) = 1^3 + 2 \times 1^2 - 3 = 0$

$P(x)$ is divisible by $(x-1)$.

Therefore,

$$P(x) = (x-1)(x^2 + 3x + 3)$$

$$\begin{array}{r} x^2 + 3x + 3 \\ x-1 \overline{) x^3 + 2x^2 + 0x - 3} \\ \underline{x^3 - x^2} \\ 3x^2 - 3x \\ \underline{3x - 3} \\ 3x - 3 \\ \underline{3x - 3} \\ 0 \end{array}$$

(3) $P(x) = x^3 + 6x^2 + 11x + 6$

$P(-1) = (-1)^3 + 6 \times (-1)^2 + 11 \times (-1) + 6 = 0$

$P(x)$ is divisible by $(x+1)$.

Therefore,

$$\begin{aligned} P(x) &= (x+1)(x^2 + 5x + 6) \\ &= (x+1)(x+3)(x+2) \end{aligned}$$

$$\begin{array}{r} x^2 + 5x + 6 \\ x+1 \overline{) x^3 + 6x^2 + 11x + 6} \\ \underline{x^3 + x^2} \\ 5x^2 + 11x \\ \underline{5x^2 + 5x} \\ 6x + 6 \\ \underline{6x + 6} \\ 0 \end{array}$$

(4) $P(x) = x^3 + 3x^2 - x - 3$

$P(1) = 1^3 + 3 \times 1^2 - 1 - 3 = 0$

$P(x)$ is divisible by $(x-1)$.

Therefore,

$$\begin{aligned} P(x) &= (x-1)(x^2 + 4x + 3) \\ &= (x-1)(x+3)(x+1) \end{aligned}$$

$$\begin{array}{r} x^2 + 4x + 3 \\ x-1 \overline{) x^3 + 3x^2 - x - 3} \\ \underline{x^3 - x^2} \\ 4x^2 - x \\ \underline{4x^2 - 4x} \\ 3x - 3 \\ \underline{3x - 3} \\ 0 \end{array}$$

Factor Theorem

Factorise $P(x)$, using the *Factor Theorem*.

Ex.

$$P(x) = x^3 + 3x^2 - 71x - 10$$

$$\begin{aligned} \text{[Sol]} \quad P(-10) &= (-10)^3 + 3 \times (-10)^2 - 71 \times (-10) - 10 \\ &= 0 \end{aligned}$$

This is a factor of the constant term -10 .

Therefore,

$$P(x) = (x+10)(x^2 - 7x - 1)$$

► It is unnecessary to substitute numbers other than the factors of 10.

$$\begin{array}{r} x^2 - 7x - 1 \\ x+10 \overline{) x^3 + 3x^2 - 71x - 10} \\ \underline{x^3 + 10x^2} \\ -7x^2 - 71x \\ \underline{-7x^2 - 70x} \\ -x - 10 \\ \underline{-x - 10} \\ 0 \end{array}$$

Note: If $P(x) = x^3 + 3x^2 - 71x - 10$ has a factor $x - a$ (where a is an integer), then a is a factor of the constant term, -10 . Try $\pm 1, \pm 2, \pm 5, \pm 10$.

$$(1) \quad P(x) = x^3 - 4x^2 - 4x - 5$$

$$P(5) = 5^3 - 4 \times 5^2 - 4 \times 5 - 5 = 0$$

Therefore,

$$P(x) = (x-5)(x^2 + x + 1)$$

$$\begin{array}{r} x^2 + x + 1 \\ x-5 \overline{) x^3 - 4x^2 - 4x - 5} \\ \underline{x^3 - 5x^2} \\ x^2 - 4x \\ \underline{x^2 - 5x} \\ x - 5 \\ \underline{x - 5} \\ 0 \end{array}$$

$$(2) \quad P(x) = x^3 - 7x^2 + 7x - 6$$

$$P(6) = 6^3 - 7 \times 6^2 + 7 \times 6 - 6 = 0$$

Therefore,

$$P(x) = (x-6)(x^2 - x + 1)$$

$$\begin{array}{r} x^2 - x + 1 \\ x-6 \overline{) x^3 - 7x^2 + 7x - 6} \\ \underline{x^3 - 6x^2} \\ -x^2 + 7x \\ \underline{-x^2 + 6x} \\ x - 6 \\ \underline{x - 6} \\ 0 \end{array}$$

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$$(3) \quad P(x) = x^3 + 4x^2 - 8$$

$$P(-2) = (-2)^3 + 4 \times (-2)^2 - 8 = 0$$

Therefore,

$$P(x) = (x+2)(x^2+2x-4)$$

$$\begin{array}{r} x^2+2x-4 \\ x+2 \overline{) x^3+4x^2+0x-8} \\ \underline{x^3+2x^2} \\ 2x^2+4x \\ \underline{-4x-8} \\ -4x-8 \\ \underline{-4x-8} \\ 0 \end{array}$$

$$(4) \quad P(x) = x^3 - 4x + 15$$

$$P(-3) = (-3)^3 - 4 \times (-3) + 15 = 0$$

Therefore,

$$P(x) = (x+3)(x^2-3x+5)$$

$$\begin{array}{r} x^2-3x+5 \\ x+3 \overline{) x^3+0x^2-4x+15} \\ \underline{x^3+3x^2} \\ -3x^2-4x \\ \underline{-3x^2-9x} \\ 5x+15 \\ \underline{5x+15} \\ 0 \end{array}$$

Factor Theorem

Factorise by applying the *Factor Theorem*.

Ex.

$$P(x) = 2x^3 - 3x^2 - x - 10 \quad \rightarrow \quad \text{This may be factorised as } (2x - \square)(x^2 + \Delta x + \square).$$

$$[\text{Sol}] \quad P\left(\frac{5}{2}\right) = 2 \times \left(\frac{5}{2}\right)^3 - 3 \times \left(\frac{5}{2}\right)^2 - \frac{5}{2} - 10 \quad \rightarrow \quad \text{Try } \pm \frac{1}{2}, \pm \frac{5}{2}.$$

$$= 0$$

Therefore,

$$P(x) = (2x - 5)(x^2 + x + 2)$$

$$\begin{array}{r} x^2 + x + 2 \\ 2x - 5 \overline{) 2x^3 - 3x^2 - x - 10} \\ \underline{2x^3 - 5x^2} \\ 2x^2 - x \\ \underline{2x^2 - 5x} \\ 4x - 10 \\ \underline{4x - 10} \\ 0 \end{array}$$

Note: The numerator of the fraction substituted is the 'O' in $(2x - \square)(x^2 + \Delta x + \square)$, and must be a factor of 10.

$$(1) \quad P(x) = 2x^3 + x^2 + x - 1$$

$$P\left(\frac{1}{2}\right) = 2 \times \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^2 + \frac{1}{2} - 1 = 0$$

Therefore,

$$P(x) = (2x - 1)(x^2 + x + 1)$$

$$\begin{array}{r} x^2 + x + 1 \\ 2x - 1 \overline{) 2x^3 + x^2 + x - 1} \\ \underline{2x^3 - x^2} \\ 2x^2 + x \\ \underline{2x^2 - x} \\ 2x - 1 \\ \underline{2x - 1} \\ 0 \end{array}$$

$$(2) \quad P(x) = 2x^3 - x^2 + 9$$

$$P\left(-\frac{3}{2}\right) = 2 \times \left(-\frac{3}{2}\right)^3 - \left(-\frac{3}{2}\right)^2 + 9 = 0$$

Therefore,

$$P(x) = (2x + 3)(x^2 - 2x + 3)$$

$$\begin{array}{r} x^2 - 2x + 3 \\ 2x + 3 \overline{) 2x^3 - x^2 + 0x + 9} \\ \underline{2x^3 + 3x^2} \\ -4x^2 \\ \underline{-4x^2 - 6x} \\ 6x + 9 \\ \underline{6x + 9} \\ 0 \end{array}$$

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(3) $P(x) = 2x^3 + 3x^2 + 2x - 2$

$$P\left(\frac{1}{2}\right) = 2 \times \left(\frac{1}{2}\right)^3 + 3 \times \left(\frac{1}{2}\right)^2 + 2 \times \frac{1}{2} - 2 = 0$$

Therefore,

$$P(x) = (2x - 1)(x^2 + 2x + 2)$$

$$\begin{array}{r}
 2x-1 \overline{) 2x^3 + 3x^2 + 2x - 2} \\
 \underline{2x^3 - x^2} \\
 4x^2 + 2x \\
 \underline{4x^2 - 2x} \\
 4x - 2 \\
 \underline{4x - 2} \\
 0
 \end{array}$$

This coefficient is 3,
so try $\pm \frac{1}{3}, \pm \frac{2}{3}$.

(4) $P(x) = 3x^3 - 4x^2 - 7x - 2$

$$P\left(-\frac{2}{3}\right) = 3 \times \left(-\frac{2}{3}\right)^3 - 4 \times \left(-\frac{2}{3}\right)^2 - 7 \times \left(-\frac{2}{3}\right) - 2 = 0$$

Therefore,

$$P(x) = (3x + 2)(x^2 - 2x - 1)$$

$$\begin{array}{r}
 3x+2 \overline{) 3x^3 - 4x^2 - 7x - 2} \\
 \underline{3x^3 + 2x^2} \\
 -6x^2 - 7x \\
 \underline{-6x^2 - 4x} \\
 -3x - 2 \\
 \underline{-3x - 2} \\
 0
 \end{array}$$

Factor Theorem

Solve the following equations.


Ex.

$$3x^3 + 7x^2 - 4 = 0$$

[Sol] Let $P(x) = 3x^3 + 7x^2 - 4$,

$$P(-1) = 0, \text{ so}$$

$$\begin{aligned} P(x) &= (x+1)(3x^2 + 4x - 4) \\ &= (x+1)(3x-2)(x+2) \end{aligned}$$

 $P(x)$ is divisible by $x+1$.

$$\text{Solving } (x+1)(3x-2)(x+2) = 0,$$


$$x = -1, \frac{2}{3}, -2$$

(1) $3x^3 - 2x^2 - 12x + 8 = 0$

Let $P(x) = 3x^3 - 2x^2 - 12x + 8$

$$P(2) = 0, \text{ so}$$

$$\begin{aligned} P(x) &= (x-2)(3x^2 + 4x - 4) \\ &= (x-2)(3x-2)(x+2) \end{aligned}$$

 $P(x)$ is divisible by $x-2$.

$$\text{Solving } (x-2)(3x-2)(x+2) = 0,$$

$$x = 2, \frac{2}{3}, -2$$

$$\begin{array}{r} + 4x - 4 \\ x-2 \overline{) 3x^3 - 2x^2 - 12x + 8} \\ \underline{3x^3 - 6x^2} \\ 4x^2 - 12x \\ \underline{4x^2 - 8x} \\ -4x + 8 \\ \underline{-4x + 8} \\ 0 \end{array}$$

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$$(2) \quad 12x^3 + 4x^2 - 3x - 1 = 0$$

$$\text{Let } P(x) = 12x^3 + 4x^2 - 3x - 1$$

$$P\left(\frac{1}{2}\right) = 0, \text{ so}$$

\curvearrowright $P(x)$ is divisible by $2x - 1$.

$$\begin{aligned} P(x) &= (2x - 1)(6x^2 + 5x + 1) \\ &= (2x - 1)(3x + 1)(2x + 1) \end{aligned}$$

$$\begin{array}{r} 6x^2 + 5x + 1 \\ 2x - 1 \overline{) 12x^3 + 4x^2 - 3x - 1} \\ \underline{12x^3 - 6x^2} \\ 10x^2 - 3x \\ \underline{10x^2 - 5x} \\ 2x - 1 \\ \underline{2x - 1} \\ 0 \end{array}$$

$$\text{Solving } (2x - 1)(3x + 1)(2x + 1) = 0,$$

$$x = \frac{1}{2}, -\frac{1}{3}, -\frac{1}{2}$$

$$(3) \quad 4x^3 - 4x^2 - 7x + 6 = 0$$

$$\text{Let } P(x) = 4x^3 - 4x^2 - 7x + 6$$

$$P\left(\frac{3}{2}\right) = 0, \text{ so}$$

\curvearrowright $P(x)$ is divisible by $2x - 3$.

$$P(x) = (2x - 3)(2x^2 + x - 2)$$

$$\text{Solving } (2x - 3)(2x^2 + x - 2) = 0,$$

$$x = \frac{3}{2}, \frac{-1 \pm \sqrt{17}}{4}$$

$$\begin{array}{r} 2x^2 + x - 2 \\ 2x - 3 \overline{) 4x^3 - 4x^2 - 7x + 6} \\ \underline{4x^3 - 6x^2} \\ 2x^2 - 7x \\ \underline{2x^2 - 3x} \\ -4x + 6 \\ \underline{-4x + 6} \\ 0 \end{array}$$

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Factor Theorem

Solve the following equations.

$$(1) \quad 3x^3 + 5x^2 - 4x - 4 = 0$$

$$\text{Let } P(x) = 3x^3 + 5x^2 - 4x - 4$$

$$P(1) = 0, \text{ so}$$

$$\begin{aligned} P(x) &= (x-1)(3x^2 + 8x + 4) \\ &= (x-1)(3x+2)(x+2) \end{aligned}$$

$$\text{Solving } (x-1)(3x+2)(x+2) = 0,$$

$$x = 1, -\frac{2}{3}, -2$$

$$\begin{array}{r} x-1 \overline{) 3x^3 + 5x^2 - 4x - 4} \\ \underline{3x^3 - 3x^2} \\ 8x^2 - 4x \\ \underline{8x^2 - 8x} \\ 4x - 4 \\ \underline{4x - 4} \\ 0 \end{array}$$

$$(2) \quad 2x^3 - x^2 - 15x + 18 = 0$$

$$\text{Let } P(x) = 2x^3 - x^2 - 15x + 18$$

$$P(2) = 0, \text{ so}$$

$$\begin{aligned} P(x) &= (x-2)(2x^2 + 3x - 9) \\ &= (x-2)(2x-3)(x+3) \end{aligned}$$

$$\text{Solving } (x-2)(2x-3)(x+3) = 0,$$

$$x = 2, \frac{3}{2}, -3$$

$$\begin{array}{r} x-2 \overline{) 2x^3 - x^2 - 15x + 18} \\ \underline{2x^3 + 3x - 9} \\ 3x^2 - 4x^2 - 15x + 18 \\ \underline{3x^2 - 6x} \\ -9x + 18 \\ \underline{-9x + 18} \\ 0 \end{array}$$

Ex.

$$x^4 + x^3 - 2x^2 - x + 1 = 0$$

[Sol] Let $P(x) = x^4 + x^3 - 2x^2 - x + 1$

$$P(1) = 0, \text{ so}$$

$$P(x) = (x-1)(x^3 + 2x^2 - 1) \dots \textcircled{1}$$

Let $Q(x) = x^3 + 2x^2 - 1$  Factorise $x^3 + 2x^2 - 1$ using the **Factor Theorem**.

$$Q(-1) = 0, \text{ so}$$

$$Q(x) = (x+1)(x^2 + x - 1) \dots \textcircled{2}$$

From $\textcircled{1}$ and $\textcircled{2}$,

$$P(x) = (x-1)Q(x) \\ = (x-1)(x+1)(x^2 + x - 1)$$

Solving $(x-1)(x+1)(x^2 + x - 1) = 0$,

$$x = 1, -1, \frac{-1 \pm \sqrt{5}}{2}$$

$$\begin{array}{r} x^3 + 2x^2 - 1 \\ x-1 \overline{) x^4 + x^3 - 2x^2 - x + 1} \\ \underline{x^4 - x^3} \\ 2x^3 - 2x^2 \\ \underline{2x^3 - 2x^2} \\ -x + 1 \\ \underline{-x + 1} \\ 0 \end{array}$$

(3) $x^4 + x^3 - 8x^2 + 2x + 4 = 0$

Let $P(x) = x^4 + x^3 - 8x^2 + 2x + 4$

$$P(1) = 0, \text{ so}$$

$$P(x) = (x-1)(x^3 + 2x^2 - 6x - 4) \dots \textcircled{1}$$

Let $Q(x) = x^3 + 2x^2 - 6x - 4$

$$Q(2) = 0, \text{ so}$$

$$Q(x) = (x-2)(x^2 + 4x + 2) \dots \textcircled{2}$$

From $\textcircled{1}$ and $\textcircled{2}$,

$$P(x) = (x-1)Q(x) \\ = (x-1)(x-2)(x^2 + 4x + 2)$$

Solving $(x-1)(x-2)(x^2 + 4x + 2) = 0$,

$$x = 1, 2, -2 \pm \sqrt{2}$$

$$\begin{array}{r} x^3 + 2x^2 - 6x - 4 \\ x-1 \overline{) x^4 + x^3 - 8x^2 + 2x + 4} \\ \underline{x^4 - x^3} \\ 2x^3 - 8x^2 \\ \underline{2x^3 - 2x^2} \\ -6x^2 + 2x \\ \underline{-6x^2 + 6x} \\ -4x + 4 \\ \underline{-4x + 4} \\ 0 \end{array}$$

$$\begin{array}{r} x^2 + 4x + 2 \\ x-2 \overline{) x^3 + 2x^2 - 6x - 4} \\ \underline{x^3 - 2x^2} \\ 4x^2 - 6x \\ \underline{4x^2 - 8x} \\ 2x - 4 \\ \underline{2x - 4} \\ 0 \end{array}$$

Factor Theorem

1. Solve the following equations.

Ex.

$$(x-2)(x-5)(x-7) = 8 \cdot 5 \cdot 3$$

[Sol] Let $P(x) = (x-2)(x-5)(x-7) - 8 \cdot 5 \cdot 3 \dots \textcircled{1}$

$$P(10) = 8 \cdot 5 \cdot 3 - 8 \cdot 5 \cdot 3 = 0$$

Therefore, $x-10$ is a factor of $P(x)$.

$$P(x) = x^3 - 14x^2 + 59x - 70 - 120 \quad \text{Expanding } \textcircled{1}.$$

$$= x^3 - 14x^2 + 59x - 190$$

$$= (x-10)(x^2 - 4x + 19) \quad \text{Find the other factor by long division.}$$

Solving $(x-10)(x^2 - 4x + 19) = 0$,

$$x = 10, 2 \pm \sqrt{15}i$$

(1) $x(x+1)(x+2) = 3 \cdot 4 \cdot 5$

Let $P(x) = x(x+1)(x+2) - 3 \cdot 4 \cdot 5$

$$P(3) = 3 \cdot 4 \cdot 5 - 3 \cdot 4 \cdot 5 = 0$$

Therefore, $x-3$ is a factor of $P(x)$.

$$P(x) = x^3 + 3x^2 + 2x - 60$$

$$= (x-3)(x^2 + 6x + 20)$$

Solving $(x-3)(x^2 + 6x + 20) = 0$,

$$x = 3, -3 \pm \sqrt{11}i$$

$$\begin{array}{r} x^2 + 6x + 20 \\ x-3 \overline{) x^3 + 3x^2 + 2x - 60} \\ \underline{x^3 - 3x^2} \\ 6x^2 + 2x \\ \underline{6x^2 - 18x} \\ 20x - 60 \\ \underline{20x - 60} \\ 0 \end{array}$$

Ex.

One root of $x^3 - 6x^2 + ax - 6 = 0$ is 3. Find the value of a and then the other roots.

[Sol] Let $P(x) = x^3 - 6x^2 + ax - 6$

$$P(3) = 3a - 33 = 0, \text{ so } \Rightarrow \text{Substitute } x = 3.$$

$$a = 11$$

Therefore,

$$P(x) = x^3 - 6x^2 + 11x - 6 \Rightarrow \text{Substitute } a = 11.$$

$$= (x-3)(x^2 - 3x + 2) \Rightarrow x-3 \text{ is a factor of } P(x).$$

$$= (x-3)(x-2)(x-1)$$

Solving $(x-3)(x-2)(x-1) = 0$,

$$x = 3, 2, 1$$

Ans. $a = 11$ and the other roots are $x = 2, 1$

2. One root of $x^3 + ax^2 + 6x - 2 = 0$ is 1. Find the value of a and then the other roots.

[Sol] Let $P(x) = x^3 + ax^2 + 6x - 2$

$$P(1) = a + 5 = 0, \text{ so}$$

$$a = -5$$

Therefore,

$$P(x) = x^3 - 5x^2 + 6x - 2$$

$$= (x-1)(x^2 - 4x + 2)$$

Solving $(x-1)(x^2 - 4x + 2) = 0$,

$$x = 1, 2 \pm \sqrt{2}$$

$$\begin{array}{r} x^2 - 4x + 2 \\ x-1 \overline{) x^3 - 5x^2 + 6x - 2} \\ \underline{x^3 - x^2} \\ -4x^2 + 6x \\ \underline{-4x^2 + 4x} \\ 2x - 2 \\ \underline{2x - 2} \\ 0 \end{array}$$

Ans. $a = -5$ and other roots are $x = 2 \pm \sqrt{2}$

Factor Theorem

1. Two of the roots of $x^4 + ax^3 + ax^2 + 11x + b = 0$ are 3 and -2 .

(1) Find the values of a and b .

$$\text{Let } P(x) = x^4 + ax^3 + ax^2 + 11x + b,$$

$$P(3) = 0, \text{ so } 36a + b = -114 \dots \textcircled{1}$$

$$P(-2) = 0, \text{ so } 4a - b = -6 \dots \textcircled{2}$$

From $\textcircled{1}$ and $\textcircled{2}$,

$$a = -3, b = -6$$

Ans. $a = -3, b = -6$

(2) Find the other roots.

Substituting a and b into the original equation,

$$P(x) = x^4 - 3x^3 - 3x^2 + 11x - 6$$

$$= (x-3)(x+2)(x^2 - 2x + 1) \quad \rightarrow$$

$$= (x-3)(x+2)(x-1)^2$$

Solving $(x-3)(x+2)(x-1)^2 = 0$,

$$x = 3, -2, 1$$

Using the known roots, 3 and -2 ,
 $(x-3)(x+2) = x^2 - x - 6$

$$\begin{array}{r} x^2 - 2x + 1 \\ x^2 - x - 6 \overline{) x^4 - 3x^3 - 3x^2 + 11x - 6} \\ \underline{x^4 - x^3 - 6x^2} \\ -2x^3 + 3x^2 + 11x \\ \underline{-2x^3 + 2x^2 + 12x} \\ x^2 - x - 6 \\ \underline{x^2 - x - 6} \\ 0 \end{array}$$

Ans. The other root is $x = 1$

J 178b

2. Two of the roots of $x^4 + ax^3 + (a+3)x^2 + 16x + b = 0$ are 1 and 3.
Find the values of a and b and the other roots.

[Sol] Let $P(x) = x^4 + ax^3 + (a+3)x^2 + 16x + b$

$$P(1) = 0, \text{ so } 2a + b = -20 \quad \dots \textcircled{1}$$

$$P(3) = 0, \text{ so } 36a + b = -156 \quad \dots \textcircled{2}$$

From $\textcircled{1}$ and $\textcircled{2}$,

$$a = -4, b = -12$$

$$P(x) = x^4 - 4x^3 - x^2 + 16x - 12$$

$$= (x-1)(x-3)(x^2-4)$$

$$= (x-1)(x-3)(x+2)(x-2)$$



Using the known roots, 1 and 3,
 $(x-1)(x-3) = x^2 - 4x + 3$

Solving $(x-1)(x-3)(x+2)(x-2) = 0$,

$$x = 1, 3, \pm 2$$

$$\begin{array}{r} x^2 \quad -4 \\ x^2 - 4x + 3 \overline{) x^4 - 4x^3 - x^2 + 16x - 12} \\ \underline{x^4 - 4x^3 + 3x^2} \\ -4x^2 + 16x - 12 \\ \underline{-4x^2 + 16x - 12} \\ 0 \end{array}$$

Ans. $a = -4, b = -12$, and the other roots are $x = \pm 2$

Factor Theorem

1. Solve $2x^4 - 3x^3 - x^2 - 3x + 2 = 0$ using the following method.

(1) Dividing both sides by x^2 ($x \neq 0$) gives $2x^2 - 3x - 1 - \frac{3}{x} + \frac{2}{x^2} = 0$.

Express the left hand side in terms of t , where $t = x + \frac{1}{x}$.

$$\begin{aligned}
 \text{[Sol]} \quad & 2x^2 - 3x - 1 - \frac{3}{x} + \frac{2}{x^2} \\
 &= 2\left(x^2 + \frac{1}{x^2}\right) - 3\left(x + \frac{1}{x}\right) - 1 \\
 &= 2\left[\left(x + \frac{1}{x}\right)^2 - 2\right] - 3\left(x + \frac{1}{x}\right) - 1 \\
 &= 2\left(x + \frac{1}{x}\right)^2 - 3\left(x + \frac{1}{x}\right) - \boxed{5} \\
 &= 2t^2 - 3t - \boxed{5}
 \end{aligned}$$

(2) Find the value of t first, and then x .

$$\begin{aligned}
 \text{[Sol]} \quad & 2t^2 - 3t - \boxed{5} \\
 &= (2t - \boxed{5})(t + \boxed{1}) = 0 \\
 t &= \frac{\boxed{5}}{2}, \boxed{-1}
 \end{aligned}$$

(i) If $x + \frac{1}{x} = \frac{\boxed{5}}{2}$

Multiply both sides by $2x$,

$$\begin{aligned}
 2x^2 - 5x + 2 &= 0 \\
 (2x - 1)(x - 2) &= 0 \\
 x &= \frac{1}{2}, 2
 \end{aligned}$$

(ii) If $x + \frac{1}{x} = \boxed{-1}$

Multiply both sides by x ,

$$\begin{aligned}
 x^2 + x + 1 &= 0 \\
 x &= \frac{-1 \pm \sqrt{3}i}{2}
 \end{aligned}$$

Ans. $x = \frac{1}{2}, 2, \frac{-1 \pm \sqrt{3}i}{2}$

J 179b

2. Solve the equation $2x^4 - x^3 - 6x^2 - x + 2 = 0$. (Hint: Let $x + \frac{1}{x} = t$)

[Sol] Divide both sides by x^2 ($x \neq 0$),

$$2x^2 - x - 6 - \frac{1}{x} + \frac{2}{x^2} = 0$$

Express the left hand side in terms of t ,

$$\begin{aligned} & 2x^2 - x - 6 - \frac{1}{x} + \frac{2}{x^2} \\ &= 2\left(x + \frac{1}{x}\right)^2 - \left(x + \frac{1}{x}\right) - 10 \\ &= 2t^2 - t - 10 \end{aligned}$$

Find the value of t first,

$$\begin{aligned} & 2t^2 - t - 10 \\ &= (2t - 5)(t + 2) = 0 \\ & t = \frac{5}{2}, -2 \end{aligned}$$

(i) If $x + \frac{1}{x} = \frac{5}{2}$

Multiply both sides by $2x$,

$$\begin{aligned} & 2x^2 - 5x + 2 = 0 \\ & (2x - 1)(x - 2) = 0 \\ & x = \frac{1}{2}, 2 \end{aligned}$$

(ii) If $x + \frac{1}{x} = -2$

Multiply both sides by x ,

$$\begin{aligned} & x^2 + 2x + 1 = 0 \\ & (x + 1)^2 = 0 \\ & x = -1 \end{aligned}$$

Ans. $x = \frac{1}{2}, 2, -1$

J180a KUMON

Factor Theorem

1. Factorise using the *Factor Theorem*.

$$(1) \quad P(x) = x^3 + 6x^2 + 11x + 6$$

$$P(-1) = 0, \text{ so}$$

$$\begin{aligned} P(x) &= (x+1)(x^2 + 5x + 6) \\ &= (x+1)(x+3)(x+2) \end{aligned}$$

$$\begin{array}{r} x+1 \overline{)x^3 + 6x^2 + 11x + 6} \\ \underline{x^3 + x^2} \\ 5x^2 + 11x \\ \underline{5x^2 + 5x} \\ 6x + 6 \\ \underline{6x + 6} \\ 0 \end{array}$$

$$(2) \quad P(x) = 2x^3 + 3x^2 + 2x - 2$$

$$P\left(\frac{1}{2}\right) = 0, \text{ so}$$

$$P(x) = (2x-1)(x^2 + 2x + 2)$$

$$\begin{array}{r} 2x-1 \overline{)2x^3 + 3x^2 + 2x - 2} \\ \underline{2x^3 - x^2} \\ 4x^2 + 2x \\ \underline{4x^2 - 2x} \\ 4x - 2 \\ \underline{4x - 2} \\ 0 \end{array}$$

J 180b

2. Solve the following equations.

(1) $x^3 - 2x^2 + 3x - 6 = 0$

Let $P(x) = x^3 - 2x^2 + 3x - 6$

$P(2) = 0$, so

$P(x) = (x-2)(x^2+3)$

Solving $(x-2)(x^2+3) = 0$,

$x = 2, \pm\sqrt{3}i$

$$\begin{array}{r} x-2 \overline{)x^3 - 2x^2 + 3x - 6} \\ \underline{x^3 - 2x^2} \\ 3x - 6 \\ \underline{ 3x - 6} \\ 0 \end{array}$$

(2) $x^4 - 3x^3 + 3x^2 + x - 6 = 0$

Let $P(x) = x^4 - 3x^3 + 3x^2 + x - 6$

$P(-1) = 0$, so

$P(x) = (x+1)(x^3 - 4x^2 + 7x - 6) \dots \textcircled{1}$

Let $Q(x) = x^3 - 4x^2 + 7x - 6$

$Q(2) = 0$, so

$Q(x) = (x-2)(x^2 - 2x + 3) \dots \textcircled{2}$

From $\textcircled{1}$ and $\textcircled{2}$,

$P(x) = (x+1)Q(x)$

$= (x+1)(x-2)(x^2 - 2x + 3)$

Solving $(x+1)(x-2)(x^2 - 2x + 3) = 0$,

$x = -1, 2, 1 \pm \sqrt{2}i$

$$\begin{array}{r} x^3 - 4x^2 + 7x - 6 \\ x+1 \overline{)x^4 - 3x^3 + 3x^2 + x - 6} \\ \underline{x^4 + x^3} \\ -4x^3 + 3x^2 \\ \underline{ -4x^3 - 4x^2} \\ 7x^2 + x \\ \underline{ 7x^2 + 7x} \\ -6x - 6 \\ \underline{ -6x - 6} \\ 0 \end{array}$$

$$\begin{array}{r} x^2 - 2x + 3 \\ x-2 \overline{)x^3 - 4x^2 + 7x - 6} \\ \underline{x^3 - 2x^2} \\ -2x^2 + 7x \\ \underline{ -2x^2 + 4x} \\ 3x - 6 \\ \underline{ 3x - 6} \\ 0 \end{array}$$

Proof of Identities

Find the values of a , b and c so that the following equations are true for any value of x .

Ex.

$$(ax+b)(x+1) = 3x^2 + 5x + 2 \quad \text{Expand the left hand side, and arrange in powers of } x.$$

$$[\text{Sol}] ax^2 + (a+b)x + b = 3x^2 + 5x + 2$$

Comparing the coefficients of the left and right hand sides,

$$\begin{cases} a = 3 & \dots \textcircled{1} \\ a + b = 5 & \dots \textcircled{2} \\ b = 2 & \dots \textcircled{3} \end{cases}$$

From the coefficients of x^2 .From the coefficients of x .

From the constants.

From $\textcircled{1}$, $\textcircled{2}$ and $\textcircled{3}$,

$$a = 3, b = 2$$

$$(1) (ax+2b)(x-1) = 2x^2 - x - 1$$

$$ax^2 + (-a+2b)x - 2b = 2x^2 - x - 1$$

Comparing the coefficients of the left and right hand sides,

$$\begin{cases} a = 2 & \dots \textcircled{1} \\ -a + 2b = -1 & \dots \textcircled{2} \\ -2b = -1 & \dots \textcircled{3} \end{cases}$$

From $\textcircled{1}$, $\textcircled{2}$ and $\textcircled{3}$,

$$a = 2, b = \frac{1}{2}$$

J181b

$$(2) \quad x^3 - 6x^2 + 15x - 7 = (x + a)^3 + bx + c$$

Use the formula
 $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
 [J7a]

$$\begin{aligned} x^3 - 6x^2 + 15x - 7 &= x^3 + 3x^2a + 3xa^2 + a^3 + bx + c \\ &= x^3 + 3ax^2 + (3a^2 + b)x + (a^3 + c) \end{aligned}$$

Comparing the coefficients of the left and right hand sides,

$$\begin{cases} 3a = -6 & \dots \textcircled{1} \\ 3a^2 + b = 15 & \dots \textcircled{2} \\ a^3 + c = -7 & \dots \textcircled{3} \end{cases}$$

From $\textcircled{1}$, $a = -2$

Substituting $a = -2$ into $\textcircled{2}$,

$$3(-2)^2 + b = 15$$

$$12 + b = 15$$

$$b = 3$$

Substituting $a = -2$ into $\textcircled{3}$,

$$(-2)^3 + c = -7$$

$$-8 + c = -7$$

$$c = 1$$

Therefore, $\mathbf{a = -2}$, $\mathbf{b = 3}$, $\mathbf{c = 1}$

Note:

- $\textcircled{1}$ From the example, when $a = 3$ and $b = 2$,

$$(3x + 2)(x + 1) = 3x^2 + 5x + 2$$

We can substitute any value of x into both sides, e.g. taking $x = 4$,

$$\text{LHS} = (3 \cdot 4 + 2)(4 + 1) = 70$$

$$\text{RHS} = 3 \cdot 4^2 + 5 \cdot 4 + 2 = 70$$

We find $\text{LHS} = \text{RHS}$.

In fact this equation is true for any value of x : we call this an *identity*.

- $\textcircled{2}$ The equation $ax^2 + bx + c = a'x^2 + b'x + c'$ becomes an *identity* if and only if $a = a'$, $b = b'$, $c = c'$.

(Similar conditions apply for a cubic equation.)

Proof of Identities

Find the values of a , b and c so that the following equations become identities.

Ex.

$$3x^2 + 4x + a = b(x + c)^2$$



Expand the RHS and arrange in powers of x .

[Sol] $3x^2 + 4x + a = bx^2 + 2bcx + bc^2$

$$\begin{cases} 3 = b & \dots \textcircled{1} \\ 4 = 2bc & \dots \textcircled{2} \\ a = bc^2 & \dots \textcircled{3} \end{cases}$$

From $\textcircled{1}$, $\textcircled{2}$ and $\textcircled{3}$,

$$a = \frac{4}{3}, \quad b = 3, \quad c = \frac{2}{3}$$

(1) $x^3 + ax + 2 = (x + 1)(x^2 + bx + c)$

$$x^3 + ax + 2 = x^3 + (b + 1)x^2 + (b + c)x + c$$

$$\begin{cases} 0 = b + 1 & \dots \textcircled{1} \\ a = b + c & \dots \textcircled{2} \\ 2 = c & \dots \textcircled{3} \end{cases}$$

There is no x^2 term on the LHS.


From $\textcircled{1}$, $\textcircled{2}$ and $\textcircled{3}$,


$$\mathbf{a = 1, \quad b = -1, \quad c = 2}$$

Ex.

$$x^2 = a(x-1)(x-2) + b(x-1) + c$$

[Sol] When $x = 1$, $1 = c$...①  Substitute $x = 1$ in both sides.

When $x = 2$, $4 = b + c$...②  Substitute $x = 2$ in both sides.

When $x = 3$, $9 = 2a + 2b + c$...③  Substitute $x = 3$ in both sides.

From ①, ② and ③,

$$a = 1, b = 3, c = 1$$

(This method is called the *Value Substitution Method*.)

(2) $a(x-1)^2 + b(x+1)^2 + c = x^2$

When $x = 1$, $4b + c = 1$...①

When $x = -1$, $4a + c = 1$...②

When $x = 0$, $a + b + c = 0$...③

From ① - ②,

$$4b - 4a = 0$$

$$a = b$$

Substituting into ③,

$$2a + c = 0 \quad \dots\text{④}$$

From ② - ④,

$$2a = 1$$

$$a = \frac{1}{2}$$

Since $a = b$, $b = \frac{1}{2}$

Substituting $a = \frac{1}{2}$ into ④,

$$2 \cdot \frac{1}{2} + c = 0$$

$$c = -1$$

Therefore, $a = \frac{1}{2}$, $b = \frac{1}{2}$, $c = -1$

(3) $ax^2 + bx + c = 0$ (Use the values $x = 0$, $x = 1$, $x = -1$)

When $x = 0$, $c = 0$...①

When $x = 1$, $a + b + c = 0$...②

When $x = -1$, $a - b + c = 0$...③

From ①, ② and ③,

$$a = 0, b = 0, c = 0$$

Note: In (3), in order for $ax^2 + bx + c = 0$ to be an identity, the following must be true:
 $a = 0, b = 0, c = 0$.

J183a KUMON

Proof of Identities

Determine the values of a , b , and c so that the following equations become identities. (Consider carefully whether to use the *Coefficient Comparison Method* or the *Value Substitution Method*.)

$$(1) \quad (ax+b)(x+c) = 3x^2 + 5x + 2$$

Expanding the LHS,

$$ax^2 + (ac+b)x + bc = 3x^2 + 5x + 2$$

Comparing coefficients,

$$\begin{cases} a = 3 & \dots \textcircled{1} \\ ac + b = 5 & \dots \textcircled{2} \\ bc = 2 & \dots \textcircled{3} \end{cases}$$

Substituting $\textcircled{1}$ into $\textcircled{2}$,

$$3c + b = 5$$

$$b = 5 - 3c \quad \dots \textcircled{4}$$

Substituting $\textcircled{4}$ into $\textcircled{3}$,

$$(5 - 3c) \cdot c = 2$$

$$3c^2 - 5c + 2 = 0$$

$$(3c - 2)(c - 1) = 0$$

$$c = \frac{2}{3}, 1$$

$$(i) \quad \text{When } c = \frac{2}{3},$$

$$b = 5 - 3 \times \frac{2}{3} = 3$$

$$(ii) \quad \text{When } c = 1,$$

$$b = 5 - 3 = 2$$

Therefore, $\mathbf{a = 3, b = 3, c = \frac{2}{3}}$, or $\mathbf{a = 3, b = 2, c = 1}$

J 183b

$$(2) \quad ax^3 - x^2 - 4x + 5 = (2x - 1)(bx^2 + cx - 5)$$

Expanding the right hand side,

$$ax^3 - x^2 - 4x + 5 = 2bx^3 + (2c - b)x^2 - (10 + c)x + 5$$

Comparing coefficients,

$$\begin{cases} a = 2b & \dots \textcircled{1} \\ -1 = 2c - b & \dots \textcircled{2} \\ 4 = 10 + c & \dots \textcircled{3} \end{cases}$$

Solving $\textcircled{3}$ for c ,

$$c = -6 \quad \dots \textcircled{4}$$

Substituting $\textcircled{4}$ into $\textcircled{2}$,

$$-12 - b = -1$$

$$b = -11 \quad \dots \textcircled{5}$$

Substituting $\textcircled{5}$ into $\textcircled{1}$,

$$a = -22$$

Therefore, $\mathbf{a = -22}$, $\mathbf{b = -11}$, $\mathbf{c = -6}$

$$(3) \quad a(x-1)(x-2) + b(x-2)(x-3) + c(x-3)(x-1) = x$$

$$\text{When } x = 1, \quad 2b = 1 \quad \dots \textcircled{1}$$

$$\text{When } x = 2, \quad -c = 2 \quad \dots \textcircled{2}$$

$$\text{When } x = 3, \quad 2a = 3 \quad \dots \textcircled{3}$$

From $\textcircled{1}$, $\textcircled{2}$ and $\textcircled{3}$,

$$\mathbf{a = \frac{3}{2}}, \quad \mathbf{b = \frac{1}{2}}, \quad \mathbf{c = -2}$$

Note Summary

There are 2 ways to find the value of the coefficients in an identity.

① The *Coefficient Comparison Method*

② The *Value Substitution Method*

Both methods give the same results.

Proof of Identities

Ex.

Determine the values of a and b for which $x^3 + ax^2 + bx + 1$ is divisible by $x^2 + x + 1$.

[Sol] Let the quotient be $x + c$.  **Quotient must be a linear expression.**

$$\begin{aligned} x^3 + ax^2 + bx + 1 &= (x^2 + x + 1)(x + c) \\ &= x^3 + (c + 1)x^2 + (c + 1)x + c \end{aligned}$$

Expand RHS and arrange in powers of x .

Comparing coefficients,

$$\begin{cases} a = c + 1 & \dots \textcircled{1} \\ b = c + 1 & \dots \textcircled{2} \\ c = 1 & \dots \textcircled{3} \end{cases}$$

From $\textcircled{1}$, $\textcircled{2}$ and $\textcircled{3}$,

$$a = 2, \quad b = 2$$

1. Determine the values of a and b for which $x^3 + ax + b$ is divisible by $(x - 2)^2$.

[Sol] Let the quotient be $x + c$.

$$\begin{aligned} x^3 + ax + b &= (x - 2)^2(x + c) \\ &= (x^2 - 4x + 4)(x + c) \\ &= x^3 + (c - 4)x^2 + (4 - 4c)x + 4c \end{aligned}$$

Comparing coefficients,

$$\begin{cases} c - 4 = 0 & \dots \textcircled{1} \\ 4 - 4c = a & \dots \textcircled{2} \\ 4c = b & \dots \textcircled{3} \end{cases}$$

From $\textcircled{1}$, $c = 4$

Substituting into $\textcircled{2}$ and $\textcircled{3}$,

$$a = -12, \quad b = 16$$

2. Find the values of a and b so that $x^4 + 6x^3 + 7x^2 + ax + b$ is a perfect square.

Hint

[Sol] Let $x^4 + 6x^3 + 7x^2 + ax + b = (x^2 + cx + d)^2$.

$$\begin{aligned} x^4 + 6x^3 + 7x^2 + ax + b &= [(x^2 + cx) + d]^2 \\ &= (x^2 + cx)^2 + 2d(x^2 + cx) + d^2 \\ &= x^4 + 2cx^3 + (c^2 + 2d)x^2 + 2cdx + d^2 \end{aligned}$$

Comparing coefficients,

$$\begin{cases} 2c = 6 & \dots \textcircled{1} \\ c^2 + 2d = 7 & \dots \textcircled{2} \\ 2cd = a & \dots \textcircled{3} \\ d^2 = b & \dots \textcircled{4} \end{cases}$$

From $\textcircled{1}$, $c = 3$

Substituting into $\textcircled{2}$,

$$9 + 2d = 7$$

$$d = -1$$

Substituting into $\textcircled{3}$,

$$2 \times 3 \times (-1) = a$$

$$a = -6$$

Substituting into $\textcircled{4}$,

$$(-1)^2 = b$$

$$b = 1$$

Therefore, $a = -6$, $b = 1$

Hint

$$\text{Let } x^4 + 6x^3 + 7x^2 + ax + b = (x^2 + cx + d)^2.$$

Proof of Identities

Find the values of x and y so that the following equations are true for any value of k .

Ex.

$$(k+2)x + (2k+3)y - 3k + 2 = 0$$

$$[\text{Sol}] (x+2y-3)k + (2x+3y+2) = 0 \quad \text{Arrange in powers of } k.$$

Therefore,

$$\begin{cases} x+2y-3 = 0 \quad \dots \textcircled{1} \\ 2x+3y+2 = 0 \quad \dots \textcircled{2} \end{cases}$$

Compare coefficients in
 $(x+2y-3)k + (2x+3y+2) = 0 \times k + 0.$

From $\textcircled{1}$ and $\textcircled{2}$,

$$\begin{cases} x = -13 \\ y = 8 \end{cases}$$

$$(1) \quad (k+1)x - (2k+3)y - 3k - 5 = 0$$

$$kx + x - 2ky - 3y - 3k - 5 = 0$$

$$(x-2y-3)k + (x-3y-5) = 0$$

Therefore,

$$\begin{cases} x-2y-3 = 0 \quad \dots \textcircled{1} \\ x-3y-5 = 0 \quad \dots \textcircled{2} \end{cases}$$

From $\textcircled{1} - \textcircled{2}$,

$$y = -2$$

Substituting into $\textcircled{1}$,

$$x + 4 - 3 = 0$$

$$x = -1$$

Therefore,

$$\begin{cases} x = -1 \\ y = -2 \end{cases}$$

J 185b

$$(2) \quad x^2 + y^2 + (k-2)x - 2ky - k - 4 = 0$$

$$x^2 + y^2 + kx - 2x - 2ky - k - 4 = 0$$

$$(x - 2y - 1)k + (x^2 + y^2 - 2x - 4) = 0$$

Therefore,

$$\begin{cases} x - 2y - 1 = 0 & \dots \textcircled{1} \\ x^2 + y^2 - 2x - 4 = 0 & \dots \textcircled{2} \end{cases}$$

Solving $\textcircled{1}$ for x ,

$$x = 2y + 1 \quad \dots \textcircled{3}$$

Substituting into $\textcircled{2}$,

$$(2y + 1)^2 + y^2 - 2(2y + 1) - 4 = 0$$

$$4y^2 + 4y + 1 + y^2 - 4y - 2 - 4 = 0$$

$$5y^2 - 5 = 0$$

$$y^2 = 1$$

$$y = \pm 1$$

Substituting into $\textcircled{3}$,

$$\text{When } y = 1, \quad x = 3$$

$$\text{When } y = -1, \quad x = -1$$

Therefore,


$$\begin{cases} \mathbf{x = 3} \\ \mathbf{y = 1} \end{cases} \quad \begin{cases} \mathbf{x = -1} \\ \mathbf{y = -1} \end{cases}$$

Proof of Identities

Ex.

Find the values of a and b so that the following equation is an identity.

$$\frac{3x-1}{(x-2)(x+3)} = \frac{a}{x-2} + \frac{b}{x+3}$$

[Sol] $\frac{3x-1}{(x-2)(x+3)} = \frac{a(x+3)+b(x-2)}{(x-2)(x+3)}$  Simplify the RHS.

$$= \frac{(a+b)x+3a-2b}{(x-2)(x+3)}$$

 Arrange the numerator.

Comparing coefficients in the numerators,

$$\begin{cases} a+b=3 & \dots \textcircled{1} \\ 3a-2b=-1 & \dots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}$,


$$a=1, b=2$$

1. Find the values of a and b so that the following equation is an identity.

$$\frac{x-3}{x^2-3x+2} = \frac{a}{x-1} + \frac{b}{x-2}$$

[Sol] $\frac{x-3}{(x-1)(x-2)} = \frac{a(x-2)+b(x-1)}{(x-1)(x-2)}$

$$= \frac{(a+b)x-2a-b}{(x-1)(x-2)}$$

 Factorise the LHS, and simplify the RHS.

Comparing coefficients in the numerators,

$$\begin{cases} a+b=1 & \dots \textcircled{1} \\ -2a-b=-3 & \dots \textcircled{2} \end{cases}$$

From $\textcircled{1}$, $b=-a+1$ $\dots \textcircled{3}$

Substituting $\textcircled{3}$ into $\textcircled{2}$,

$$-2a-(-a+1)=-3$$

$$-a-1=-3$$

$$a=2$$

Substituting into $\textcircled{3}$,

$$b=-1$$

Therefore, $a=2$, $b=-1$

J 186b

2. Find the values of a , b and c so that the following equation is true for any value of x .

$$\frac{4x-2}{x^3-3x^2+2x} = \frac{a}{x} + \frac{b}{x-1} + \frac{c}{x-2}$$

Factorise the LHS.

[Sol] LHS = $\frac{4x-2}{x(x^2-3x+2)} = \frac{4x-2}{x(x-1)(x-2)}$

Simplify the RHS.

$$\begin{aligned} \text{RHS} &= \frac{a(x-1)(x-2) + bx(x-2) + cx(x-1)}{x(x-1)(x-2)} \\ &= \frac{ax^2 - 3ax + 2a + bx^2 - 2bx + cx^2 - cx}{x(x-1)(x-2)} \\ &= \frac{(a+b+c)x^2 - (3a+2b+c)x + 2a}{x(x-1)(x-2)} \end{aligned}$$

Comparing coefficients in the numerators,

$$\begin{cases} a+b+c = 0 & \dots \textcircled{1} \\ -(3a+2b+c) = 4 & \dots \textcircled{2} \\ 2a = -2 & \dots \textcircled{3} \end{cases}$$

From $\textcircled{3}$, $a = -1$

Substituting into $\textcircled{1}$ and $\textcircled{2}$,

$$-1 + b + c = 0 \quad \dots \textcircled{4}$$

$$-(-3 + 2b + c) = 4 \quad \dots \textcircled{5}$$

From $\textcircled{4}$, $b = -c + 1 \quad \dots \textcircled{6}$

Substituting $\textcircled{6}$ into $\textcircled{5}$,

$$-[-3 + 2(-c + 1) + c] = 4$$

$$-(-c - 1) = 4$$

$$c + 1 = 4$$

$$c = 3$$

Substituting into $\textcircled{6}$,

$$b = -2$$

Therefore, $a = -1$, $b = -2$, $c = 3$

Proof of Identities

Prove the following identities.

Ex.

$$(x+y)^2 + (x-y)^2 = 2(x^2 + y^2)$$

$$\begin{aligned} \text{[Sol] LHS} &= x^2 + 2xy + y^2 + x^2 - 2xy + y^2 \\ &= 2x^2 + 2y^2 \end{aligned}$$

$$\text{RHS} = 2x^2 + 2y^2$$

$$\text{Thus, LHS} = \text{RHS}$$

$$(1) \quad (a^2 + b^2)(c^2 + d^2) = (ac + bd)^2 + (ad - bc)^2$$

$$\text{LHS} = a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2$$

$$\begin{aligned} \text{RHS} &= a^2c^2 + 2abcd + b^2d^2 + a^2d^2 - 2abcd + b^2c^2 \\ &= a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 \end{aligned}$$

$$\text{Thus, LHS} = \text{RHS}$$

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$$(2) \quad (x^2 - 1)(y^2 - 1) = (xy + 1)^2 - (x + y)^2$$

$$\text{LHS} = x^2y^2 - x^2 - y^2 + 1$$

$$\begin{aligned} \text{RHS} &= (x^2y^2 + 2xy + 1) - (x^2 + 2xy + y^2) \\ &= x^2y^2 - x^2 - y^2 + 1 \end{aligned}$$

Thus, LHS = RHS

$$(3) \quad (a + b + c)(ab + bc + ca) = (a + b)(b + c)(c + a) + abc$$

$$\begin{aligned} \text{LHS} &= a^2b + abc + a^2c + ab^2 + b^2c + abc + abc + bc^2 + ac^2 \\ &= a^2b + a^2c + ab^2 + b^2c + bc^2 + ac^2 + 3abc \end{aligned}$$

$$\begin{aligned} \text{RHS} &= (ab + ac + b^2 + bc)(c + a) + abc \\ &= abc + a^2b + ac^2 + a^2c + b^2c + ab^2 + bc^2 + abc + abc \\ &= a^2b + a^2c + ab^2 + b^2c + bc^2 + ac^2 + 3abc \end{aligned}$$

Thus, LHS = RHS

$$\left[\begin{array}{l} \text{Alternative Solution} \\ \text{LHS} = [(a + b) + c][ab + c(a + b)] \\ \quad = ab(a + b) + c(a + b)^2 + abc + c^2(a + b) \\ \quad = (a + b)[ab + (a + b)c + c^2] + abc \\ \quad = (a + b)(a + c)(b + c) + abc \\ \quad = \text{RHS} \end{array} \right]$$

Proof of Identities

Ex.

Given $x + y = 1$, prove that $x^2 + y = y^2 + x$.

[Sol] If $x + y = 1$, then $y = -x + 1 \dots \textcircled{1}$

$$\text{LHS} = x^2 - x + 1 \quad \rightarrow \text{Substitute } \textcircled{1}.$$

$$\text{RHS} = (-x + 1)^2 + x \quad \rightarrow \text{Substitute } \textcircled{1}.$$

$$= x^2 - x + 1$$

Thus, LHS = RHS

1. Given $a + b = 1$, prove that $a^2 + b^2 + 1 = 2(a + b - ab)$.

[Sol] If $a + b = 1$, then $b = -a + 1 \dots \textcircled{1}$


$$\text{LHS} = a^2 + (-a + 1)^2 + 1 = 2a^2 - 2a + 2$$

$$\text{RHS} = 2[a + (-a + 1) - a(-a + 1)]$$

$$= 2(a^2 - a + 1)$$

$$= 2a^2 - 2a + 2$$

Thus, LHS = RHS

 Substitute $\textcircled{1}$.

Ex.

Given $x + y = 1$, prove that $x^2 + y = y^2 + x$.

$$\begin{aligned}
 \text{[Sol] (LHS) - (RHS)} &= x^2 + y - y^2 - x \\
 &= (x + y)(x - y) - (x - y) \\
 &= (x - y)(x + y - 1) \\
 &= 0
 \end{aligned}$$

\curvearrowright $x + y = 1$, so
 $x + y - 1 = 0$.

Thus, LHS = RHS

2. Given $a + b + c = 0$, prove that $a^2 - bc = b^2 - ca$.

$$\begin{aligned}
 \text{[Sol] (LHS) - (RHS)} &= a^2 - bc - b^2 + ca \\
 &= (a + b)(a - b) + c(a - b) \\
 &= (a - b)(a + b + c) \\
 &= 0
 \end{aligned}$$

Thus, LHS = RHS

Proof of Identities

Ex.

Given $a + b + c = 0$, prove that $(a + b)(b + c)(c + a) + abc = 0$.

[Sol] From $a + b + c = 0$,

$$\begin{cases} a + b = -c \dots \textcircled{1} \\ b + c = -a \dots \textcircled{2} \\ c + a = -b \dots \textcircled{3} \end{cases}$$

Substituting $\textcircled{1}$, $\textcircled{2}$ and $\textcircled{3}$ into the LHS,

$$\begin{aligned} \text{LHS} &= (-c)(-a)(-b) + abc \\ &= -abc + abc \\ &= 0 \end{aligned}$$

Thus, LHS = RHS

1. Given $a + b + c = 0$, prove that $\frac{1}{a}(b + c) + \frac{1}{b}(c + a) + \frac{1}{c}(a + b) + 3 = 0$.

[Sol] From $a + b + c = 0$,

$$\begin{cases} a + b = -c \dots \textcircled{1} \\ b + c = -a \dots \textcircled{2} \\ c + a = -b \dots \textcircled{3} \end{cases}$$

Substituting $\textcircled{1}$, $\textcircled{2}$ and $\textcircled{3}$ into the LHS,

$$\begin{aligned} \text{LHS} &= \frac{1}{a}(-a) + \frac{1}{b}(-b) + \frac{1}{c}(-c) + 3 \\ &= -1 - 1 - 1 + 3 = 0 \end{aligned}$$

Thus, LHS = RHS

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2. Given $a + b + c = 0$, prove that $a^3 + b^3 + c^3 = 3abc$.

Hint

[Sol] If $a + b + c = 0$, then $c = -(a + b) \dots \textcircled{1}$

$$\begin{aligned} \text{LHS} &= a^3 + b^3 - (a + b)^3 \quad \text{Substitute } \textcircled{1}. \\ &= a^3 + b^3 - a^3 - 3a^2b - 3ab^2 - b^3 \\ &= -3a^2b - 3ab^2 \end{aligned}$$

Use the formula
 $(a + b)^3$
 $= a^3 + 3a^2b + 3ab^2 + b^3$

$$\begin{aligned} \text{RHS} &= -3ab(a + b) \quad \text{Substitute } \textcircled{1}. \\ &= -3a^2b - 3ab^2 \end{aligned}$$

Thus, LHS = RHS

Hint

Since $a + b + c = 0$, then $c = -(a + b)$.

Proof of Identities

1. Find the values of a , b , c and d so that the following equation is an identity.

$$x^3 + 1 = a(x-1)^3 + b(x-1)^2 + c(x-1) + d$$

[Sol] RHS = $ax^3 + (-3a+b)x^2 + (3a-2b+c)x + (-a+b-c+d)$

Comparing coefficients,

$$\begin{cases} a = 1 \\ -3a + b = 0 \\ 3a - 2b + c = 0 \\ -a + b - c + d = 1 \end{cases}$$

Therefore,

$$\mathbf{a = 1, \quad b = 3, \quad c = 3, \quad d = 2}$$

2. Given $xy = 1$, prove that $\frac{1}{x+1} + \frac{1}{y+1} = 1$.

[Sol] LHS = $\frac{y+1+x+1}{(x+1)(y+1)} = \frac{x+y+2}{xy+x+y+1} \dots \textcircled{1}$

Substituting $xy = 1$ into $\textcircled{1}$,

$$\text{LHS} = \frac{x+y+2}{1+x+y+1} = \frac{x+y+2}{x+y+2} = 1$$

Thus, LHS = RHS

Alternative Solution

$$\begin{aligned} (\text{LHS}) - (\text{RHS}) &= \frac{1}{x+1} + \frac{1}{y+1} - 1 \\ &= \frac{y+1+x+1 - (x+1)(y+1)}{(x+1)(y+1)} \\ &= \frac{y+1+x+1 - xy - x - y - 1}{(x+1)(y+1)} \\ &= \frac{1 - xy}{(x+1)(y+1)} = 0 \quad \text{☞ Substitute } xy = 1. \end{aligned}$$

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3. Given $a+b+c=0$, prove that $a\left(\frac{1}{b} + \frac{1}{c}\right) + b\left(\frac{1}{c} + \frac{1}{a}\right) + c\left(\frac{1}{a} + \frac{1}{b}\right) + 3 = 0$.

$$\begin{aligned} \text{[Sol] LHS} &= \frac{a}{b} + \frac{a}{c} + \frac{b}{c} + \frac{b}{a} + \frac{c}{a} + \frac{c}{b} + 3 \\ &= \frac{b+c}{a} + \frac{a+c}{b} + \frac{a+b}{c} + 3 \quad \dots \textcircled{1} \end{aligned}$$

Since $a+b+c=0$,

$$\begin{cases} b+c = -a \\ a+c = -b \\ a+b = -c \end{cases}$$

Substituting into $\textcircled{1}$,

$$\begin{aligned} \text{LHS} &= \frac{-a}{a} + \frac{-b}{b} + \frac{-c}{c} + 3 \\ &= -1 - 1 - 1 + 3 = 0 \end{aligned}$$

Thus, LHS = RHS

Proof of Inequalities

Given the proportion $\frac{a}{b} = \frac{c}{d}$, prove the following.

Ex.

$$\frac{a-b}{a+b} = \frac{c-d}{c+d} \quad \dots \textcircled{1}$$

[Sol] Let $\frac{a}{b} = \frac{c}{d} = k$,  Let the given proportion equal k .

$$a = bk, \quad c = dk \quad \dots \textcircled{2}$$

Substituting $\textcircled{2}$ into $\textcircled{1}$,

$$\text{LHS} = \frac{bk-b}{bk+b} = \frac{b(k-1)}{b(k+1)} = \frac{k-1}{k+1}$$

$$\text{RHS} = \frac{dk-d}{dk+d} = \frac{d(k-1)}{d(k+1)} = \frac{k-1}{k+1}$$

Thus, LHS = RHS

$$(1) \quad \frac{a+b}{a+2b} = \frac{c+d}{c+2d} \quad \dots \textcircled{1}$$

$$\text{Let } \frac{a}{b} = \frac{c}{d} = k,$$

$$a = bk, \quad c = dk \quad \dots \textcircled{2}$$

Substituting $\textcircled{2}$ into $\textcircled{1}$,

$$\text{LHS} = \frac{bk+b}{bk+2b} = \frac{b(k+1)}{b(k+2)} = \frac{k+1}{k+2}$$

$$\text{RHS} = \frac{dk+d}{dk+2d} = \frac{d(k+1)}{d(k+2)} = \frac{k+1}{k+2}$$

Thus, LHS = RHS

Note: An equation that states that two ratios are equal is called a *proportion*.

Proportions can be written as

$$\frac{a}{b} = \frac{c}{d} \quad \text{or} \quad a:b = c:d$$

Both are equivalent.

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$$(2) \quad \frac{a(a+c)}{b(b+d)} = \frac{c^2}{d^2} \quad \dots \textcircled{1}$$

$$\text{Let } \frac{a}{b} = \frac{c}{d} = k,$$

$$a = bk, \quad c = dk \quad \dots \textcircled{2}$$

Substituting $\textcircled{2}$ into $\textcircled{1}$,

$$\text{LHS} = \frac{bk(bk+dk)}{b(b+d)} = \frac{bk^2(b+d)}{b(b+d)} = k^2$$

$$\text{RHS} = \frac{(dk)^2}{d^2} = \frac{d^2k^2}{d^2} = k^2$$

Thus, LHS = RHS

$$(3) \quad \frac{b^2+d^2}{a^2+c^2} = \frac{b^2-d^2}{a^2-c^2} \quad (b^2 \neq d^2) \quad \dots \textcircled{1}$$

$$\text{Let } \frac{a}{b} = \frac{c}{d} = k,$$

$$a = bk, \quad c = dk \quad \dots \textcircled{2}$$

Substituting $\textcircled{2}$ into $\textcircled{1}$,

$$\text{LHS} = \frac{b^2+d^2}{b^2k^2+d^2k^2} = \frac{b^2+d^2}{k^2(b^2+d^2)} = \frac{1}{k^2}$$

$$\text{RHS} = \frac{b^2-d^2}{b^2k^2-d^2k^2} = \frac{b^2-d^2}{k^2(b^2-d^2)} = \frac{1}{k^2}$$

Thus, LHS = RHS

Proof of Inequalities

Ex.

Given the proportion $\frac{x}{2} = \frac{y}{3} = \frac{z}{4}$, find the value of $\frac{x-y+z}{x+y+z}$.

[Sol] Let $\frac{x}{2} = \frac{y}{3} = \frac{z}{4} = k$,

$$\begin{cases} x = 2k & \dots \textcircled{1} \\ y = 3k & \dots \textcircled{2} \\ z = 4k & \dots \textcircled{3} \end{cases}$$

Substituting $\textcircled{1}$, $\textcircled{2}$ and $\textcircled{3}$,

$$\frac{x-y+z}{x+y+z} = \frac{2k-3k+4k}{2k+3k+4k} = \frac{3k}{9k} = \frac{1}{3}$$

1. Given the proportion $\frac{x}{3} = \frac{y}{4} = \frac{z}{5}$, find the value of $\frac{xy+yz+zx}{x^2+y^2+z^2}$.

[Sol] Let $\frac{x}{3} = \frac{y}{4} = \frac{z}{5} = k$,

$$\begin{cases} x = 3k & \dots \textcircled{1} \\ y = 4k & \dots \textcircled{2} \\ z = 5k & \dots \textcircled{3} \end{cases}$$

Substituting $\textcircled{1}$, $\textcircled{2}$ and $\textcircled{3}$,

$$\begin{aligned} \frac{xy+yz+zx}{x^2+y^2+z^2} &= \frac{3k \cdot 4k + 4k \cdot 5k + 5k \cdot 3k}{(3k)^2 + (4k)^2 + (5k)^2} \\ &= \frac{47k^2}{50k^2} \\ &= \frac{47}{50} \end{aligned}$$

Ex.

Given the proportion $\frac{x+2y}{4} = \frac{2x+y}{5}$, find the value of $\frac{3xy}{x^2-xy+y^2}$.

[Sol] Let $\frac{x+2y}{4} = \frac{2x+y}{5} = k$,

$$\begin{cases} x+2y = 4k & \dots \textcircled{1} \\ 2x+y = 5k & \dots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}$, $x = 2k$ $\dots \textcircled{3}$

$$y = k \quad \dots \textcircled{4}$$

Substituting $\textcircled{3}$ and $\textcircled{4}$,

$$\frac{3xy}{x^2-xy+y^2} = \frac{3 \cdot 2k \cdot k}{(2k)^2 - 2k \cdot k + k^2} = \frac{6k^2}{3k^2} = 2$$

2. Given the proportion $\frac{3x+2y}{7} = \frac{2x+3y}{8}$, find the value of $\frac{xy^2-x^2y}{x^3+y^3}$.

[Sol] Let $\frac{3x+2y}{7} = \frac{2x+3y}{8} = k$,

$$\begin{cases} 3x+2y = 7k & \dots \textcircled{1} \\ 2x+3y = 8k & \dots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}$, $x = k$ $\dots \textcircled{3}$

$$y = 2k \quad \dots \textcircled{4}$$


Substituting $\textcircled{3}$ and $\textcircled{4}$,

$$\frac{xy^2-x^2y}{x^3+y^3} = \frac{k \cdot (2k)^2 - k^2 \cdot 2k}{k^3 + (2k)^3} = \frac{2k^3}{9k^3} = \frac{2}{9}$$

Proof of Inequalities

Ex.Given $a+b+c \neq 0$, find the value of the proportion

$$\frac{b+c}{a} = \frac{c+a}{b} = \frac{a+b}{c}.$$

[Sol] Let $\frac{b+c}{a} = \frac{c+a}{b} = \frac{a+b}{c} = k$,  Find the value of k .

$$\begin{cases} b+c = ak & \dots \textcircled{1} \\ c+a = bk & \dots \textcircled{2} \\ a+b = ck & \dots \textcircled{3} \end{cases}$$

From $\textcircled{1} + \textcircled{2} + \textcircled{3}$,

$$2(a+b+c) = k(a+b+c)$$

Since $a+b+c \neq 0$, $k = 2$

Thus, the proportion equals 2.

$a+b+c \neq 0$
This must be true in
order to divide by
 $(a+b+c)$.

$$\left[\frac{b+c}{a} = \frac{c+a}{b} = \frac{a+b}{c} = 2 \right]$$

1. Given $a+b+c \neq 0$, find the value of the proportion

$$\frac{c}{a+3b} = \frac{a}{b+3c} = \frac{b}{c+3a}.$$

[Sol] Let $\frac{c}{a+3b} = \frac{a}{b+3c} = \frac{b}{c+3a} = k$,

$$\begin{cases} c = k(a+3b) & \dots \textcircled{1} \\ a = k(b+3c) & \dots \textcircled{2} \\ b = k(c+3a) & \dots \textcircled{3} \end{cases}$$

From $\textcircled{1} + \textcircled{2} + \textcircled{3}$,

$$\begin{aligned} a+b+c &= k(4a+4b+4c) \\ &= 4k(a+b+c) \end{aligned}$$

Since $a+b+c \neq 0$, $k = \frac{1}{4}$ Thus, the proportion equals $\frac{1}{4}$. $\left[\frac{c}{a+3b} = \frac{a}{b+3c} = \frac{b}{c+3a} = \frac{1}{4} \right]$

2. Find the value of the proportion $\frac{bc+ca}{ab} = \frac{ca+ab}{bc} = \frac{ab+bc}{ca}$, in each of the following cases.

(1) When $ab+bc+ca \neq 0$

Let $\frac{bc+ca}{ab} = \frac{ca+ab}{bc} = \frac{ab+bc}{ca} = k,$

$$\begin{cases} bc+ca = kab & \dots \textcircled{1} \\ ca+ab = kbc & \dots \textcircled{2} \\ ab+bc = kca & \dots \textcircled{3} \end{cases}$$

From $\textcircled{1} + \textcircled{2} + \textcircled{3},$

$$2(ab+bc+ca) = k(ab+bc+ca)$$

Since $ab+bc+ca \neq 0,$ $k = 2$

Thus, the proportion equals **2**.

$$\left[\frac{bc+ca}{ab} = \frac{ca+ab}{bc} = \frac{ab+bc}{ca} = \mathbf{2} \right]$$

(2) When $ab+bc+ca = 0$

[Sol] From $ab+bc+ca = 0$

$$\begin{cases} bc+ca = -ab & \dots \textcircled{1} \\ ca+ab = -bc & \dots \textcircled{2} \\ ab+bc = -ca & \dots \textcircled{3} \end{cases}$$


Substituting $\textcircled{1}, \textcircled{2}$ and $\textcircled{3}$ into $\frac{bc+ca}{ab} = \frac{ca+ab}{bc} = \frac{ab+bc}{ca},$

$$\frac{-ab}{ab} = \frac{-bc}{bc} = \frac{-ca}{ca} = -1$$

Thus, the proportion equals **-1**.

$$\left[\frac{bc+ca}{ab} = \frac{ca+ab}{bc} = \frac{ab+bc}{ca} = \mathbf{-1} \right]$$

Proof of Inequalities

Ex.Given $x > y$, prove $4x + 2y > 2x + 4y$.[Sol] (LHS) - (RHS) = $(4x + 2y) - (2x + 4y)$  Prove that LHS - RHS > 0.

$$= 2x - 2y$$

$$= 2(x - y)$$

As $x > y$,

$$x - y > 0,$$

Therefore, (LHS) - (RHS) > 0

Thus, $4x + 2y > 2x + 4y$ is true.1. Given $x > y$, prove $\frac{x+y}{2} > \frac{2x+3y}{5}$.

$$[\text{Sol}] (\text{LHS}) - (\text{RHS}) = \frac{x+y}{2} - \frac{2x+3y}{5}$$

$$= \frac{x-y}{10}$$

As $x > y$,

$$x - y > 0$$

Therefore, (LHS) - (RHS) > 0

Thus, $\frac{x+y}{2} > \frac{2x+3y}{5}$ is true.

Ex.Prove $x^2 > x - 1$.

$$[\text{Sol}] \text{ (LHS)} - \text{(RHS)} = x^2 - x + 1$$

$$= \left(x - \frac{1}{2}\right)^2 + \frac{3}{4}$$

$$\text{Since } \left(x - \frac{1}{2}\right)^2 \geq 0,$$

$$\left(x - \frac{1}{2}\right)^2 + \frac{3}{4} \geq \frac{3}{4} > 0$$

Therefore, $x^2 > x - 1$

Complete the square.

2. Prove $x^2 > 2x - 3$.

$$[\text{Sol}] \text{ (LHS)} - \text{(RHS)} = x^2 - 2x + 3$$

$$= (x - 1)^2 + 2$$

$$\text{Since } (x - 1)^2 \geq 0,$$

$$(x - 1)^2 + 2 \geq 2 > 0$$

Therefore, $x^2 > 2x - 3$

Proof of Inequalities

1. Prove the following inequalities. Then state for which values the LHS equals the RHS.

Ex.

$$a^2 + b^2 \geq 2ab$$

$$\text{[Sol]} \text{ (LHS)} - \text{(RHS)} = a^2 + b^2 - 2ab$$

$$= (a - b)^2 \geq 0$$

Changes to an equal sign
when $a - b = 0$.

$$\text{Therefore, } a^2 + b^2 \geq 2ab$$

$$\text{LHS} = \text{RHS when } a = b$$

$$(1) \quad (a^2 + b^2)(x^2 + y^2) \geq (ax + by)^2$$

$$\text{(LHS)} - \text{(RHS)}$$

$$= (a^2 + b^2)(x^2 + y^2) - (ax + by)^2$$

$$= (a^2x^2 + a^2y^2 + b^2x^2 + b^2y^2) - (a^2x^2 + 2abxy + b^2y^2)$$

$$= a^2y^2 - 2abxy + b^2x^2$$

$$= (ay - bx)^2 \geq 0$$

Changes to an equal sign
when $ay - bx = 0$.

$$\text{Thus, } (a^2 + b^2)(x^2 + y^2) \geq (ax + by)^2$$

$$\text{LHS} = \text{RHS when } ay - bx = 0, \text{ i.e. when } ay = bx$$

2. Prove the following inequalities. Then state for which values the LHS equals the RHS.

Ex.

$$a^2 + b^2 \geq ab$$

$$[\text{Sol}] (\text{LHS}) - (\text{RHS}) = a^2 + b^2 - ab$$

$$= \left(a - \frac{1}{2}b\right)^2 + \frac{3}{4}b^2$$

$$\text{Since } \left(a - \frac{1}{2}b\right)^2 \geq 0, \text{ and } \frac{3}{4}b^2 \geq 0,$$

$$\left(a - \frac{1}{2}b\right)^2 + \frac{3}{4}b^2 \geq 0$$

Changes to an equal sign when

$$a = \frac{1}{2}b, \text{ and } \frac{3}{4}b^2 = 0.$$

$$\text{Therefore, } a^2 + b^2 \geq ab$$

$$\text{LHS} = \text{RHS} \text{ when } a = b = 0$$

$$\frac{3}{4}b^2 = 0 \text{ if and only if } b = 0$$

$$a = \frac{1}{2}b, \text{ so } a = 0.$$

$$(1) \quad a^2 + b^2 \geq 2(a - b - 1)$$

$$(\text{LHS}) - (\text{RHS}) = a^2 + b^2 - 2(a - b - 1)$$

$$= a^2 - 2a + b^2 + 2b + 2$$

$$= (a^2 - 2a + 1) + (b^2 + 2b + 1)$$

$$= (a - 1)^2 + (b + 1)^2$$

$$\text{Since } (a - 1)^2 \geq 0, \text{ and } (b + 1)^2 \geq 0,$$

$$(a - 1)^2 + (b + 1)^2 \geq 0$$

$$\text{Therefore, } a^2 + b^2 \geq 2(a - b - 1)$$

$$\text{LHS} = \text{RHS} \text{ when } a - 1 = 0 \text{ and } b + 1 = 0, \text{ i.e. when}$$

$$a = 1, b = -1.$$

Proof of Inequalities

1. Prove the following inequalities. Then state the conditions for which the LHS equals the RHS.

Ex.

Given $a + b = 1$, prove the inequality $\frac{1}{2} \geq 2ab$.

[Sol] $a + b = 1$, so $b = 1 - a$

$$(\text{LHS}) - (\text{RHS}) = \frac{1}{2} - 2a(1 - a) \quad \text{Substitute } b = 1 - a.$$

$$= 2a^2 - 2a + \frac{1}{2}$$

$$= 2\left(a - \frac{1}{2}\right)^2 \geq 0 \quad \Rightarrow$$

Changes to an equal sign
when $a - \frac{1}{2} = 0$.

Therefore, $\frac{1}{2} \geq 2ab$

Thus, LHS = RHS when $a = b = \frac{1}{2} \Rightarrow$

If $a - \frac{1}{2} = 0$ and $a + b = 1$,
then $a = \frac{1}{2}$ and $b = \frac{1}{2}$.

- (1) Given $a + b = 1$, prove the inequality $a^2 + b^2 \geq \frac{1}{2}$.

$a + b = 1$, so $b = 1 - a$

$$(\text{LHS}) - (\text{RHS}) = a^2 + (1 - a)^2 - \frac{1}{2} \quad \text{Substitute } b = 1 - a.$$

$$= 2a^2 - 2a + \frac{1}{2}$$

$$= 2\left(a - \frac{1}{2}\right)^2 \geq 0$$

Therefore, $a^2 + b^2 \geq \frac{1}{2}$

Thus, LHS = RHS when $a = b = \frac{1}{2} \Rightarrow$

If $a - \frac{1}{2} = 0$ and $a + b = 1$,
then $a = \frac{1}{2}$ and $b = \frac{1}{2}$.

J 196b

2. Prove the following inequalities.

(1) Given $a + b = 1$, prove $a^2 + b^2 > ab$.

$$a + b = 1, \text{ so } b = 1 - a$$

$$\begin{aligned}(\text{LHS}) - (\text{RHS}) &= a^2 + (1 - a)^2 - a(1 - a) && \text{Substitute } b = 1 - a. \\ &= 3a^2 - 3a + 1 \\ &= 3\left(a - \frac{1}{2}\right)^2 + \frac{1}{4} \geq \frac{1}{4} > 0\end{aligned}$$

Thus, $a^2 + b^2 > ab$

(2) Given $a + b = 1$, prove $a + b > ab$.

$$a + b = 1, \text{ so } b = 1 - a$$

$$\begin{aligned}(\text{LHS}) - (\text{RHS}) &= a + (1 - a) - a(1 - a) \\ &= a^2 - a + 1 \\ &= \left(a - \frac{1}{2}\right)^2 + \frac{3}{4} \geq \frac{3}{4} > 0\end{aligned}$$

Thus, $a + b > ab$

Proof of Inequalities

1. Given that $a > 0$ and $b > 0$, prove the following inequalities.

Then state the conditions for which the LHS equals the RHS.

Ex.

$$\frac{a+b}{2} \geq \sqrt{ab}$$

$$\begin{aligned} \text{[Sol] (LHS) - (RHS)} &= \frac{a+b-2\sqrt{ab}}{2} \\ &= \frac{(\sqrt{a})^2 + (\sqrt{b})^2 - 2\sqrt{ab}}{2} \\ &= \frac{(\sqrt{a} - \sqrt{b})^2}{2} \geq 0 \end{aligned}$$

Changes to an equal sign
when $\sqrt{a} - \sqrt{b} = 0$.

Therefore, $\frac{a+b}{2} \geq \sqrt{ab}$

LHS = RHS when $a = b$

If $\sqrt{a} - \sqrt{b} = 0$, then $a = b$.

(1) $\sqrt{ab} \geq \frac{2ab}{a+b}$

$$\begin{aligned} \text{(LHS) - (RHS)} &= \frac{\sqrt{ab}(a+b) - 2ab}{a+b} \\ &= \frac{\sqrt{ab}(a+b-2\sqrt{ab})}{a+b} \\ &= \frac{\sqrt{ab}(\sqrt{a} - \sqrt{b})^2}{a+b} \geq 0 \end{aligned}$$

Therefore, $\sqrt{ab} \geq \frac{2ab}{a+b}$

LHS = RHS when $a = b$

If $\sqrt{a} - \sqrt{b} = 0$, then $a = b$.

Given two positive numbers, a and b , $\frac{a+b}{2}$ is called the **arithmetic mean**, and \sqrt{ab} is called the **geometric mean**.

Relationship between arithmetic mean and geometric mean

When $a > 0$ and $b > 0$, then $\frac{a+b}{2} \geq \sqrt{ab}$

LHS = RHS when $a = b$.

$$\left(\begin{array}{c} \text{arithmetic} \\ \text{mean} \end{array} \right) \geq \left(\begin{array}{c} \text{geometric} \\ \text{mean} \end{array} \right)$$

You may use this relationship in the following proofs.

Ex.

Given that $x > 0$, prove $x + \frac{1}{x} \geq 2$. Then state the conditions for which the LHS equals the RHS.

[Sol] $\frac{\text{LHS}}{2} = \frac{x + \frac{1}{x}}{2} \geq \sqrt{x \cdot \frac{1}{x}}$

$$\frac{x + \frac{1}{x}}{2} \geq 1$$

Thus, $x + \frac{1}{x} \geq 2$

LHS = RHS when $x = 1$

Use $\left(\begin{array}{c} \text{arithmetic} \\ \text{mean} \end{array} \right) \geq \left(\begin{array}{c} \text{geometric} \\ \text{mean} \end{array} \right)$

i.e. $\frac{a+b}{2} \geq \sqrt{ab}$,
with x as a , and $\frac{1}{x}$ as b .

From $x = \frac{1}{x}$.

2. Given that $x > 0$ and $y > 0$, prove $\frac{x}{y} + \frac{y}{x} \geq 2$. Then state the conditions for which the LHS equals the RHS.

[Sol] $\frac{\text{LHS}}{2} = \frac{\frac{x}{y} + \frac{y}{x}}{2} \geq \sqrt{\frac{x}{y} \cdot \frac{y}{x}}$

$$\frac{\frac{x}{y} + \frac{y}{x}}{2} \geq 1$$

Thus, $\frac{x}{y} + \frac{y}{x} \geq 2$

LHS = RHS when $x = y$

Use $\frac{a+b}{2} \geq \sqrt{ab}$,

with $\frac{x}{y}$ as a , and $\frac{y}{x}$ as b .

From $\frac{x}{y} = \frac{y}{x}$.

Proof of Inequalities

1. Prove the following inequalities when $a > 0$. Then state the conditions for which the LHS equals the RHS.

$$(1) \quad a + \frac{9}{a} \geq 6$$

$$\frac{\text{LHS}}{2} = \frac{a + \frac{9}{a}}{2} \geq \sqrt{a \cdot \frac{9}{a}} \quad \Rightarrow \quad \text{Use } \left(\begin{array}{l} \text{arithmetic} \\ \text{mean} \end{array} \right) \geq \left(\begin{array}{l} \text{geometric} \\ \text{mean} \end{array} \right).$$

$$\frac{a + \frac{9}{a}}{2} \geq 3$$

$$\text{Therefore, } a + \frac{9}{a} \geq 6$$

$$\text{LHS} = \text{RHS when } a = 3 \quad \Rightarrow \quad \text{From } a = \frac{9}{a}.$$

$$(2) \quad 2a + \frac{1}{8a} \geq 1$$

$$\frac{\text{LHS}}{2} = \frac{2a + \frac{1}{8a}}{2} \geq \sqrt{2a \cdot \frac{1}{8a}} \quad \Rightarrow \quad \text{Use } \left(\begin{array}{l} \text{arithmetic} \\ \text{mean} \end{array} \right) \geq \left(\begin{array}{l} \text{geometric} \\ \text{mean} \end{array} \right).$$

$$\frac{2a + \frac{1}{8a}}{2} \geq \frac{1}{2}$$

$$\text{Therefore, } 2a + \frac{1}{8a} \geq 1$$

$$\text{LHS} = \text{RHS when } a = \frac{1}{4} \quad \Rightarrow \quad \text{From } 2a = \frac{1}{8a}.$$

2. Prove the following inequalities when $a > 0$ and $b > 0$. Then state the conditions for which the LHS equals the RHS.

Ex.

$$\left(a + \frac{1}{b}\right)\left(b + \frac{1}{a}\right) \geq 4$$

[Sol] LHS = $2 + ab + \frac{1}{ab}$

$a > 0$ and $b > 0$, so $ab > 0$ and $\frac{1}{ab} > 0$

Therefore,

$$\begin{aligned} 2 + ab + \frac{1}{ab} &\geq 2 + 2\sqrt{ab \cdot \frac{1}{ab}} \\ &= 2 + 2 \\ &= 4 \end{aligned}$$

Use $\left(\begin{smallmatrix} \text{arithmetic} \\ \text{mean} \end{smallmatrix}\right) \geq \left(\begin{smallmatrix} \text{geometric} \\ \text{mean} \end{smallmatrix}\right)$
in the form $A + B \geq 2\sqrt{AB}$
with ab as A and $\frac{1}{ab}$ as B .

Thus, $\left(a + \frac{1}{b}\right)\left(b + \frac{1}{a}\right) \geq 4$

LHS = RHS when $ab = 1$ From $ab = \frac{1}{ab}$.

(1) $\left(a + \frac{2}{b}\right)\left(b + \frac{8}{a}\right) \geq 18$

LHS = $\left(a + \frac{2}{b}\right)\left(b + \frac{8}{a}\right) = 10 + ab + \frac{16}{ab}$

$a > 0$ and $b > 0$, so $ab > 0$ and $\frac{16}{ab} > 0$

Therefore,

$$\begin{aligned} 10 + ab + \frac{16}{ab} &\geq 10 + 2\sqrt{ab \cdot \frac{16}{ab}} \\ &= 10 + 8 \\ &= 18 \end{aligned}$$

Use $\left(\begin{smallmatrix} \text{arithmetic} \\ \text{mean} \end{smallmatrix}\right) \geq \left(\begin{smallmatrix} \text{geometric} \\ \text{mean} \end{smallmatrix}\right)$.

Thus, $\left(a + \frac{2}{b}\right)\left(b + \frac{8}{a}\right) \geq 18$

LHS = RHS when $ab = 4$ From $ab = \frac{16}{ab}$.

Proof of Inequalities

1. Prove the following inequality when $a > 0$ and $b > 0$. Then state the conditions for which the LHS equals the RHS.

$$(a+b)\left(\frac{1}{a} + \frac{1}{b}\right) \geq 4$$

[Sol] LHS = $(a+b)\left(\frac{1}{a} + \frac{1}{b}\right) = 2 + \frac{a}{b} + \frac{b}{a}$

$a > 0$ and $b > 0$, so $\frac{a}{b} > 0$ and $\frac{b}{a} > 0$

Therefore,

$$2 + \frac{a}{b} + \frac{b}{a} \geq 2 + 2\sqrt{\frac{a}{b} \cdot \frac{b}{a}} = 4 \quad \Rightarrow \quad \text{Use } \left(\text{arithmetic mean}\right) \geq \left(\text{geometric mean}\right).$$

Thus, $(a+b)\left(\frac{1}{a} + \frac{1}{b}\right) \geq 4$

LHS = RHS when $a = b \quad \Rightarrow \quad \text{From } \frac{a}{b} = \frac{b}{a}.$

2. Prove the following inequality when $a > 0$, $b > 0$, $c > 0$ and $d > 0$. Then state the conditions for which the LHS equals the RHS.

$$\left(\frac{b}{a} + \frac{d}{c}\right)\left(\frac{a}{b} + \frac{c}{d}\right) \geq 4$$

[Sol] LHS = $\left(\frac{b}{a} + \frac{d}{c}\right)\left(\frac{a}{b} + \frac{c}{d}\right) = 2 + \frac{bc}{ad} + \frac{ad}{bc}$

$a > 0$, $b > 0$, $c > 0$, $d > 0$, so $\frac{bc}{ad} > 0$ and $\frac{ad}{bc} > 0$

Therefore,

$$2 + \frac{bc}{ad} + \frac{ad}{bc} \geq 2 + 2\sqrt{\frac{bc}{ad} \cdot \frac{ad}{bc}} = 4 \quad \Rightarrow \quad \text{Use } \left(\text{arithmetic mean}\right) \geq \left(\text{geometric mean}\right).$$

Thus, $\left(\frac{b}{a} + \frac{d}{c}\right)\left(\frac{a}{b} + \frac{c}{d}\right) \geq 4$

LHS = RHS when $ad = bc \quad \Rightarrow \quad \text{From } \frac{bc}{ad} = \frac{ad}{bc}.$

Let's think about this!

We can prove that $\left(a + \frac{1}{b}\right)\left(b + \frac{1}{a}\right) \geq 4$ is the example on J198b, in another way, as follows.

From $a > 0$ and $b > 0$, $\frac{1}{a} > 0$ and $\frac{1}{b} > 0$

$$a + \frac{1}{b} \geq 2\sqrt{a \cdot \frac{1}{b}} = 2\sqrt{\frac{a}{b}} \text{ and, from } a = \frac{1}{b}, \text{ LHS} = \text{RHS when } ab = \boxed{1}$$

$$b + \frac{1}{a} \geq 2\sqrt{b \cdot \frac{1}{a}} = 2\sqrt{\frac{b}{a}} \text{ and, from } b = \frac{1}{a}, \text{ LHS} = \text{RHS when } ab = \boxed{1}$$

Multiplying the above inequalities gives,

$$\left(a + \frac{1}{b}\right)\left(b + \frac{1}{a}\right) \geq 2\sqrt{\frac{a}{b}} \cdot 2\sqrt{\frac{b}{a}} = 4\sqrt{\frac{a}{b} \cdot \frac{b}{a}} = 4$$

Therefore, $\left(a + \frac{1}{b}\right)\left(b + \frac{1}{a}\right) \geq 4$ and LHS = RHS when $ab = \boxed{1}$

Let's consider if we can prove that $\left(a + \frac{1}{b}\right)\left(b + \frac{4}{a}\right) \geq 9$ using the method shown above.

$$a + \frac{1}{b} \geq 2\sqrt{\frac{a}{b}} \text{ and, from } a = \frac{1}{b}, \text{ LHS} = \text{RHS when } ab = \boxed{1}$$

$$b + \frac{4}{a} \geq 2\sqrt{b \cdot \frac{4}{a}} = 4\sqrt{\frac{b}{a}} \text{ and, from } b = \frac{4}{a}, \text{ LHS} = \text{RHS when } ab = \boxed{4}$$

If we multiply both sides of the above inequalities,

$$\left(a + \frac{1}{b}\right)\left(b + \frac{4}{a}\right) \geq 2\sqrt{\frac{a}{b}} \cdot 4\sqrt{\frac{b}{a}} = \boxed{8}$$

This is different from the original inequality. Therefore, we cannot prove the inequality using this method.

In this case, we should use the method shown on J198b, i.e. expand first, then use (*arithmetic mean*) \geq (*geometric mean*).

J200a KUMON

Proof of Inequalities

1. Given the proportion $\frac{x+y}{3} = \frac{y+z}{4} = \frac{z+x}{5}$, find the value of

$$\frac{xy + yz + zx}{x^2 + y^2 + z^2}.$$

[Sol] Let $\frac{x+y}{3} = \frac{y+z}{4} = \frac{z+x}{5} = k$,

$$\begin{cases} x+y = 3k & \dots \textcircled{1} \\ y+z = 4k & \dots \textcircled{2} \\ z+x = 5k & \dots \textcircled{3} \end{cases}$$

From $\textcircled{1}$, $\textcircled{2}$ and $\textcircled{3}$, $x = 2k$, $y = k$, $z = 3k$,

Substituting $x = 2k$, $y = k$, $z = 3k$,

$$\frac{xy + yz + zx}{x^2 + y^2 + z^2} = \frac{2k^2 + 3k^2 + 6k^2}{4k^2 + k^2 + 9k^2} = \frac{11k^2}{14k^2} = \frac{11}{14}$$

2. Given that $a > 0$, $b > 0$, prove that $(a+b)\left(\frac{1}{a} + \frac{4}{b}\right) \geq 9$. Then state the conditions for which the LHS equals the RHS.

[Sol] $(a+b)\left(\frac{1}{a} + \frac{4}{b}\right) = 5 + \frac{4a}{b} + \frac{b}{a}$

$a > 0$, $b > 0$, so $\frac{4a}{b} > 0$ and $\frac{b}{a} > 0$

$$\begin{aligned} 5 + \frac{4a}{b} + \frac{b}{a} &\geq 5 + 2\sqrt{\frac{4a}{b} \cdot \frac{b}{a}} && \text{Use } \left(\text{arithmetic mean}\right) \geq \left(\text{geometric mean}\right). \\ &= 5 + 4 \\ &= 9 \end{aligned}$$

Therefore, $(a+b)\left(\frac{1}{a} + \frac{4}{b}\right) \geq 9$

LHS = RHS when $2a = b$ From $\frac{4a}{b} = \frac{b}{a}$.

3. Prove that $a^2 + b^2 + c^2 \geq ab + bc + ca$. Then state the conditions for which the LHS equals the RHS.

$$\begin{aligned}
 \text{[Sol]} \quad (\text{LHS}) - (\text{RHS}) &= a^2 + b^2 + c^2 - (ab + bc + ca) \\
 &= a^2 - a(b+c) + (b^2 - bc + c^2) \\
 &= \left(a - \frac{b+c}{2}\right)^2 - \frac{(b+c)^2}{4} + b^2 - bc + c^2 \\
 &= \left(a - \frac{b+c}{2}\right)^2 + \frac{3b^2 - 6bc + 3c^2}{4} \\
 &= \left(a - \frac{b+c}{2}\right)^2 + \frac{3}{4}(b^2 - 2bc + c^2) \\
 &= \left(a - \frac{b+c}{2}\right)^2 + \frac{3}{4}(b-c)^2 \geq 0
 \end{aligned}$$

Thus, $a^2 + b^2 + c^2 \geq ab + bc + ca$

LHS = RHS when $a - \frac{b+c}{2} = 0$ and $b - c = 0$, i.e. when $a = b = c$.

Let's think about this!

Alternative solution to prove the inequality $a^2 + b^2 + c^2 \geq ab + bc + ca$

[Sol]

$$\begin{aligned}
 (\text{LHS}) - (\text{RHS}) &= a^2 + b^2 + c^2 - (ab + bc + ca) \\
 &= \frac{1}{2} [(a^2 - 2ab + b^2) + (b^2 - 2bc + c^2) + (c^2 - 2ca + a^2)] \\
 &= \frac{1}{2} [(a-b)^2 + (b-c)^2 + (c-a)^2] \geq 0
 \end{aligned}$$

Thus, $a^2 + b^2 + c^2 \geq ab + bc + ca$

LHS = RHS when $a - b = 0$, $b - c = 0$, $c - a = 0$, which means when $a = b = c$.